NEW SOLUTIONS FOR FINITE DEFORMATION

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1. Introduction

In recent years the theory of finite deformation has received fresh impetus from a number of exact solutions that have been obtained for imcompressible elastic bodies without assuming any form of the strain energy function. The exact solutions obtained for compressible bodies have not been very large, and in general they have been of the form obtained by B. R. Seth. Extensive references to existing literature on the subject are given in the Presidential Address to the Section of Mathematics at the 42nd Session of the Indian Science Congress.[1]

Very few attempts have been made to obtain solutions for the finite bending of plates into shells. The earliest exact solution on the basis of a linear stress strain tensor law was obtained by Seth[2] for a rectangular plate bent into a right cylindrical shell. The form of this solution has been used by many workers. In the present paper an attempt has been made to discuss solutions which may be used to discuss the finite bending of plates into spherical shells. A linear stress strain tensor law is again assumed, and this gives sufficiently good results for technical applications.

In cylindrical co-ordinates it is found that, if we take

$$u = r - \phi(r, z), \quad v = \theta - A\theta, \quad w = z - f(r^2 + z^2),$$

all the equations of finite deformation and boundary conditions on a spherical shell can be satisfied only if

$$\phi = ar, \; f(r^2 + z^2) = -\frac{2c_1}{1 - c}(r^2 + z^2)^{\frac{1}{2}(1-c)},$$

where a, A, c_1 are known constants and $c = (1 - 2\sigma)/(1 - \sigma)$ is an elastic constant.

2. Finite Strain Components in Cylindrical Co-ordinates

For all applications finite strain components should be referred to the strained state of the body. Let ${}^{i}x$ be the unstrained and x^i the strained co-ordinates of the deformed body with reference to a fixed frame with ${}_{ij}g$ and g_{ij} as metric tensors. The strain tensor e_{ij} is given by[3]

$$2\epsilon_{ij} = g_{ij} - {}_{\alpha\beta}g\,({}^{\alpha}x,_{i})\,({}^{\beta}x,_{j}). \tag{2.1}$$

Putting $u^i = x^i - {}^ix$, (2.1) becomes

$$2\epsilon_{ij} = g_{ij} - {}_{ij}g + {}_{\alpha j}g u^\alpha{}_{,i} + {}_{i\beta}g u^\beta{}_{,j} - {}_{\alpha\beta}g u^\alpha{}_{,i} u^\beta{}_{,j}. \tag{2.2}$$

In cylindrical co-ordinates

$${}^ix = r_0,\ \theta_0,\ z_0,\ x^i = r,\ \theta,\ z,$$

$$u = r - r_0,\ v = \theta - \theta_0,\ w = z - z_0,$$

$${}_{ij}g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r_0^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.3}$$

The tensor components ϵ_{ij} can be replaced by the physical components e_{ij} by the relation[4]

$$e_{ij} = (g^{ii}g^{jj})^{\frac{1}{2}}\epsilon_{ij}. \tag{2.4}$$

Thus we get the components of finite strain in cylindrical co-ordinates as[5]

$$2e_{rr} = 2\frac{\partial u}{\partial r} - \left[\left(\frac{\partial u}{\partial r}\right)^2 + r^2\left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2\right]$$
$$+ 2ru\left(\frac{\partial v}{\partial r}\right)^2 - u^2\left(\frac{\partial v}{\partial r}\right)^2,$$

$$2e_{\theta\theta} = 2\frac{u}{r} + 2\frac{\partial v}{\partial \theta} - \frac{u^2}{r^2} - 4\frac{u}{r}\frac{\partial v}{\partial \theta}$$
$$-\left[\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 + \left(\frac{\partial v}{\partial \theta}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2\right]$$
$$+ 2\frac{u^2}{r^2}\frac{\partial v}{\partial \theta} - 2\frac{u}{r}\left(\frac{\partial v}{\partial \theta}\right)^2 + \frac{u^2}{r^2}\left(\frac{\partial v}{\partial \theta}\right)^2,$$

$$2e_{zz} = 2\frac{\partial w}{\partial z} - \left[\left(\frac{\partial u}{\partial z}\right)^2 + r^2\left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]$$
$$+ 2ru\left(\frac{\partial v}{\partial z}\right)^2 - u^2\left(\frac{\partial v}{\partial z}\right)^2,$$

$$2e_{\theta z} = \frac{1}{r}\frac{\partial w}{\partial \theta} + r\frac{\partial v}{\partial z} - 2u\frac{\partial v}{\partial z}$$

$$- \left[\frac{1}{r}\frac{\partial u}{\partial \theta}\frac{\partial u}{\partial z} + r\frac{\partial v}{\partial \theta}\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}\frac{\partial w}{\partial z}\right]$$

$$- \frac{u^2}{r^2}\frac{\partial v}{\partial \theta} + 2u\frac{\partial v}{\partial \theta}\frac{\partial v}{\partial z} - \frac{u^2}{r}\frac{\partial v}{\partial \theta}\frac{\partial v}{\partial z},$$

$$2e_{zr} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} - \left[\frac{\partial u}{\partial z}\frac{\partial u}{\partial r} + r^2\frac{\partial v}{\partial z}\frac{\partial v}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial r}\right]$$

$$+ 2ru\frac{\partial v}{\partial z}\frac{\partial v}{\partial r} - u^2\frac{\partial v}{\partial z}\frac{\partial v}{\partial r},$$

$$2e_{r\theta} = r\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial r} - 2u\frac{\partial v}{\partial r}$$

$$- \frac{1}{r}\left[\frac{\partial u}{\partial r}\frac{\partial u}{\partial \theta} + r\frac{\partial v}{\partial r}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}\right]$$

$$+ \frac{u^2}{r}\frac{\partial v}{\partial r} + 2u\frac{\partial v}{\partial r}\frac{\partial v}{\partial \theta} - \frac{u^2}{r}\frac{\partial v}{\partial r}\frac{\partial v}{\partial \theta}. \qquad (2.5)$$

We have now to use the semi-inverse method of assuming a possible set of displacements. For the type of problem in view we should take

$$u = r - \phi(r, z),\ v = \theta - A\theta,\ w = z - f(r^2 + z^2), \qquad (2.6)$$

where f and ϕ are arbitrary functions to be determined and A is a constant. The corresponding strain components are found to be

$$e_{rr} = \tfrac{1}{2}(1 - \phi_r^2) - 2r^2 f'^2,\ e_{\theta\theta} = K_0 - \tfrac{1}{2}K\frac{\phi^2}{r^2},$$

$$e_{zz} = \tfrac{1}{2}(1 - \phi_z^2 - 4z^2 f'^2),\ e_{rz} = -\tfrac{1}{2}(\phi_r\phi_z + 4rzf'^2),$$

$$e_{\theta z} = 0, \quad e_{r\theta} = 0, \qquad (2.7)$$

where

$$K_0 = \tfrac{1}{2} - (1 - A)^2,\ \ K_1 = 1 - 2(1 - A) - (1 - A)^2, \qquad (2.8)$$

$$\phi_r = \frac{\partial \phi}{\partial r},\ \phi_z = \frac{\partial \phi}{\partial z},\ f' = \frac{\partial f}{\partial R^2},\ R^2 = r^2 + z^2.$$

3. STRESS-STRAIN RELATIONS AND BODY-STRESS EQUATIONS

We assume that

$$\tau_{ij} = \lambda \delta_{ij} \Delta + 2\mu e_{ij}. \tag{3.1}$$

The body-stress equations of equilibrium are

$$\tau_{ij,\, j} = 0, \tag{3.2}$$

and

$$\Delta = 1 - K_0 - 2(r^2 + z^2) f'^2 - \tfrac{1}{2} K_1 \frac{\phi^2}{r^2} - \tfrac{1}{2}(\phi_r^2 + \phi_z^2). \tag{3.3}$$

Substituting the value of e_{ij} from (2.7) we see that the first equation gives

$$\frac{\partial}{\partial r}[\lambda\Delta - 8\mu \int f'^2 dR^2 - 8\mu \int R^2 f' f'' dR^2 - \mu(\phi_r^2 + \phi_z^2)]$$

$$+ \frac{\mu}{r}\left(1 - 2K_0 + K_1 \frac{\phi^2}{r^2}\right) - \mu\phi_r^2 \left[\frac{\partial}{\partial z}\left(\frac{\phi_z}{\phi_r}\right) + \frac{1}{r}\right] = 0. \tag{3.4}$$

The second is identically satisfied and the third gives

$$\frac{\partial}{\partial z}[\lambda\Delta - 8\mu \int f'^2 dR^2 - 8\mu \int R^2 f' f'' dR^2 - \mu(\phi_r^2 + \phi_z^2)]$$

$$+ \mu\phi_r^2 \left[\frac{\partial}{\partial r}\left(\frac{\phi_z}{\phi_r}\right) - \frac{1}{r}\frac{\phi_z}{\phi_r}\right] = 0. \tag{3.5}$$

These will be satisfied by

$$P \equiv \lambda\Delta - 8\mu \int f'^2 dR^2 - 8\mu \int R^2 f' f'' dR^2 - \mu(\phi_r^2 + \phi_z^2)$$

$$= a \text{ constant}, \qquad R^2 = r^2 + z^2. \tag{3.6}$$

provided

$$\frac{\partial}{\partial r}\left(\frac{\phi_z}{\phi_r}\right) - \frac{1}{r}\frac{\phi_z}{\phi_r} = 0. \tag{3.7}$$

$$\frac{1}{r}\left(1 - 2K_0 + \frac{K_1\phi^2}{r^2}\right) - \phi_r^2\left[\frac{\partial}{\partial z}\left(\frac{\phi_z}{\phi_r}\right) + \frac{1}{r}\right] = 0. \tag{3.8}$$

From (3.7) we get

$$\frac{\phi_z}{\phi_r} = r\chi'(z), \tag{3.9}$$

which gives

$$\phi = F[\chi(z) + \log r], \tag{3.10}$$

where χ and F are arbitrary functions of z and $\chi(z) + \log r$ respectively. Substituting this value of ϕ in (3.8) we get

$$(1 - 2K_0) r^2 + K_1 F^2 - r^2 F'^2 \chi''(z) - F'^2 = 0. \tag{3.11}$$

From (3.6) we see that

$$\tfrac{1}{2}\lambda K_1 \frac{\phi^2}{r^2} + (\tfrac{1}{2}\lambda + \mu)(\phi_r^2 + \phi_z^2) = M(r^2 + z^2), \tag{3.12}$$

M being an arbitrary function of $(r^2 + z^2)$. Substituting from (3.10) we get

$$K_1 F^2 + \frac{\lambda + 2\mu}{\lambda}\left[r^2 F'^2 \chi'^2(z) + F'^2\right] = \frac{2r^2 M}{\lambda}. \tag{3.13}$$

Comparing (3.11) with (3.13) we get

$$\left.\begin{aligned} M &= -\tfrac{1}{2}\lambda(1 - 2K_0) \\ \chi'' &= \chi'^2, \end{aligned}\right\} \tag{3.14}$$

Thus

$$\chi = -\log(cz + D),$$

and

$$\phi = F\left[\log \frac{r}{cz + D}\right]. \tag{3.15}$$

Also

$$\left[\frac{r^2 c^2}{(cz + D)^2} + 1\right] F'^2 = (1 - 2K_0) r^2 + K_1 F^2,$$

which shows that $c = 0$ and

$$\begin{aligned} F &= a e^{\log r}, \quad \text{omitting } a \text{ constant}, \\ &= ar, \end{aligned} \tag{3.16}$$

where

$$1 - 2K_0 + K_1 a^2 - a^2 = 0,$$

or

$$(1 - A)[(1 - A) - (3 - A) a^2] = 0,$$

$$A = 1, \quad a^2 = \frac{1 - A}{3 - A}. \tag{3.17}$$

Thus we see that the value of ϕ is

$$\phi = ar = \sqrt{\frac{1 - A}{3 - A}}\, r, \quad \text{if } A \neq 1. \tag{3.18}$$

4. DETERMINATION OF f

Substituting the value of ϕ from (3.18) in (3.6) we get,

$$\lambda\,[1 - K_0 - \tfrac{1}{2}K_1 a^2 - \tfrac{1}{2}a^2 - 2R^2 f'^2] - \mu a^2 - 8\mu \int f'^2 dR^2$$
$$- 8\mu \int R^2 f' f'' dR^2 = a \text{ constant.} \tag{4.1}$$

By differentiating (4.1) with respect to $X = R^2$ we get

$$X(\lambda + 2\mu)\frac{df'^2}{dX} + (\lambda + 4\mu) f'^2 = 0, \tag{4.2}$$

which gives

$$f'^2 = \frac{c_1{}^2}{(r^2 + z^2)^{1+c}}, \quad c = \frac{2\mu}{\lambda + 2\mu} \tag{4.3}$$

$$f = \frac{2c_1}{1 - c}(r^2 + z^2)^{\frac{1}{2}(1-c)}, \tag{4.4}$$

neglecting a constant.

The components of displacements now take the form

$$u = r - \alpha r, \; v = \theta - A\theta, \; w = z - \frac{2c_1}{1 - c}(r^2 + z^2)^{\frac{1}{2}(1-c)}, \tag{4.5}$$

where c is the elastic constant $(1 - 2\sigma)/(1 - \sigma)$ and α, A, c_1 are constants to be determined from the boundary conditions and equation (3.17). The non-vanishing stress components are

$$\tau_{rr} = \lambda\Delta + 2\mu e_{rr}$$
$$= \lambda(1 - K_0 - \tfrac{1}{2}K_1 a^2 - \tfrac{1}{2}a^2) + \mu(1 - a^2)$$
$$- 2\lambda c_1{}^2 R^{-2} - 4\mu r^2 c_1{}^2 R^{-2(1+c)},$$
$$\tau_{\theta\theta} = \lambda(1 - K_0 - \tfrac{1}{2}K_1 a^2 - \tfrac{1}{2}a^2) + \mu(K_0 - \tfrac{1}{2}K a^2) - 2\lambda c_1{}^2 R^{-2},$$
$$\tau_{zz} = \lambda(1 - K_0 - \tfrac{1}{2}K_1 a^2 - \tfrac{1}{2}a^2) + \mu - 2\lambda c_1{}^2 R^{-2}$$
$$- 4\mu z^2 c_1{}^2 R^{-2(1+c)},$$
$$\tau_{rz} = -4\mu r z c_1{}^2 R^{-2(1+c)},$$
$$\tau_{r\theta} = 0, \quad \tau_{\theta z} = 0. \tag{4.6}$$

5. BOUNDARY CONDITIONS FOR A SPHERICAL SHELL

If N denotes the direction of the normal at any point of the shell, the boundary tractions in the direction of r, θ, z corresponding to the general set of displacements given in (2.6) are

$$N_r = \frac{r}{\sqrt{r^2+z^2}}[\lambda\Delta + \mu - 4\mu(r^2+z^2)f'^2]$$

$$- \frac{\mu\phi_r}{\sqrt{r^2+z^2}}(r\phi_r + z\phi_z), \tag{5.1}$$

$$N_z = \frac{z}{\sqrt{r^2+z^2}}[\lambda\Delta + \mu - 4\mu(r^2+z^2)f'^2]$$

$$- \frac{\mu\phi_z}{\sqrt{r^2+z^2}}(r\phi_r + z\phi_z), \tag{5.2}$$

$$N_\theta = 0. \tag{5.3}$$

Substituting the values of ϕ and f'^2 from (3.18) and (4.3), we get

$$RN_r = r[\lambda(1 - K_0 - \tfrac{1}{2}K_1\alpha^2 - \tfrac{1}{2}\alpha^2) + \mu - \mu\alpha^2$$

$$- 2(\lambda + 2\mu)c_1^2R^{-2c}], \tag{5.4}$$

$$RN_z = z[\lambda(1 - K_0 - \tfrac{1}{2}K_1\alpha^2 - \tfrac{1}{2}\alpha^2) + \mu$$

$$- 2(\lambda + 2\mu)c_1^2R^{-2c}]. \tag{5.5}$$

If we assume that the inner surface $R = a$ of the shell is acted upon by a normal traction of the amount $\frac{1}{2}\mu\alpha^2$, we get

$$\lambda(1 - K_0 - \tfrac{1}{2}K_1\alpha^2 - \tfrac{1}{2}\alpha^2) + \mu - 2(\lambda + 2\mu)c_1^2a^{-2c} = \tfrac{1}{2}\mu\alpha^2. \tag{5.6}$$

Assuming the external surface $R = b$ to be acted on by a traction of amount $\mu\alpha^2 z/b$ in the direction of the axis, we get

$$\lambda(1 - K_0 - \tfrac{1}{2}K_1\alpha^2 - \tfrac{1}{2}\alpha^2) + \mu - 2(\lambda + 2\mu)c_1^2b^{-2c} = \mu\alpha^2. \tag{5.7}$$

These give

$$c_1^2 = \tfrac{1}{8}c\alpha^2\frac{(ab)^{2c}}{b^{2c} - a^{2c}}, \tag{5.8}$$

and

$$\tfrac{1}{4}\alpha^2\left[\frac{ca^{2c}}{b^{2c} - a^{2c}} + (2 - c)\right] = \frac{3}{2} - c - 2K_0(1 - c). \tag{5.9}$$

Also from (3.17) we have

$$(i)\ A = 1,\quad K_0 = \tfrac{1}{2},\quad K_1 = 1. \tag{5.10}$$

$$(ii)\ \alpha^2 = \frac{1 - A}{3 - A}.$$

$$\left.\begin{aligned} \text{In } (i) \qquad v = 0,\ \alpha^2 &= \frac{2\,(b^{2c} - a^{2c})}{ca^{2c} + (2-c)\,(b^{2c} - a^{2c})}\,, \\ c_1{}^2 &= \tfrac{1}{4}c\,\frac{(ab)^{2c}}{ca^{2c} + (2-c)\,(b^{2c} - a^{2c})}\,. \end{aligned}\right\} \tag{5.11}$$

In (*ii*) A may be determined from (5.9) and substituted in (5.8) to get $c_1{}^2$. Assuming that the plate is bent into a spherical shell with a plane rim, the tractions on the rim are

$$Z_r = \tau_{rz} = -\,4\mu rzc_1{}^2\mathrm{R}^{-2\,(1+c)}, \tag{5.12}$$

$$\begin{aligned} Z_z = \tau_{zz} &= \lambda\,(1 - \mathrm{K}_0 - \tfrac{1}{2}\mathrm{K}_1\alpha^2 - \tfrac{1}{2}\alpha^2) + \mu - 2\lambda c_1{}^2\mathrm{R}^{-2c} \\ &\quad - 4\mu c_1{}^2z^2\mathrm{R}^{-2\,(1+c)}. \end{aligned} \tag{5.13}$$

Their statical resultant per unit length of the rim $z = z_0$, is

$$\begin{aligned} \int_a^b \tau_{zz}\,.\,rdr. &= \tfrac{1}{2}\lambda\,(1 - \mathrm{K}_0 - \tfrac{1}{2}\mathrm{K}_1\alpha^2 - \tfrac{1}{2}\alpha^2)\,(b^2 - a^2) \\ &\quad - \frac{\lambda c_1{}^2}{1-c}\,(b^{1-c} - a^{1-c}) + \frac{4\mu z_0{}^2c_1{}^2}{c}(b^c - a^c). \end{aligned}$$

This is the force which must be applied on the rim to keep the plate in the spherical form.

6. SUMMARY

The use of finite components of strain shows that a plate can be bent into a spherical shell if suitable tractions are applied to the inner and outer surfaces and to the plane rim. The corresponding components of displacement have been obtained.

7. REFERENCES

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