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Existence of dicritical divisors revisited

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Abstract. We characterize the dicriticals of special pencils. We also initiate higher dimensional dicritical theory.

Keywords. Dicritical; special pencils; normal varieties.

1. Introduction

The analytical (topological) theory of dicritical divisors was developed in [15–19]. It was algebracized in [8,9]. The algebraic theory was furthered in [10,11,13,14]. In this paper we shall make further progress in this theory. In particular, in Theorem 8.2 we shall prove a converse of Proposition 3.5 of [11] characterizing the dicritical set of a special pencil on a nonsingular surface; see §8 for the statement of this and other related results. In §9 we shall initiate the dicritical theory of higher dimensional normal varieties, which may be algebraic or arithmetical, and we shall indicate how this could be used in attacking the higher dimensional Jacobian conjecture.

We shall use the notation and terminology introduced in [3–9] and more specifically in [10,11]. Relevant background material can be found in [1,2,20–22].

It may be pointed out that the theory of graded rings, say as found on pages 206–215, 236–241, 272–277, 399–408 and 585–587 of [4], is a cornerstone of this paper. The basis of that theory is the idea of collecting terms of like degree in a polynomial coming from classical algebra. This idea is used in geometry in the aphorism which says that the factors of the highest degree terms of a bivariate polynomial give points at infinity while the factors of its lowest degree term give tangents at the origin. Another cornerstone of this paper consists of the theories of blowing up and Veronese embedding as developed on pages 7–45, 155–192, 262–283 of [3] and reproduced on pages 146–161, 534–552, 553–577 of [4].

In §2 to §5 we shall review some notations to be used frequently. In §6, which is the heart of the paper, we study natural extensions of valuations, which are sometimes called Gauss extensions. Behind all this are Rees rings and their suitable homomorphic images which we call form rings and which are sometimes called fiber rings. In §7 we make connections with extended Rees rings.

2. Quasilocal rings

Recall that a ring is commutative with 1. A quasilocal ring is a ring S with a unique maximal ideal $M(S)$; by

$$H_S : S \to H(S) = S/M(S),$$

we denote the residue class epimorphism. A quasilocal ring S dominates a quasilocal ring T means T is a subring of S with $M(T) = T \cap M(S)$, and then we say that S is residually rational (respectively residually algebraic, residually transcendental, residually simple transcendental, residually almost simple transcendental, ...) over T if $H(S) = H_S(T)$ (respectively $H(S)$ is algebraic over $H_S(T)$, $H(S)$ is transcendental over $H_S(T)$, $H(S)$ is simple transcendental over $H_S(T)$, $H(S)$ is almost simple transcendental over $H_S(T)$, ...). Note that a field k^* is simple transcendental over a subfield k means $k^* = k(t)$ for some element t which is transcendental over k, and a field k^* is *almost simple transcendental* over a subfield k means k^* is simple transcendental over a finite algebraic field extension of k. A local ring is a noetherian quasilocal ring.

As usual $\mathbb{N}$ (respectively $\mathbb{N}_+$) denotes the set of all nonnegative (respectively positive) integers. The set of all nonzero elements in a ring A is denoted by $A^\times$.

For any set U of quasilocal domains and any $i \in \mathbb{N}$, U_i denotes the set of all i-dimensional members of U.

For the definitions of $\dim(R)$, $\mathrm{spec}(R)$, $\mathrm{mspec}(R)$, $\mathrm{ht}_R J$, $\mathrm{dpt}_R J$, $\mathrm{vspec}_R J$, $\mathrm{mvspec}_R J$ and $\mathrm{nvspec}_R J$, for a ring R and an ideal J in it, see pages 115 and 127 of [4]. Note that $\mathrm{nvspec}_R J$ stands for the minimal spectrum of J in R, and its members are called the minimal primes of J in R. For the definition of vector space dimension

$$[L : K] = \dim_K L,$$

see page 9 of [4].

For the definition of coefficient set, coefficient ring, and coefficient field of a quasilocal ring, see the third paragraph of §3 of [10].

3. Modelic blowups

Referring to pages 145–161 of [4], for the foundations of models, recall that: $\mathfrak{V}(A) = \{A_P : P \in \mathrm{spec}(A)\}$ = the modelic spec of a domain A. $S^{\mathfrak{N}}$ = the set of all members of $\mathfrak{V}(\bar{S})$ which dominate S where $\bar{S}$ is the integral closure of a quasilocal domain S in its quotient field QF(S). $U^{\mathfrak{N}} = \cup_{B \in U} B^{\mathfrak{N}}$ for any set U of quasilocal domains. $\mathfrak{W}(A, J) = \cup_{0 \neq x \in J} \mathfrak{V}(A[Jx^{-1}])$ = the modelic blowup of A at a nonzero ideal J in a domain A; note that $Jx^{-1} = \{y/x : y \in J\}$; see pages 152–160 of [4]. $\mathfrak{W}(S, J)_i^{\Delta}$ = the set of all i-dimensional members of $\mathfrak{W}(S, J)$ which dominates S where J is a nonzero ideal in a quasilocal domain S and $i \in \mathbb{N}$. $\mathfrak{D}(S, J) = (\mathfrak{W}(S, J)_1^{\Delta})^{\mathfrak{N}}$ = the *dicritical set* of a nonzero ideal J in a quasilocal domain S; members of this set are called *dicritical divisors* of J in S. $\bar{C}(A)$ = the set of all nonzero complete ideals in a normal domain A. See the second paragraph after (2.3) of [11] for the definitions of normal domain, simple ideal, valuation ideal, complete ideal, normal ideal and completion of an ideal. $C(S)$ = the set of all $M(S)$-primary simple complete ideals in a normal local domain S. $\mathrm{ord}_S a = \max\{d \in \mathbb{N} : a \in M(S)^d \text{ or } a \subset M(S)^d\}$ where S is a local ring and $a \in S$ or $a \subset S$, with the understanding that if $a = 0$ or $a \subset \{0\}$ then $\mathrm{ord}_S a = \infty$. $\mathrm{ord}_S(\alpha/\beta) =$

$(\mathrm{ord}_S\alpha) - (\mathrm{ord}_S\beta)$ for $\alpha \neq 0 \neq \beta$ in a regular local domain S. DVR (discrete valuation ring) = one dimensional regular local domain. $o(S)$ = the natural DVR of a positive dimensional regular local domain S, i.e., the DVR V with $\mathrm{QF}(V) = L = \mathrm{QF}(S)$ such that $\mathrm{ord}_V(a) = \mathrm{ord}_S(a)$ for all $a \in L$. $V(a) = v(a)$ for all $a \in L$ where $v : L \to G \cup \{\infty\}$ is a valuation of a field L and V is its valuation ring. Note that $V(a) = \infty$ or $V(a) \in v(L^\times)$ according as $a = 0$ or $a \neq 0$; see page 41 of [4]. Now let $I \subset S$ where S is a noetherian subring of L with $S \subset V$. If $I \subset \{0\}$ then we let $V(I) = \infty$. If $I \not\subset \{0\}$ then we let $V(I)$ be the unique element of $v(L^\times)$ such that for some $0 \neq x \in I$ we have $V(I) = v(x) \leq v(y)$ for all $y \in I$. Note that the noetherianness guarantees the existence of x. Also note that the equation $V(I) = V(x)$ is equivalent to the equation $IV = xV$.

4. Graded rings

Referring to pages 206–215 of [4], for the foundations of graded rings, recall that: $E_R(I) = R[IZ]$ = the Rees ring of an ideal I in a nonnull ring R relative to R with variable Z. Note that $R[Z]$ is the univariate polynomial ring as a naturally graded homogeneous ring with $R[Z]_n$ = the set of all homogeneous polynomials of degree n including the zero polynomial, and n varying over $\mathbb{N}$. Now $E_R(I)$ is a graded subring of $R[Z]$ with $E_R(I)_n = \{gZ^n : g \in I^n\}$, and every $f \in E_R(I)$ can uniquely be written as a finite sum

$$f = \sum_{n\in\mathbb{N}} f_n Z^n \quad \text{with} \quad f_n \in I^n. \tag{4.1}$$

Details are in the third paragraph after (2.3) of [11] where you can also find the the definitions of: an element or subset of a nonnull ring S to be integral over an ideal J in a subring of S, the integral closure of J in S, and the reduction of an ideal.

$F_R(I) = E_R(I)/ME_R(I)$ = the form ring of an ideal $I \subset M = M(R)$ in a local ring R relative to R with variable Z. Note that $ME_R(I)$ is a homogeneous ideal in $E_R(I)$ and hence $F_R(I)$ is a naturally graded homogeneous ring over the field R/M, and for its homogeneous n-th component $F_R(I)_n$ we have a canonical R-epimorphism

$$\mu_n : I^n \to F_R(I)_n \tag{4.2}$$

with kernel MI^n. Details are in the third paragraph after (2.3) of [11], where we slightly generalized the matter by letting I to be an ideal in a nonnull ring R with $I \subset M$ = a nonunit ideal in R, and denoting the form ring by $F_{(R,M)}(I)$. Note that $F_{(R,M)}(I)$ is isomorphic as a graded ring to the associated graded ring $\mathrm{grad}(R, I, M)$ of Definition (D3) on page 586 of [4].

5. Quadratic transformations

Let R be a two dimensional regular local domain with quotient field $\mathrm{QF}(R) = L$. Recall that:

$D(R)^\Delta$ = the set of all DVRs V with quotient field L such that V dominates R and is residually transcendental over R; by [1,2] it follows that $H(V)$ is almost simple transcendental over $H_V(R)$. Members of $D(R)^\Delta$ are called prime divisors of R. QDT = Quadratic transformation or quadratic transform. $Q_j(R)$ = the set of all two dimensional QDTs of R. $Q(R) = \coprod_{j\in\mathbb{N}} Q_j(R)$ = the set of all two dimensional regular local domains which birationally dominate R, i.e., whose quotient field is L and which dominate R; proof of the second equality in [1]. $o_R : Q(R) \to D(R)^\Delta$ is the bijective map given by $S \mapsto o(S)$. The QDT sequence of R along $o(S)$ is the finite sequence $(R_j)_{0\le j\le \nu}$ with $R_0 = R$ and $R_\nu = S$ such that $R_{j+1} \in Q_1(R_j)$ for $0 \le j < \nu$. A finite QDT sequence of R means the QDT sequence of R along some $V \in D(R)^\Delta$. See proofs in [1]. $\zeta_R : D(R)^\Delta \to C(R)$ is the bijective map given in Appendix 5 of [22] which we call the Zariski map. Details are in §2 of [11]. $a_R(z)$ (respectively $b_R(z)$, $J_R(z)$, $I_R(z)$) = the numerator (respectively denominator, first asociated, second associated) ideal of z in R. Details are in the fourth paragraph after (2.3) of [11]. $\mathfrak{D}(R, z)^\sharp \subset \mathfrak{D}(R, z)^\flat \subset \mathfrak{D}(R, z) = (\mathfrak{W}(R, J_R(z))_1^\Delta)^{\mathfrak{N}}$, where $\mathfrak{D}(R, z)^\sharp$ (respectively $\mathfrak{D}(R, z)^\flat$, $\mathfrak{D}(R, z)$) is the set of all *sharp dicritical divisors* (respectively *flat dicritical divisors, dicritical divisors*) of z in R, i.e., the set of those DVRs $V \in D(R)^\Delta$ at which the element $z \in L^\times$ is a residual transcendental generator (respectively residually a polynomial, residually transcendental) over R. We call $\mathfrak{D}(R, z)^\sharp$ (respectively $\mathfrak{D}(R, z)^\flat$, $\mathfrak{D}(R, z)$) the *sharp dicritical set* (respectively the *flat dicritical set*, the *dicritical set*) of z in R. See the material starting with the second display in §2 of [10].

Geometrically speaking, we may visualize R to be the local ring of a simple point of an algebraic or arithmetical surface, and think of z as a *rational function* at that simple point which corresponds to the *local pencil* of curves $a = ub$ at that point. We say that z generates a *special pencil at* R to mean that b can be chosen so that $b = x^m$ for some $x \in M(R)\backslash M(R)^2$ and $m \in \mathbb{N}$, i.e., $zx^m \in R$ for some $x \in M(R)\backslash M(R)^2$ and $m \in \mathbb{N}$. We say that z generates a *semispecial pencil at* R to mean that b can be chosen so that $b = x^m y^n$ for some x, y in $M(R)$ and m, n in $\mathbb{N}$ with $M(R) = (x, y)R$, i.e., $zx^m y^n \in R$ for some x, y in $M(R)$ and m, n in $\mathbb{N}$ with $M(R) = (x, y)R$. We say that z generates a *polynomial* or *nonpolynomial* pencil in R according as $\mathfrak{D}(R, z) = \mathfrak{D}(R, z)^\flat$ or $\mathfrak{D}(R, z) \neq \mathfrak{D}(R, z)^\flat$. We say that z generates a *generating* or *nongenerating* pencil in R according as $\mathfrak{D}(R, z) = \mathfrak{D}(R, z)^\sharp$ or $\mathfrak{D}(R, z) \neq \mathfrak{D}(R, z)^\sharp$. See the material starting with the second display in §2 of [10].

For a moment let J be a nonzero ideal in R. We call J a *pencil* (in R) if $J = yJ_R(z)$ for some $y \in R^\times$ and $z \in L^\times$, and we note that $\mathfrak{D}(R, J) = \mathfrak{D}(R, z)$. If J is a pencil with $J = yJ_R(z)$ then we let $\mathfrak{D}(R, J)^\sharp = \mathfrak{D}(R, z)^\sharp$ and $\mathfrak{D}(R, J)^\flat = \mathfrak{D}(R, z)^\flat$, and if J is not a pencil then we let $\mathfrak{D}(R, J)^\sharp = \mathfrak{D}(R, J)^\flat = \emptyset$. Note that now we have

$$\mathfrak{D}(R, J)^\sharp \subset \mathfrak{D}(R, J)^\flat \subset \mathfrak{D}(R, J) = \left(\mathfrak{W}(R, J)_1^\Delta\right)^{\mathfrak{N}}.$$

We say that J is a *polynomial* or *nonpolynomial* ideal in R according as $J = yJ_R(z)$ for some $y \in R^\times$ and $z \in L^\times$ such that z generates a polynomial or nonpolynomial pencil in R. We say that J is a *generating* or *nongenerating* ideal in R according as $J = yJ_R(z)$ for some $y \in R^\times$ and $z \in L^\times$ such that z generates a generating or nongenerating pencil in R. In the last two sentences we may say *pencil* instead of ideal. Regardless of whether J is a pencil or not, we say that J is *primary* to mean that the ideal J is $M(R)$-primary. We say that J is *special* (respectively *semispecial*) *at* R if $J = yJ_R(z)$ for some $y \in R^\times$ and $z \in L^\times$ such that z generates a special (respectively semispecial) pencil at R. We put

$$\begin{cases} \mathfrak{B}(R, J)^\sharp = \{o_R^{-1}(V) : V \in \mathfrak{D}(R, J)^\sharp\} \\ \mathfrak{B}(R, J)^\flat = \{o_R^{-1}(V) : V \in \mathfrak{D}(R, J)^\flat\} \\ \mathfrak{B}(R, J) = \{o_R^{-1}(V) : V \in \mathfrak{D}(R, J)\} \\ \mathfrak{Q}(R, J) = \{T \in Q(R) : (R, T)(J) \text{ is not principal}\} \end{cases}$$

and we note that $\mathfrak{Q}(R, J)$ is a finite set (for a proof see Proposition 2 on page 367 of [22]) with

$$\mathfrak{B}(R, J)^\sharp \subset \mathfrak{B}(R, J)^\flat \subset \mathfrak{B}(R, J) \subset \mathfrak{Q}(R, J) \subset Q(R).$$

We say that J *goes through* the members of $\mathfrak{Q}(R, J)$ but not through the members of $Q(R) \setminus \mathfrak{Q}(R, J)$. See the middle of §2 of [10].

6. Natural extensions of valuations

Let Y be an indeterminate over a field L. Let $v : L \to G \cup \{\infty\}$ be a valuation of L. By (J4) to (J9) on pages 79–80 of [4] we get a unique valuation $w : L(Y) \to G \cup \{\infty\}$ of $L(Y)$ such that for all $\sum_{i\in\mathbb{N}} a_i Y^i \in L[Y]$ with $a_i \in L$ we have

$$w\left(\sum_{i\in\mathbb{N}} a_i Y^i\right) = \min\{v(a_i) : i \in \mathbb{N}\} \tag{6.1}$$

with the understanding that $w(0) = \infty$. We call w the *natural extension* of v to $L(Y)$; see [12]. We rename the valuation rings R_v and R_w by putting

$$V = R_v \quad \text{and} \quad W = R_w$$

and we call W the *natural extension* of V to $L(Y)$. Note that if v is real discrete, i.e., if $G = v(L^\times) = \mathbb{Z}$, then V and W are DVRs.

Let R be a noetherian domain with quotient field L. Let I be a nonzero ideal in R. Let

$$E = E_R(I) = R[IZ] = \sum_{n\in\mathbb{N}} E_n = \text{the Rees ring of } I.$$

Assume that $R \subset V$. We claim that there exists a unique valuation

$$w' : L(Z) \to G \cup \{\infty\}$$

such that, for all $0 \neq f \in E$, in the notation of (4.1) we have

$$w'(f) = \min\{v(f_n) - nV(I) : n \in \mathbb{N} \text{ with } f_n \neq 0\}. \tag{6.2}$$

Namely, we can take $0 \neq x \in I$ with $V(x) = V(I)$, and for any such x, upon letting $Y = xZ$, by (6.1) and (6.2) we get

$$w' = w. \tag{6.3}$$

So we call w the (R, I)-*extension* of v to $L(Z)$ and we call W the (R, I)-*extension* of V to $L(Z)$.

Let A be a noetherian domain with quotient field L, and let us consider the meromorphic polynomial ring $B = A[Y, Y^{-1}]$. Also consider the multiplicative set $M^* = \{1, Y, Y^2, \ldots\}$ in the usual polynomial ring $B^* = A[Y]$. By T(30.1) on page 233 of [4] we see that

$$\begin{cases} \text{for any } P \in \operatorname{spec}(A) \text{ we have:} \\ \quad PB^* \in \operatorname{spec}(B^*) \text{ with } M^* \cap (PB^*) = \emptyset \\ \quad \text{and } P = A \cap (PB^*) \text{ with } \operatorname{ht}_A P = \operatorname{ht}_{B^*}(PB^*). \end{cases} \tag{6.4}$$

Clearly B equals the localization $B^*_{M^*}$ and hence by taking $(B^*, M^*) = (R, S)$ in (T12) on page 139 of [4] we get

$$\begin{cases} \text{for any } Q^* \in \operatorname{spec}(B^*) \text{ with } M^* \cap Q^* = \emptyset \text{ we have:} \\ \quad Q^*B \in \operatorname{spec}(B) \text{ with } M^* \cap (Q^*B) = \emptyset \\ \quad \text{and } Q^* = B^* \cap (Q^*B) \text{ with } \operatorname{ht}_{B^*} Q^* = \operatorname{ht}_B(Q^*B). \end{cases} \tag{6.5}$$

Taking $Q^* = PB^*$ in (6.5), by (6.4) and (6.5) we see that

$$\begin{cases} \text{for any } P \in \operatorname{spec}(A) \text{ we have:} \\ \quad PB \in \operatorname{spec}(B) \text{ with } M^* \cap (PB) = \emptyset \\ \quad \text{and } P = A \cap (PB) \text{ with } \operatorname{ht}_A P = \operatorname{ht}_B(PB). \end{cases} \tag{6.6}$$

Since every element of $L[Y, Y^{-1}]$ lands in $L[Y]$ after we multiply it by a high enough power of Y, by (6.1) it follows that for any $\sum_{i\in\mathbb{Z}} a_i Y^i \in L[Y, 1/Y]^\times$ with $a_i \in L$ we have

$$w\left(\sum_{i\in\mathbb{Z}} a_i Y^i\right) = \min\{v(a_i) : i \in \mathbb{Z}\}. \tag{6.7}$$

Note that if $A \subset V$ then, upon letting $P = A \cap M(V)$ and $Q = B \cap M(W)$, we clearly have $P = A \cap Q \in \operatorname{spec}(A)$ with $Q \in \operatorname{spec}(B)$, and by (6.7) we get $B \subset W$ with $PB = Q$. Therefore by (6.6) it follows that

$$\begin{cases} \text{if } A \subset V \text{ then,} \\ \text{upon letting } P = A \cap M(V) \text{ and } Q = B \cap M(W), \text{ we have:} \\ \quad B \subset W \text{ with } P = A \cap Q \in \operatorname{spec}(A) \text{ and } M^* \cap Q = \emptyset \\ \quad \text{and } Q = PB \in \operatorname{spec}(B) \text{ with } \operatorname{ht}_A P = \operatorname{ht}_B Q. \end{cases} \tag{6.8}$$

Having separately dealt with the strands of the two subdomains R and A of L, let us now weave them together.

Lemma 6.9. *Assume that* $Y = xZ$ *and* $A = R[Ix^{-1}]$ *where* $0 \neq x \in I$ *is such that* $V(x) = V(I)$. *Then*

$$A \subset V \text{ and } E \subset B \subset W \tag{1}$$

and

$$M^* \text{ is a multiplicative set in } E \text{ with } E_{M^*} = B \tag{2}$$

and upon letting

$$P = A \cap M(V) \text{ and } Q = B \cap M(W) \text{ and } P^* = E \cap M(W)$$

we have

$$P^* \in \operatorname{spec}(E) \tag{3}$$

and

$$P = A \cap Q \in \operatorname{spec}(A) \text{ with } Q \in \operatorname{spec}(B) \tag{4}$$

and

$$M^* \cap Q = \emptyset \tag{5}$$

and

$$PB = Q = P^*B \ \ \textit{with} \ \ \mathrm{ht}_A P = \mathrm{ht}_B Q = \mathrm{ht}_E P^*. \tag{6}$$

Finally, if $V \in (\mathfrak{W}(R, I)_1)^{\mathfrak{N}}$ *then we have*

$$ht_A P = \mathrm{ht}_B Q = \mathrm{ht}_E P^* = 1. \tag{7}$$

Proof. Clearly $A \subset V$ and hence by (6.8) we get $B \subset W$.

Now $Y = xZ \in E_1 \subset E$ and hence M^* is a multiplicative set in E. Every element of E_1 can be written as yZ with $y \in I$, and we have $yZ = (yx^{-1})Y$ with $yx^{-1} \in A$ and hence $E_1 \subset A[Y]$. Also $E_0 = R \subset A$ and therefore $E \subset A[Y]$. Consequently $E_{M^*} = E[Y^{-1}] \subset A[Y, Y^{-1}] = B$.

To prove the reverse inclusion, note that $R \subset E$ and for every $y \in I$ we have $(yx^{-1})Y = yZ \in E_1 \subset E$. Consequently, for every $a \in A$ we have $aY^m \in E$ for some $m \in \mathbb{N}$. Therefore $B = A[Y, Y^{-1}] \subset E[Y^{-1}] = E_{M^*}$.

This proves (1) and (2). Now (3) and (4) are obvious.

By (6.8) we get (5). By (6.8) we also get the first equalities in the two assertions

$$PB = Q = Q^*B \quad \text{and} \quad \mathrm{ht}_A P = \mathrm{ht}_B Q = \mathrm{ht}_E Q^*$$

of (6), whereas the second equalities in these assertions follow from (2), (4) and (5), by invoking (T12) on page 139 of [4]. This proves (6).

If $V \in (\mathfrak{W}(R, I)_1)^{\mathfrak{N}}$ then clearly $\mathrm{ht}_A P = 1$, and hence by (3) and (6) we get (7).

Remark 6.9. The above proof is not difficult but, because of the mixing of two strands, it is certainly subtle. Thus first we go up the ladder $A \subset B^* \subset B$ with prime ideals $P \subset Q^* \subset Q$ and then down the ladder $R \subset E \subset B$ with prime ideals $R \cap M(V) \subset P^* \subset Q$. This enables us to compare the ideal theories of the two seemingly uncomparable rings A and E, neither of which is contained in the other. This subtlety is accentuated in the proof of the following Lemma 6.11. The subtlety of these proofs reminds me of the engraving which I had seen in Fine Hall of Princeton University Mathematics Department citing Einstein's quotation "Raffiniert ist der Herr Gott, aber boshaft ist er nicht."

DEFINITION–OBSERVATION 6.10

Inspired by Lemma 6.9(7), for any nonzero ideals $J \subset M$ in a domain S and any $i \in \mathbb{N}$ we put $\mathfrak{W}(S, J, M)_i^{\Delta}$ = the set of all i-dimensional members T of $\mathfrak{W}(S, J)$ such that $M \subset M(T)$. We also put $\mathfrak{D}(S, J, M) = (\mathfrak{W}(S, J, M)_1^{\Delta})^{\mathfrak{N}}$ which we call the *dicritical set* of (J, M) in S, and we call its members the *dicritical divisors* of (J, M) in S.

We make the following observations concerning these concepts.

(I) If N is an ideal in S with $J \subset N$ such that $\mathrm{rad}_S N = \mathrm{rad}_S M$ then we have $\mathfrak{W}(S, J, N)_i^{\Delta} = \mathfrak{W}(S, J, M)_i^{\Delta}$ and $\mathfrak{D}(S, J, N) = \mathfrak{D}(S, J, M)$. If $M = S$ then $\mathfrak{W}(S, J, M)_i^{\Delta} = \emptyset$ and $\mathfrak{D}(S, J, M) = \emptyset$.

(II) If S is quasilocal and $M = M(S)$ then $\mathfrak{W}(S, J, M)_i^\Delta = \mathfrak{W}(S, J)_i^\Delta$ and $\mathfrak{D}(S, J, M) = \mathfrak{D}(S, J)$.

(III) If R is a noetherian domain and $I \subset M$ are nonzero nonunit ideals in R then $\mathfrak{D}(R, I, M)$ is a finite set of DVRs. As in (5.6)(†*) of [8], this follows from (33.10) on page 118 of [20] or (33.2) on page 115 of [20]. Upon letting $V_1, \ldots, V_h$ be all the distinct members of $\mathfrak{D}(R, I, M)$ and upon letting $W_1, \ldots, W_h$ be their respective (R, I)-extension to $L(Z)$, it follows that $W_1, \ldots, W_h$ are distinct DVRs with $E \subset W_j$ for $1 \le j \le h$. Note that if $M \subset \mathrm{rad}_R I$ then clearly $h > 0$.

Lemma 6.11. *Assume that R is a normal noetherian domain and let I be any nonzero nonunit normal ideal in R. Then E is a normal noetherian domain with $\mathfrak{W}(R, I)^{\mathfrak{N}} = \mathfrak{W}(R, I)$, $\mathfrak{D}(R, I, I)$ is a nonempty finite set of DVRs, and upon letting $V_1, \ldots, V_h$ be all the distinct members of $\mathfrak{D}(R, I, I)$, and upon letting $W_1, \ldots, W_h$ be their respective (R, I)-extensions to $L(Z)$, we have that $W_1, \ldots, W_h$ are distinct DVRs with $E \subset W_j$ for $1 \le j \le h$.*

Moreover, upon letting

$$P_j^* = E \cap M(W_j), \quad \bar{P}_j^* = E \cap (IW_j) \quad \textit{and} \quad J_j = R \cap (IV_j)$$

we have that $P_1^*, \ldots, P_h^*$ are all the distinct members of $\mathrm{nvspec}_E(IE)$ *and*

$$\mathrm{ht}_E P_j^* = 1 \quad \textit{with} \quad E_{P_j^*} = W_j \quad \textit{and} \quad \bar{P}_j^* \textit{ is } P_j^*\textit{-primary.}$$

for $i \le j \le h$. Furthermore,

$$IE = \bar{P}_1^* \cap \cdots \cap \bar{P}_h^*$$

is the unique irredundant primary decomposition of IE in E, and we have

$$I = J_1 \cap \cdots \cap J_h \quad \textit{with} \quad J_j = R \cap \bar{P}_j^* \quad \textit{for} \quad 1 \le j \le h.$$

Now assume that $Y = xZ$ and $A = R[Ix^{-1}]$ where $0 \ne x \in R$ is such that $V_j(x) = V_j(I)$ for all j in a nonempty subset Λ of $\{1, \ldots, h\}$. Then A and B are normal noetherian domains with $A \subset V_j$ and $E \subset B \subset W_j$ for all $j \in \Lambda$. Moreover, upon letting $P_j = A \cap M(V_j)$ with $\bar{P}_j = A \cap (IV_j)$ and $Q_j = B \cap M(W_j)$ with $\bar{Q}_j = B \cap (IW_j)$, for all $j \ne j'$ in Λ we have

$$P_j \ne P_{j'} \quad \textit{with} \quad \bar{P}_j \ne \bar{P}_{j'} \quad \textit{and} \quad Q_j \ne Q_{j'} \quad \textit{with} \quad \bar{Q}_j \ne \bar{Q}_{j'}.$$

Furthermore, for all $j \in \Lambda$ we have that

$$xA = IA \subset P_j \in \mathrm{nvspec}_A(IA) \quad \textit{and} \quad xB = IB \subset Q_j \in \mathrm{nvspec}_B(IB)$$

and

$$\mathrm{ht}_A P_j = 1 \quad \textit{with} \quad A_{P_j} = V_j \quad \textit{and} \quad \bar{P}_j \textit{ is } P_j\textit{-primary}$$

and

$$\mathrm{ht}_B Q_j = 1 \quad \textit{with} \quad B_{Q_j} = W_j \quad \textit{and} \quad \bar{Q}_j \textit{ is } Q_j\textit{-primary}$$

and

$$P_jB = Q_j = P_j^*B \quad \textit{with} \quad P_j = A \cap Q_j \quad \textit{and} \quad P_j^* = E \cap Q_j$$

and

$$\bar{P}_jB = \bar{Q}_j = \bar{P}_j^*B \quad \textit{with} \quad \bar{P}_j = A \cap \bar{Q}_j \quad \textit{and} \quad \bar{P}_j^* = E \cap \bar{Q}_j.$$

Finally, if for all $l \in \{1, \dots, h\} \setminus \Lambda$ *we have* $V_l(x) \neq V_l(I)$, *then*

$$IA = \cap_{j\in\Lambda}\bar{P}_j \quad \textit{and} \quad IB = \cap_{j\in\Lambda}\bar{Q}_j$$

are the unique irredundant primary decompositions of IA *and* IB *in* A *and* B *respectively, and we have*

$$\text{nvspec}_A(IA) = \{P_j : j \in \Lambda\} \quad \textit{and} \quad \text{nvspec}_B(IB) = \{Q_j : j \in \Lambda\}.$$

Proof. By (8.1)(VI) of [13] and the above Lemma 6.9(III) we see that E is a normal noetherian domain with $\mathfrak{W}(R, I)^{\mathfrak{N}} = \mathfrak{W}(R, I)$, $\mathfrak{D}(R, I, I)$ is a nonempty finite set of DVRs, and upon letting $V_1, \dots, V_h$ be all the distinct members of $\mathfrak{D}(R, I, I)$, and upon letting $W_1, \dots, W_h$ be their respective (R, I)-extensions to $L(Z)$, we have that $W_1, \dots, W_h$ are distinct DVRs with $E \subset W_j$ for $1 \leq j \leq h$.

For $1 \leq j \leq h$, let

$$P_j^* = E \cap M(W_j), \quad \bar{P}_j^* = E \cap (IW_j) \quad \text{and} \quad J_j = R \cap (IV_j)$$

but let us postpone considering the rest of the second paragraph.

Turning to the third paragraph: Now assume that $Y = xZ$ and $A = R[Ix^{-1}]$ where $0 \neq x \in R$ is such that $V_j(x) = V_j(I)$ for all j in a nonempty subset Λ of $\{1, \dots, h\}$. Then for all $j \in \Lambda$ we clearly have $A \subset V_j$ and hence $B \subset W_j$. For all $j \in \Lambda$, let

$$P_j = A \cap M(V_j) \text{ with } \bar{P}_j = A \cap (IV_j)$$

and

$$Q_j = B \cap M(W_j) \text{ with } \bar{Q}_j = B \cap (IW_j).$$

Given any $j \in \Lambda$, by taking $V = V_j$ in (6.9) we see that

$$M^* \text{ is a multiplicative set in } E \text{ with } E_{M^*} = B \text{ and } E \subset B \subset W_j \tag{1}$$

and

$$\begin{cases} P_j = A \cap Q_j \in \text{spec}(A) \text{ with } Q \in \text{spec}(B) \text{ and } P_j^* = E \cap Q \in \text{spec}(E) \\ \text{and } P_jB = Q_j = P_j^*B \text{ with } \text{ht}_A P_j = \text{ht}_B Q_j = \text{ht}_E P_j^* = 1. \end{cases} \tag{2}$$

Since E is a normal noetherian domain, by (1) we see that B is also a normal noetherian domain. Now $B = A[Y, Y^{-1}]$ tells us that $\text{QF}(A) = L \subset L(Y) = \text{QF}(B)$ with $L \cap B = A$, and hence the normality of B yields the normality of A. Thus

$$\text{all the three rings } A, B, E \text{ are normal noetherian domains.} \tag{3}$$

By (2) and (3) we see that for all $j \in \Lambda$ we have

$$\begin{cases} xA = IA \subset P_j \in \text{nvspec}_A(IA), \\ xB = IB \subset Q_j \in \text{nvspec}_B(IB), \\ P_j^* \in \text{nvspec}_E(IE), \\ A_{P_j} = V_j \text{ and } B_{Q_j} = E_{P_j^*} = W_j. \end{cases} \tag{4}$$

By the last line of (4) we see that for all $j \neq j'$ in Λ we have

$$P_j \neq P_{j'}, \quad Q_j \neq Q_{j'} \quad \text{and} \quad P_j^* \neq P_{j'}^*. \tag{5}$$

If $p, \bar{p}$ is a prime-primary pair (i.e., if p is a prime ideal and $\bar{p}$ is a p-primary ideal) in a ring, then their contractions $q, \bar{q}$ to a subring constitute a prime-primary pair in that subring; moreover, every nonzero nonunit ideal in a DVR is primary for the maximal ideal; consequently, for all $j \in \Lambda$,

$$\begin{cases} \text{the ideals } \bar{P}_j, \bar{Q}_j, \bar{P}_j^* \text{ are } P_j\text{-primary, } Q_j\text{-primary,} \\ P_j^*\text{-primary ideals in } A, B, E \text{ respectively} \end{cases} \tag{6}$$

and, because ideals which are primary for distinct prime ideals are obviously distinct, by (5) we see that for all $j \neq j'$ in Λ we have

$$\bar{P}_j \neq \bar{P}_{j'}, \quad \bar{Q}_j \neq \bar{Q}_{j'} \quad \text{and} \quad \bar{P}_j^* \neq \bar{P}_{j'}^*. \tag{7}$$

In view of (2)–(7) together with (T82) on page 355 of [4], the equation $\mathfrak{W}(R, I)^{\mathfrak{N}} = \mathfrak{W}(R, I)$ implies that

$$\begin{cases} \text{if for all } l \in \{1, \ldots, h\} \setminus \Lambda \text{ we have } V_l(x) \neq V_l(I), \\ \text{then nvspec}_A(IA) = \{P_j : j \in \Lambda\} \text{ and } IA = \cap_{j \in \Lambda} \bar{P}_j \\ \text{is the unique irredundant primary decomposition of } IA \text{ in } A. \end{cases} \tag{8}$$

In view of (2)–(8) together with (T17) on page 145 and (T30) on page 235 of [4], the equation $B = A[Y]_{M^*}$ tells us that

$$\begin{cases} \text{if for all } l \in \{1, \ldots, h\} \setminus \Lambda \text{ we have } V_l(x) \neq V_l(I) \\ \text{then nvspec}_B(IB) = \{Q_j : j \in \Lambda\} \text{ and } IA = \cap_{j \in \Lambda} \bar{Q}_j \\ \text{is the unique irredundant primary decomposition of } IB \text{ in } B. \end{cases} \tag{9}$$

Thus we have proved everything in the third paragraph.

Now let us prove the assertion in the second paragraph which says that

$$\begin{cases} P_1^*, \ldots, P_h^* \text{ are all the distinct members of nvspec}_E(IE), \\ \text{ht}_E P_j^* = 1 \text{ with } E_{P_j^*} = W_j \text{ and } \bar{P}_j^* \text{ is } P_j^*\text{-primary for } i \leq j \leq h, \\ IE = \bar{P}_1^* \cap \cdots \cap \bar{P}_h^* \\ \text{is the unique irredundant primary decomposition of } IE \text{ in } E \end{cases} \tag{†}$$

and

$$I = J_1 \cap \cdots \cap J_h \text{ with } J_j = R \cap \bar{P}_j^* \text{ for } 1 \leq j \leq h. \tag{‡}$$

Recall that $R \subset E$ are normal noetherian domains and for $1 \le j \le h$ we have

$$\begin{cases} E \subset W_j \text{ with } P_j^* = E \cap M(W_j) \text{ and } \bar{P}_j^* = E \cap (IW_j) \\ \text{and } R \subset V_j \text{ with } J_j = R \cap (IV_j). \end{cases} \tag{1*}$$

Given any j with $1 \le j \le h$, we can clearly find $0 \ne x \in I$ with $V_j(x) = V_j(I)$, and then upon taking $Y = xZ$ and $A = R[Ix^{-1}]$ with $\Lambda = \{j\}$, by (2), (3), (4) and (6) we see that

$$\begin{cases} P_j^* \in \text{nvspec}_E(IE) \text{ with } \text{ht}_E P_j^* = 1 \text{ and } E_{P_j^*} = W_j \\ \text{and } \bar{P}_j^* \text{ is a } P_j^*\text{-primary ideal in } E. \end{cases} \tag{2*}$$

In (‡) it is clear that $I \subset J_1 \cap \cdots \cap J_h$ with $J_j = R \cap \bar{P}_j^*$ for $1 \le j \le h$. We shall show that $J_1 \cap \cdots \cap J_h \subset I$ and this will complete the proof of (‡). Since I is a complete ideal in the normal noetherian domain R, by definition we have

$$I = \cap_{V \in \bar{D}(L/R)} (R \cap I(V)),$$

where for each $V \in \bar{D}(L/R)$, $I(V)$ is some ideal in V; recall that $\bar{D}(L/R) =$ the set of all valuation rings V with $\text{QF}(V) = L$ and $R \subset V$. It follows that

$$I = \cap_{V \in \bar{D}(L/R)} (R \cap (IV)).$$

Given any $V \in \bar{D}(L/R)$ with $I \subset M(V)$ we shall show that $J_1 \cap \cdots \cap J_h \subset R \cap (IV)$ and this will complete the proof of (‡). We may assume that $Y = xZ$ and $A = R[Ix^{-1}]$ where $0 \ne x \in R$ is such that $V(x) = V(I)$. Then $A \subset V$ and $xA = IA \subset A \cap (IV)$. Since $\mathfrak{W}(R, I)^{\mathfrak{N}} = \mathfrak{W}(R, I)$, it follows that $V_j(x) = V_j(I)$. So we may assume that $\Lambda = \{j : 1 \le j \le h \text{ and } V_j(x) = V_j(I)\}$. By (8) we have $IA = \cap_{j \in \Lambda} \bar{P}_j$ and hence

$$\cap_{j \in \Lambda} \bar{P}_j \subset A \cap (IV).$$

By definition $\bar{P}_j = A \cap (IV_j)$ and clearly $\cap_{1 \le j \le h} (A \cap (IV_j)) \subset \cap_{j \in \Lambda} (A \cap (IV_j))$; therefore by the above display we get

$$\cap_{1 \le j \le h} (A \cap (IV_j)) \subset A \cap (IV).$$

Intersecting both sides with R we conclude that $J_1 \cap \cdots \cap J_h \subset R \cap (IV)$. This completes the proof of (‡). Since I is a normal ideal, for every $n \in \mathbb{N}$, the ideal I^{n+1} is a normal ideal; consequently by (‡) we see that for every $n \in \mathbb{N}$ we have

$$I^{n+1} = \cap_{1 \le j \le h} \big(R \cap \big(I^{n+1} V_j\big)\big). \tag{‡$_n$}$$

Given any $f \in \cap_{1 \le j \le h} \bar{P}_j^*$, by using (‡$_n$) we shall show that $f \in IE$ and, in view of (1*) and (2*), this will prove (†) which will complete the proof of Lemma 6.11. By (4.1) we can express f as a finite sum $f = \sum_{n \in \mathbb{N}} f_n Z^n$ with $f_n \in I^n$. By the definition of W_j as the (R, I)-extension of V_j we see that $\bar{P}_j^*$ is a homogeneous ideal in the homogeneous ring E; alternatively this follows because $\bar{P}_j^* = E \cap (IE_{P_j^*}) =$ the primary component of the homogeneous ideal IE with respect to its minimal prime P_j^*. Therefore $f_n Z^n \in$

$\cap_{1\le j\le h}\bar{P}_j^*$ for all $n\in\mathbb{N}$, and it suffices to show that for every $n\in\mathbb{N}$ we have $f_nZ^n\in IE$. Assuming $f_n\neq 0$, for $1\le j\le h$ we clearly have

$$f_nZ^n\in\bar{P}_j^*\Rightarrow W_j\big(f_nZ^n\big)\ge V_j(I)$$

with

$$W_j\big(f_nZ^n\big)=V_j(f_n)-nV_j(I)$$

and hence

$$\begin{aligned} f_nZ^n\in\bar{P}_j^* &\Rightarrow V_j(f_n)-nV_j(I)\ge V_j(I)\Rightarrow V_j(f_n)\ge(n+1)V_j(I)\\ &= V_j(I^{n+1}). \end{aligned}$$

Therefore

$$f_nZ^n\in\cap_{1\le j\le h}\bar{P}_j^*\Rightarrow f_n\in\cap_{1\le j\le h}\big(R\cap\big(I^{n+1}V_j\big)\big)$$

and hence by $(\ddagger_n)$ we conclude that

$$f_nZ^n\in\cap_{1\le j\le h}\bar{P}_j^*\Rightarrow f_n\in I^{n+1}$$

and clearly

$$f_n\in I^{n+1}\Rightarrow f_nZ^n\in IE.$$

Remark 6.11a. Alternatively, the normality of A and B can be seen thus. A is normal because $\mathfrak{V}(A)$ is an affine piece of the normal variety $\mathfrak{W}(R,I)$. Therefore B is normal because it is a localization of $A[Y]$.

DEFINITION–OBSERVATION 6.12

For any nonunit ideal J in a ring S and any nonnegative integer i we define the *depth i portion* of $\text{nvspec}_S J$ by putting

$$(\text{nvspec}_S J)_i=\{P\in\text{nvspec}_S J:\text{dpt}_S J=i\}.$$

To use the above definition, referring to pages 206–215 and 399–408 of [4] for the details of the theory of homogeneous rings and irrelevant ideals, let

$$F=\sum_{n\in\mathbb{N}}F_n$$

be a homogeneous ring over a field F_0 with $[F_1:F_0]<\infty$. As usual, let

$$\Omega(F)=F_1F=\sum_{n\in\mathbb{N}_+}F_n=\text{the unique homogeneous maximal ideal in } F$$

and let

$$\bar{\Omega}(F)=\cup_{n\in\mathbb{N}_+}F_n.$$

Recall that a nonunit homogeneous ideal G in F is irrelevant means $\Omega(F) \subset \mathrm{rad}_F G$, i.e., equivalently, $\Omega(F) = \mathrm{rad}_F G$. Note that $\dim(F) \in \mathbb{N}$ and for any nonunit homogeneous ideal G in F we have $\dim(F/G) \in \mathbb{N}$ with $\dim(F/G) \le \dim(F)$.

Now given any nonunit homogeneous ideal G in F, upon letting $\dim(F/G) = e$, let us prove the following Observations (I)–(VI):

(I) F is integral over $F_0[G] \Leftrightarrow G$ is irrelevant $\Leftrightarrow e = 0$.

(II) $\emptyset \neq (\mathrm{nvspec}_F G)_e \subset \mathrm{nvspec}_F G$. If $e \le 1$ then $(\mathrm{nvspec}_F G)_e = \mathrm{nvspec}_F G$.

(III) For any $y \in \bar{\Omega}(F)$ we have $\dim(F/(G \cup \{y\})F) = e - 1 \Leftrightarrow y \notin P$ for all $P \in (\mathrm{nvspec}_F G)_e$.

(IV) For any $y \in \bar{\Omega}(F)$ we have $\dim(F/(G \cup \{y\})F) \ge e - 1$.

(V) For any elements $y_1, \dots, y_e$ in $\bar{\Omega}(F)$, upon letting $G_i = (G \cup \{y_1, \dots, y_i\})F$ for $0 \le i \le e$, the following conditions (1) to (4) are mutually equivalent.

(1) For $1 \le i \le e$ we have $\dim(F/G_i) = e - i$.
(2) For $1 \le i \le e$ we have $\dim(F/G_i) \le e - i$.
(3) For $1 \le i \le e$ we have $y_i \notin P$ for all $P \in (\mathrm{nvspec}_F G_{i-1})_{e-i+1}$.
(4) For $1 \le i \le e$ we have $y_i \notin P$ for all $P \in \hat{G}_{i-1}$, where $\hat{G}_{i-1} = \mathrm{nvspec}_F G_{i-1}$ or $\hat{G}_{i-1} = (\mathrm{nvspec}_F G_{i-1})_{e-i+1}$ according as $i = e$ or $i \neq e$.

(VI) There exist elements $z_1, \dots, z_e$ in F_s for some $s \in \mathbb{N}_+$ such that upon letting

$$G_e = (G \cup \{z_1, \dots, z_e\})F$$

we have that $\dim(F/G_i) = 0$ and F is integral over $F_0[G_e]$. Moreover, if F_0 is infinite then we can take $s = 1$.

Proof. Let us observe that the associated primes of any homogeneous ideal are homogeneous, and $F_1 F$ is the only maximal ideal in F which is homogeneous.

Now (I) follows from (T104) on page 401 of [4]. The proofs of (II) and (III) are straightforward.

To prove (IV), consider the homogeneous ring $F^* = F/G$. Also consider the homogeneous ring $F' = F/(G \cup \{y\})F$ and let $\dim(F') = e'$. By the homogeneous noether normalization theorem (T106) on page 408 of [4], we can find elements $z_1, \dots, z_{e'}$ in $\bar{\Omega}(F)$ whose images $z'_1, \dots, z'_{e'}$ in F' are such that F' is integral over $F'_0[z'_1, \dots, z'_{e'}]$. By (I) we see that $(z'_1, \dots, z'_{e'})F'$ is an irrelevant ideal in F', and hence upon letting $y^*, z^*_1, \dots, z^*_{e'}$ be the respective images of $y, z_1, \dots, z_{e'}$ in F^* it follows that $(y^*, z^*_1, \dots, z^*_{e'})F^*$ is an irrelevant ideal in F^*. Therefore, again by (I), F^* is integral over $F^*_0[y^*, z^*_1, \dots, z^*_{e'}]$. Consequently, say by (O10), (O11), (T45), (T47) on pages 110, 111, 247, 250 of [4], we get $1 + e' \ge e$ and hence $e' \ge e - 1$.

Turning to (V), by (IV) we get (1) $\Leftrightarrow$ (2), by (III) we get (1) $\Leftrightarrow$ (3), and by (II) and (III) we get (1) $\Leftrightarrow$ (4).

To prove (VI), again consider the homogeneous ring $F^* = F/G$. Now by the normalization theorem (T46) and the homogeneous normalization theorem (T106) respectively on pages 248 and 408 of [4], we can find elements $z_1, \dots, z_e$ in F_s for some $s \in \mathbb{N}_+$, where we can take $s = 1$ in case F_0 is infinite, such that upon letting $z^*_1, \dots, z^*_e$ be their respective images in F^* we have that F^* is integral over $F^*_0[z^*_1, \dots, z^*_e]$. By (I) it follows that $(z^*_1, \dots, z^*_e)F^*$ is an irrelevant ideal in F^*. Therefore

$$G_e = (G \cup \{z_1, \dots, z_e\})F$$

is an irrelevant ideal in F. Therefore by (I) we conclude that $\dim(F/G_e) = 0$ and F is integral over $F_0[G_e]$.

Lemma 6.13. *Let the assumptions be as in* (6.11). *Also assume that R is a d-dimensional normal local domain with $M = M(R)$, and I is a nonzero normal M-primary ideal in R. Let*

$$F = F_R(I) = E/ME = \sum_{n\in\mathbb{N}} F_n = \text{the form ring of } I$$

and let

$$\mu_n : I^n \to F_n \quad \text{with} \quad \ker(\mu_n) = MI^n$$

be the canoninal R-epimorphism. For $1 \le j \le h$ let

$$Q'_j = \mu_1\big(P^*_j\big) \quad \text{and} \quad P'_j = Q'_j F.$$

Also let

$$\Lambda' = \{j : 1 \le j \le h \text{ and } \operatorname{dpt}_E P^*_j = d\}.$$

Then we have the following:

(I) $P^*_1, \ldots, P^*_h$ *are all the distinct members of* $\operatorname{nvspec}_E(ME)$ *and we have*

$$\operatorname{ht}_E P^*_1 = \cdots = \operatorname{ht}_E P^*_h = 1$$

and

$$\max\big(\operatorname{dpt}_E P^*_1, \ldots, \operatorname{dpt}_E P^*_h\big) = d = \dim(E) - 1.$$

(II) $P'_1, \ldots, P'_h$ *are all the distinct members of* $\operatorname{nvspec}_F\{0\}$ *and we have*

$$\operatorname{ht}_F P'_1 = \cdots = \operatorname{ht}_F P'_h = 0$$

and

$$\max\big(\operatorname{dpt}_F P'_1, \ldots, \operatorname{dpt}_F P'_h\big) = d = \dim(F).$$

Also we have

$$\Lambda' = \big\{j : 1 \le j \le h \text{ and } \operatorname{dpt}_F P'_j = d\big\}.$$

(III) *Given any $j \in \{1, \ldots, h\}$ and any $x \in I$ we have*

$$V_j(x) = V_j(I) \Leftrightarrow xZ \notin P^*_j \Leftrightarrow \mu_1(x) \notin P'_j.$$

(IV) *Given any elements $x_1, \ldots, x_d$ in I let $y_1 = \mu_1(x_1), \ldots, y_d = \mu_1(x_d)$. Then $y_1, \ldots, y_d$ are elements in F_1 and we have*

$$(x_1, \ldots, x_d)R \text{ is a reduction of } I \Leftrightarrow F \text{ is integral over } F_0[y_1, \ldots, y_h].$$

(V) *If $x_1, \ldots, x_d$ are elements in I such that $(x_1, \ldots, x_d)R$ is a reduction of I then for $1 \le i \le d$ we have $V_j(x_i) = V_j(I)$ for all $j \in \Lambda'$.*

(VI) *If $d = 1$, then $h = 1$ and $\Lambda' = \{1\}$.*
(VII) *If $d = 2$, then $\Lambda' = \{1, \dots, h\}$.*
(VIII) *If $d = 1$, then for any $x_1 \in I$ we have that*

$$V_1(x) = V_1(I) \Leftrightarrow x_1R \text{ is a reduction of } I.$$

(IX) *If R/M is infinite then there exist elements $x_1, \dots, x_d$ in I such that $(x_1, \dots, x_d)R$ is a reduction of I.*
(X) *If R/M is infinite and $x_1 \in I$ is such that $V_j(x_1) = V_j(I)$ for all $j \in \Lambda'$ then there exist elements $x_2, \dots, x_d$ in I such that $(x_1, \dots, x_d)R$ is a reduction of I.*
(XI) *Given any $j \in \{1, \dots, h\}$ there exists $x'_j \in I$ such that $V_j(x'_j) = V_j(I)$. If R/M is infinite then there exists $x_1 \in I$ such that $V_j(x_1) = V_j(I)$ for $1 \le j \le h$.*

Proof. By Lemma 6.11, we know that

$$\mathrm{ht}_E P_1^* = \cdots = \mathrm{ht}_E P_h^* = 1$$

and $P_1^*, \dots, P_h^*$ are all the distinct members of $\mathrm{nvspec}_E(IE)$. Since I is M-primary, it follows that $P_1^*, \dots, P_h^*$ are all the distinct members of $\mathrm{nvspec}_E(ME)$. By (6.4) of [13] we also have $\dim(F) = d$ and hence we get (II). Now it follows that

$$\max\big(\mathrm{dpt}_E P_1^*, \dots, \mathrm{dpt}_E P_h^*\big) = d \ge \dim(E) - 1.$$

So to complete the proof of (I) and (II) we only need to show that $\dim(E) \le \dim(R) + 1$. But this follows from the Multiple Ring Extension Lemma (T55) on page 269 of [4] by noting that E is an affine domain over R and transcendence degree of $\mathrm{QF}(E)$ over the quotient field of R is 1.

The second implication in (III) follows from the fact that $\mu_1(x)$ is the image of xZ under the residue class epimorphism $E \to F$. The first implication of (III) follows by noting that for $0 \neq x \in I$ we clearly have

$$V_j(x) = V_j(I) \Leftrightarrow V_j(x) \le V_j(I)$$

and

$$xZ \notin P_j^* \Leftrightarrow W_j(xZ) \le 0$$

and

$$W_j(xZ) = V_j(x) - W_j(Z) = V_j(x) - V_j(I).$$

(IV) follows from (6.1) of [13].

In view of Definition–Observation 6.12, assertions (V) to (X) follow from assertions (I) to (IV) where we note that: in proving (IX) we take $G = \{0\}F$ in Definition–Observation 6.12(VI), while in proving (X) we take $G = \mu_1(x_1)F$ in Definition–Observation 6.12(VI).

The first part of (XI) is obvious, and from it to deduce the second part, assume that the field R/M is infinite. Let $x_1 = a_1x'_1 + \cdots + a_hx'_h$ where $a_1, \dots, a_h$ in R are to be chosen. For any $a \in R$, let $\bar{a}$ be its image in R/M. Clearly $V_j(x_1) > V_j(I) \Leftrightarrow (\bar{a}_1, \dots, \bar{a}_h)$ belongs to a certain proper subspace K_j of $(R/M)^h$. The infiniteness of R/M implies that $K_1 \cup \cdots \cup K_h \neq (R/M)^h$ and it suffices to take $a_1, \dots, a_h$ to be such that $(\bar{a}_1, \dots, \bar{a}_h) \notin K_1 \cup \cdots \cup K_h$.

Without assuming R/M to be infinite, we shall now prove the following variation of parts (IX)–(XI) of (6.13).

Lemma 6.14. *Let the assumptions be as in Lemma* 6.13. *Then, without assuming R/M to be infinite, we have the following.*

(I) *Given any $r \in \mathbb{N}_+$, there exist elements $x_1, \ldots, x_d$ in I^{rs} for some $s \in \mathbb{N}_+$ such that, for every $t \in \mathbb{N}_+$, $(x_1^t, \ldots, x_d^t)R$ is a reduction of I^{rst}.*

(II) *If $r \in \mathbb{N}_+$ and $x_1 \in I^r$ are such that $V_j(x_1) = V_j(I^r)$ for all $j \in \Lambda'$ then there exist elements $x_2, \ldots, x_d$ in I^{rs} for some $s \in \mathbb{N}_+$ such that, for every $t \in \mathbb{N}_+$, $(x_1^{st}, x_2^t, \ldots, x_d^t)R$ is a reduction of I^{rst}.*

(III) *Given any $r \in \mathbb{N}_+$, there exist $s \in \mathbb{N}_+$ and $x_1 \in I^{rs}$ such that $V_j(x_1) = V_j(I^{rs})$ for all $j \in \Lambda'$.*

Proof. Given any $q \in \mathbb{N}_+$, clearly I^q is a nonzero normal M-primary ideal in R with $\mathfrak{W}(R, I^q) = \mathfrak{W}(R, I)$. It follows that $V_1, \ldots, V_h$ are all the distinct members of $\mathfrak{D}(R, I^q, I^q)$. Moreover

$$E^{(q)} = E_R(I^q) \subset E_R(I) = E$$

and $W_1, \ldots, W_h$ are the respective (R, I^q)-extensions of $V_1, \ldots, V_h$. Furthermore

$$P_j^{(q)} = E^{(q)} \cap P_j^* = E^{(q)} \cap M(W_j) \in \operatorname{spec}\big(E^{(q)}\big) \quad \text{with} \quad \operatorname{ht}_{E^{(q)}} P_j^{(q)} = 1$$

for $1 \le j \le h$, and the prime ideals $P_1^{(q)}, \ldots, P_h^{(q)}$ are all the distinct members of $\operatorname{nvspec}_{E^{(q)}}(I^q E^{(q)})$. Finally $\Lambda' = \{j : 1 \le j \le h \text{ and } \operatorname{dpt}_{E^{(q)}} P_j^{(q)} = d\}$. Note that the above two displays include the definitions of the symbols $E^{(q)}$ and $P_j^{(q)}$. Let

$$F^{(q)} = F_R(I^q) = E^{(q)}/ME^{(q)} = \sum_{n \in \mathbb{N}} F_n^{(q)} = \text{the form ring of } I^q$$

and let

$$\mu_n^{(q)} : I^n \to F_n^{(q)} \quad \text{with} \quad \ker\big(\mu_n^{(q)}\big) = MI^n$$

be the canonical R-epimorphism. Applying (4.1) to E and $E^{(q)}$ we see that

$$ME^{(q)} = E^{(q)} \cap (ME)$$

and hence, upon letting

$$\psi^{(q)} : E^{(q)} \to E$$

be the inclusion monomorphism and

$$\nu : E \to F \quad \text{and} \quad \nu^{(q)} : E^{(q)} \to F^{(q)}$$

be the residue class epimorphisms, there exists a unique monomorphism

$$\phi^{(q)} : F^{(q)} \to F$$

such that

$$\phi^{(q)}\nu^{(q)} = \nu\psi^{(q)}.$$

Note that the restriction of $\psi^{(q)}$ to $E_n^{(q)}$ gives an isomorphism $E_n^{(q)} \to E_{nq}$, and the restriction of $\phi^{(q)}$ to $F_n^{(q)}$ gives an isomorphism $F_n^{(q)} \to F_{nq}$; these isomorphisms are the foundation of the *Veronese embedding* expounded on pages 263–283 of [3]; let it be recorded that Veronese was the param-param-guru of Abhyankar whose guru Zariski was a pupil of Castelnuovo whose guru was Veronese; going back one step further, Cremona was the guru of Veronese; this makes Cremona Abhyankar's *param-param-param-guru*. The commutative diagram

$$\begin{tikzcd}
E_{nq} \arrow[r] & E \arrow[r, "\nu"] & F & F_{nq} \arrow[l] \\
E_n^{(q)} \arrow[u] \arrow[r] & E^{(q)} \arrow[u, "\psi^{(q)}"] \arrow[r, "\nu^{(q)}"] & F^{(q)} \arrow[u, "\phi^{(q)}"] & F_n^{(q)} \arrow[l] \arrow[u]
\end{tikzcd}$$

exhibits various maps which we have discussed.

The above observations will be used tacitly.

In proving (I) to (III) we shall assume that $r = 1$; the general case will then follow by taking I^r for I.

To prove (I) and (II), by Definition–Observation 6.12(VI) we can find elements $z_1, \ldots, z_d$ in F_s for some $s \in \mathbb{N}_+$ such that the ring F is integral over the subring $F_0[z_1, \ldots, z_d]$, where in case of (I) we take $G = \{0\}F$, while in the case of (II) we take $G = \mu_1(x_1)$ and $z_1 = \mu_1(x_1)^s$. Using the monomorphism $\phi^{(s)}$ we get unique elements $y_1, \ldots, y_d$ in $F_1^{(s)}$ such that $\phi^{(s)}(y_1) = z_1, \ldots, \phi^{(s)}(y_d) = z_d$, and then the ring $F^{(s)}$ is clearly integral over the subring $F_0^{(s)}[y_1, \ldots, y_d]$. In case of (I) we can take elements $x_1, \ldots, x_d$ in I^s such that $\mu_1^{(s)}(x_1) = y_1, \ldots, \mu_1^{(s)}(x_d) = y_d$, while in the case of (II) we can take elements $x_2, \ldots, x_d$ in I^s such that $\mu_1^{(s)}(x_2) = y_2, \ldots, \mu_1^{(s)}(x_d) = y_d$. By (6.13) we see that, in case of (I), $(x_1, \ldots, x_d)R$ is a reduction of I^s, while, in the case of (II), $(x_1^s, x_2, \ldots, x_d)R$ is a reduction of I^s. Given any $t \in \mathbb{N}_+$, using Lemma 6.13(V) and the monomorphisms $\phi^{(s)}$ and $\phi^{(st)}$, we conclude that, in the case of (I), $(x_1^t, \ldots, x_d^t)R$ is a reduction of I^{st}, while, in the case of (II), $(x_1^{st}, x_2^t, \ldots, x_d^t)R$ is a reduction of I^{st}.

This proves (I) and (II). In view of Lemma 6.13(V), (III) follows from (I).

7. Extended Rees rings

An alternative way of approaching parts of (6.11)–(6.14) is provided by the theory of extended Rees rings. To introduce these rings, let I be an ideal in a nonnull ring R. The *extended Rees ring* $\hat{E}_R(I)$ of I relative to R with variable Z is defined by putting

$$\hat{E}_R(I) = R\big[Z^{-1}, IZ\big] = \sum_{n\in\mathbb{Z}} \hat{E}_R(I)_n$$

which makes it $\mathbb{Z}$-graded = an integrally graded ring. Note that $E_R(I)$ is a graded subring of $\hat{E}_R(I)$ and we have

$$\hat{E}_R(I)_n = \begin{cases} \{gZ^n : g \in R\}, & \text{if } n < 0, \\ \{gZ^n : g \in I^n\} = E_R(I)_n, & \text{if } n \geq 0. \end{cases}$$

At any rate, every $f \in \hat{E}_R(I)$ can uniquely be written as a finite sum

$$f = \sum_{n\in\mathbb{Z}} f_n Z^n \quad \text{with} \quad f_n \in \begin{cases} R, & \text{if } n < 0, \\ I^n, & \text{if } n \geq 0. \end{cases} \tag{7.1}$$

Now consider

$$E = E_R(I) = \sum_{n\in\mathbb{N}} E_n \subset \hat{E} = \hat{E}_R(I) = \sum_{n\in\mathbb{Z}} \hat{E}_n \subset R' = R\big[Z^{-1}, Z\big] = \sum_{n\in\mathbb{Z}} R'_n$$

and note that

$$R' = R^- \oplus R^+, \quad \text{where} \quad R^- = \sum_{n\in\mathbb{Z}\setminus\mathbb{N}} R_n^- \quad \text{and} \quad R^+ = \sum_{n\in\mathbb{N}} R_n^+.$$

By (4.1) and (7.1) we see that

$$\begin{cases} E \cap \big(Z^{-1}\hat{E}\big) = IE \\ \text{which induces an isomorphism} \\ \hat{E}/\big(Z^{-1}\hat{E}\big) \approx E/(IE). \end{cases} \tag{7.2}$$

Heuristically speaking, (7.2) says that Z^{-1} in $\hat{E}$ is sort of a generic element of I, but $I\hat{E}$ is properly contained in $Z^{-1}E$. Indeed by (4.1) and (7.1) we see that

$$R^- \cap \big(Z^{-1}\hat{E}\big) = R^- \quad \text{and} \quad R^+ \cap \big(Z^{-1}\hat{E}\big) = IE \tag{7.3}$$

i.e., the negative portion of $Z^{-1}\hat{E}$ equals the entire R^-, and the nonnegative portion of $Z^{-1}\hat{E}$ equals IE. By (4.1) and (7.1) we also see that

$$\begin{cases} \text{for any homogeneous ideal } J \text{ in } E, \text{ upon letting } \hat{J} = R^- \oplus J, \\ \hat{J} \text{ is a homogeneous ideal in } \hat{E} \text{ such that} \\ R^- \cap \hat{J} = R^- \quad \text{and} \quad R^+ \cap \hat{J} = J. \end{cases} \tag{7.4}$$

Taking $E = J$ in (7.4) we get

$$\hat{E} = R^- \oplus E \quad \text{with} \quad R^- \cap \hat{E} = R^- \quad \text{and} \quad R^+ \cap \hat{E} = E. \tag{7.5}$$

Let us now prove a lemma about $\hat{E}$.

Lemma 7.6. *Let the assumptions be as in Lemma* 6.11. *Then $\hat{E}$ is a normal noetherian domain and, upon letting $P'_1, \ldots, P'_{h'}$ be the minimal primes of $Z^{-1}\hat{E}$ and upon letting $W'_j = \hat{E}_{P'_j}$ for $1 \leq j \leq h'$, we have the following.* (*Note that, by Krull normality lemma* (T82) *on page* 355 *of* [4], *all associated primes of $Z^{-1}\hat{E}$ are minimal and have height one.*)

(I) *Upon letting M' be the multiplicative set* $\{1, Z^{-1}, Z^{-2}, \dots\}$ *in* $\hat{E}$ *we have*

$$\hat{E}_{M'} = R\big[Z^{-1}, Z\big] \quad \textit{and hence} \quad \hat{E} = R\big[Z^{-1}, Z\big] \cap W'_1 \cap \dots \cap W'_{h'}.$$

(II) $h = h'$ *and, after a suitable (obviously unique) relabelling, for* $1 \le j \le h$ *we have*

$$P'_j = \hat{P}^*_j \quad \textit{and} \quad W'_j = W_j.$$

Proof. The Krull normality lemma (T82) on page 355 of [4] gives us (I). Since $P'_1, \dots, P'_{h'}$ are the minimal primes of $Z^{-1}\hat{E}$ in $\hat{E}$ and $P^*_1, \dots, P^*_h$ are the minimal primes of IE in E, by (7.2) and (7.4) we see that $h' = h$ and after a suitable relabelling we have $P'_j = \hat{P}^*_j$ for $1 \le j \le h$. Now, since $\hat{E}_{P'_j} = W'_j$ and $\hat{E} \subset E_{P^*_j} = W_j$, we get $W'_j = W_j$ for $1 \le j \le h$.

Remark 7.6*a*. The purport of Lemma 7.6 is that we can either first get the P'_j and the W'_j and then obtain W_j and P^*_j satisfying (6.11), or we can first do (6.11) and then get hold of the P'_j and the W'_j belonging to $\hat{E}$ by tacking on the negative piece R^-. The ideal $Z^{-1}\hat{E}$ contains R^- and is hence quite powerful.

8. Dicriticals of two dimensional regular local domains

We are now ready to reap the harvest from the work done in the previous sections.

Let R be a d-dimensional local domain with quotient field L, let I be a nonzero ideal in R with $I \subset M = M(R)$, let Z be an indeterminate over the quotient field L of R, and let

$$E = E_R(I) = R[IZ] = \text{the Rees ring of } I \text{ relative to } R \text{ with variable } Z.$$

Let

$$\mathfrak{D}(R, I) = \big(\mathfrak{W}(R, I)^{\Delta}_1\big)^{\mathfrak{N}} = \text{the set of all dicritical divisors of } I \text{ in } R.$$

From Definition–Observation 6.12(III) recall that $\mathfrak{D}(R, I)$ is a finite set of DVRs; upon letting $V_1, \dots, V_h$ to be all the distinct members of $\mathfrak{D}(R, J)$, and upon letting $W_1, \dots, W_h$ be their respective (R, I)-extensions to $L(Z)$, we have that $W_1, \dots, W_h$ are distinct DVRs with $E \subset W_j$ for $1 \le j \le h$; for the definition of (R, I)-extension see the second paragraph of §6; note that if I is M-primary then $h > 0$. Let

$$P^*_j = E \cap M(W_j) \in \operatorname{spec}(E) \quad \text{for} \quad 1 \le j \le h$$

and let

$$\Lambda' = \{j : 1 \le j \le h \quad \text{and} \quad \operatorname{dpt}_E P^*_j = d\}.$$

Theorem 8.1. *Assuming that R is a d-dimensional normal local domain and I is a nonzero normal M-primary ideal in R, we have the following*:

(I) $P^*_1, \dots, P^*_h$ *are all the distinct members of* $\operatorname{nvspec}_E(ME)$ *and we have*

$$\operatorname{ht}_E P^*_1 = \dots = \operatorname{ht}_E P^*_h = 1$$

and

$$\max\left(\mathrm{dpt}_E P_1^*, \dots, \mathrm{dpt}_E P_h^*\right) = d = \dim(E) - 1.$$

(II) *Given any* $j \in \{1, \dots, h\}$ *and any* $x \in I$ *we have*

$$V_j(x) = V_j(I) \Leftrightarrow xZ \notin P_j^*.$$

(III) *If* $x_1, \dots, x_d$ *are elements in* I *such that* $(x_1, \dots, x_d)R$ *is a reduction of* I *then for* $1 \le i \le d$ *we have* $V_j(x_i) = V_j(I)$ *for all* $j \in \Lambda'$.

(IV) *If* $d = 1$, *then* $h = 1$ *and* $\Lambda' = \{1\}$. *If* $d = 2$, *then* $\Lambda' = \{1, \dots, h\}$.

(V) *If* $d = 1$, *then for any* $x_1 \in I$ *we have that*

$$V_1(x) = V_1(I) \Leftrightarrow x_1 R \text{ is a reduction of } I.$$

(VI) *Given any* $r \in \mathbb{N}_+$, *there exist elements* $x_1, \dots, x_d$ *in* I^{rs} *for some* $s \in \mathbb{N}_+$ *such that, for every* $t \in \mathbb{N}_+$, $(x_1^t, \dots, x_d^t)R$ *is a reduction of* I^{rst}.

(VII) *If* $r \in \mathbb{N}_+$ *and* $x_1 \in I^r$ *are such that* $V_j(x_1) = V_j(I^r)$ *for all* $j \in \Lambda'$ *then there exist elements* $x_2, \dots, x_d$ *in* I^{rs} *for some* $s \in \mathbb{N}_+$ *such that, for every* $t \in \mathbb{N}_+$, $(x_1^{st}, x_2^t, \dots, x_d^t)R$ *is a reduction of* I^{rst}.

(VIII) *Given any* $r \in \mathbb{N}_+$, *there exist* $s \in \mathbb{N}_+$ *and* $x_1 \in I^{rs}$ *such that* $V_j(x_1) = V_j(I^{rs})$ *for all* $j \in \Lambda'$.

(IX) *If* R/M *is infinite then there exist elements* $x_1, \dots, x_d$ *in* I *such that* $(x_1, \dots, x_d)R$ *is a reduction of* I.

(X) *If* R/M *is infinite and* $x_1 \in I$ *is such that* $V_j(x_1) = V_j(I)$ *for all* $j \in \Lambda'$ *then there exist elements* $x_2, \dots, x_d$ *in* I *such that* $(x_1, \dots, x_d)R$ *is a reduction of* I.

(XI) *Given any* $j \in \{1, \dots, h\}$ *there exists* $x'_j \in I$ *such that* $V_j(x'_j) = V_j(I)$. *If* R/M *is infinite then there exists* $x_1 \in I$ *such that* $V_j(x_1) = V_j(I)$ *for* $1 \le j \le h$.

Proof. By Definition–Observation 6.12 and Lemmas 6.13 and 6.14, we are done.

Remark 8.1*a*. The form ring $F_R(I)$ was used as a tool in proving Theorems 8.1 and 8.2 but does not explicitly appear in their statements. Similarly, the Rees ring $E_R(I)$ was used as a tool in proving these theorems and does appear in their statements, but it is not referred to in parts (V), (VI), (IX), (X), (XI) of Theorem 8.1 and in parts (V)–(XI) of Theorem 8.2 below; in this list we can include part (VII) of Theorem 8.1 by changing the phrase "for all $j \in \Lambda'$ then" by the phrase "for $1 \le j \le h$ then".

Theorem 8.2. *Either assume that* R *is a two dimensional regular local domain and* I *is a complete* M*-primary ideal in* R, *or assume that* R *is a two dimensional normal local domain and* I *is a normal* M*-primary ideal in* R. *Then we have the following*:

(I) $P_1^*, \dots, P_h^*$ *are all the distinct members of* $\mathrm{nvspec}_E(ME)$ *and we have*

$$\mathrm{ht}_E P_1^* = \cdots = \mathrm{ht}_E P_h^* = 1.$$

(II) *Given any* $j \in \{1, \dots, h\}$ *and any* $x \in I$ *we have*

$$V_j(x) = V_j(I) \Leftrightarrow xZ \notin P_j^*.$$

(III) *If* x_1, x_2 *are elements in* I *such that* $(x_1, x_2)R$ *is a reduction of* I *then for* $1 \leq i \leq 2$ *and* $1 \leq j \leq h$ *we have* $V_j(x_i) = V_j(I)$.

(IV) *We have*

$$\mathrm{dpt}_E P_1^* = \cdots = \mathrm{dpt}_E P_h^* = 2.$$

(V) *We have* $\dim(E) = 3$.

(VI) Given any $r \in \mathbb{N}_+$, *there exist elements* x_1, x_2 *in* I^{rs} *for some* $s \in \mathbb{N}_+$ *such that, for every* $t \in \mathbb{N}_+$, $(x_1^t, x_2^t)R$ *is a reduction of* I^{rst}.

(VII) *If* $r \in \mathbb{N}_+$ *and* $x_1 \in I^r$ *are such that* $V_j(x_1) = V_j(I^r)$ *for* $1 \leq j \leq h$ *then there exists* $x_2 \in I^{rs}$ *for some* $s \in \mathbb{N}_+$ *such that, for every* $t \in \mathbb{N}_+$, $(x_1^{st}, x_2^t)R$ *is a reduction of* I^{rst}.

(VIII) *Given any* $r \in \mathbb{N}_+$, *there exist* $s \in \mathbb{N}_+$ *and* $x_1 \in I^{rs}$ *such that* $V_j(x_1) = V_j(I^{rs})$ *for* $1 \leq j \leq h$.

(IX) *If* R/M *is infinite then there exist elements* x_1, x_2 *in* I *such that* $(x_1, x_2)R$ *is a reduction of* I.

(X) *If* R/M *is infinite and* $x_1 \in I$ *is such that* $V_j(x_1) = V_j(I)$ *for* $1 \leq j \leq h$ *then there exists* $x_2 \in I$ *such that* $(x_1, x_2)R$ *is a reduction of* I.

(XI) *Given any* $j \in \{1, \ldots, h\}$ *there exists* $x_j' \in I$ *such that* $V_j(x_j') = V_j(I)$. *If* R/M *is infinite then there exists* $x_1 \in I$ *such that* $V_j(x_1) = V_j(I)$ *for* $1 \leq j \leq h$.

Proof. Zariski's theorem (2′) on page 385 of Volume II of [22] tells us that if R is a two dimensional regular local domain then every complete M-primary ideal in R is a normal ideal in R. Consequently by Theorem 8.1, we are done.

Remark 8.2*a*. Proposition (3.5) of [11] says the following:

(†) Assume that R is a two dimensional regular local domain. For any V, W in $D(R)^\Delta$ let $c(R, V, W) = \mathrm{ord}_V \zeta_R(W)$, i.e., $c(R, V, W) = \min\{\mathrm{ord}_V \theta : \theta \in \zeta_R(W)\}$. Let J be a special primary pencil at R and assume that $J = (a, b)R$ where $b = \eta^m$ with $\eta \in M(R) \setminus M(R)^2$ and $m \in \mathbb{N}_+$. Let $\mathfrak{D}(R, J) = U$. Then clearly there exists $n(W) \in \mathbb{N}_+$ for all $W \in U$ such that

$$I = \prod_{W \in U} \zeta_R(W)^{n(W)} \tag{$\bullet$}$$

is the integral closure of J in R, and hence

$$m\ \mathrm{ord}_V \eta = \sum_{W \in U} n(W) c(R, V, W) \quad \text{for all } V \in U. \tag{$\bullet\bullet$}$$

As said in the Introductory Note (3.5)(0) of [11], the proof of (†) is contained in its statement. Now by Theorem 8.2 we get the converse of (†) stated below.

(‡) Assume that R is a two dimensional regular local domain. For any V, W in $D(R)^\Delta$ let $c(R, V, W) = \mathrm{ord}_V \zeta_R(W)$, i.e., $c(R, V, W) = \min\{\mathrm{ord}_V \theta : \theta \in \zeta_R(W)\}$. Given any nonempty finite $U \subset D(R)^\Delta$, let I be the complete M-primary ideal in R given by ($\bullet$) with $n(W) \in \mathbb{N}_+$. Then we have the following:

(1) There exists $b \in I^s$ for some $s \in \mathbb{N}_+$ such that for all $V \in U$ we have $V(b) = V(I^s)$; given any such b and s, their exists $a \in I^{st}$ for some $t \in \mathbb{N}_+$ such that the

pencil $J = (a, b^t)R$ is a reduction of I^{st}. Moreover, if R/M is infinite then we can take $s = t = 1$.

(2) If $b = \eta^m$ with $\eta \in M(R) \setminus M(R)^2$ and $m \in \mathbb{N}_+$ are such that $(\bullet\bullet)$ is satisfied, then there exists $a \in I^t$ for some $t \in \mathbb{N}_+$ such that the special pencil $J = (a, b^t)R$ is a reduction of I^t. Moreover, if R/M is infinite then we can take $t = 1$.

Example 8.3. We can ask the question: if R is a two dimensional normal local domain, then what is the cardinality of $\mathfrak{D}(R, M)$? The answer is that it can be any preassigned positive integer n. To see this let K be a field and consider the polynomial

$$g(X, Y, Z) = Z^m - f_1(X, Y) \dots f_n(X, Y) \in K[X, Y, Z],$$

where $m > n > 0$ are integers such that m is nondivisible by the characteristic of K, and $f_1(X, Y), \dots, f_n(X, Y)$ are pairwise coprime homogeneous linear polynomials; for instance $g(X, Y, Z) = Z^3 - XY$. Clearly $g = g(X, Y, Z)$ is irreducible in the polynomial ring $B = K[X, Y, Z]$. Let

$$\phi : B \to B/gB = A = K[x, y, z]$$

be the residue class epimorphism where we have identified K with $\phi(K)$ and we have let $x = \phi(X)$, $y = \phi(Y)$, $z = \phi(Z)$. Upon letting $R = A_{(x,y,z)A}$ it can easily be seen that R is a two dimensional normal local domain with coefficient field K and maximal ideal $M = M(R) = (x, y, z)R$. We claim that now $\mathfrak{D}(R, M)$ has exactly n elements. Geometrically speaking, a section of the tangent cone of the surface $g = 0$ at $(0, 0, 0)$ consists of n lines $f_1(X, Y) = 0, \dots, f_n(X, Y) = 0$ in the (X, Y)-plane passing through $(0, 0)$, and they give rise to n elements of $\mathfrak{D}(R, M)$. Algebraically it can be showed that $\mathfrak{D}(R, M)$ has exactly n elements thus. Let

$$A' = R\big[x', y'\big], \quad \text{where} \quad x' = x/z \quad \text{and} \quad y' = y/z.$$

Now

$$z^{m-n} - f_1\big(x', y'\big) \dots f_n\big(x', y'\big) = 0$$

and hence, upon letting $P_i = (z, f_i(x', y'))A'$ for $1 \le i \le n$, we see that $P_1, \dots, P_n$ are exactly all the distinct height-one prime ideals in A' which contain $MA' = zA'$ and the localizations of A' at them are exactly all the distinct members of $\mathfrak{D}(R, M)$.

Remark 8.3*a*. Turning to a two dimensional regular local domain, let us cite some of the things that have been achieved.

(1) In the joint paper of Abhyankar and Heinzer [13], the following existence theorem of dicritical divisors is proved: Let R be a two dimensional regular local domain with quotient field L. Let U be any finite set of prime divisors of R. Then there exists $z \in L^\times$ such that $\mathfrak{D}(R, z) = U$. Moreover, if the field $R/M(R)$ is infinite then there exists $z \in L^\times$ such that $\mathfrak{D}(R, z)^\sharp = \mathfrak{D}(R, z)^\flat = \mathfrak{D}(R, z) = U$.

(2) In the joint paper of Abhyankar and Luengo [14], the following fundamental theorem of special pencils is proved: Let R be a two dimensional regular local domain with

quotient field L. Let $z \in L^\times$ be such that z generates a special pencil at R. Then z generates a polynomial pencil in R.

(3) In Notes (3.5)(I)–(3.5)(IV) of [11], concrete examples are given to illustrate the above Remark 8.2a.

(4) In Propositions (4.1)–(4.3) of [10] and Propositions (3.1)–(3.4) of [11], the sets

$$\mathfrak{B}(R, J)^\sharp \subset \mathfrak{B}(R, J)^\flat \subset \mathfrak{B}(R, J) \subset \mathfrak{Q}(R, J) \subset Q(R)$$

of a special pencil J in a two dimensional regular local domain R, mentioned at the end of §5, are studied.

9. Dicriticals of higher dimensional local domains

Lemmas 6.11, 6.13 and 6.14 of §6 together with Theorem 8.1 of §8 constitute the initiation of the higher dimensional dicritical theory mentioned in the Introduction. Here is a scheme of how this is expected to be used in attacking the higher dimensional Jacobian conjecture; for a recent work on this conjecture, see [5–7].

Geometrically speaking, the possible relationship between dicritical divisors and the Jacobian conjecture which is to be exploited may be described thus. A polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ is given by

$$Y_1 = f_1(X_1, \dots, X_n), \dots, Y_n = f_n(X_1, \dots, X_n),$$

where $X_1, \dots, X_n$ are coordinates in the source $\mathbb{C}^n$ and $Y_1, \dots, Y_n$ are coordinates in the target $\mathbb{C}^n$. For $1 \le j \le n$ we have the polynomial map

$$f_j(X_1, \dots, X_n) : \mathbb{C}^n \to \mathbb{C}^1.$$

Going over to projective spaces we get the corresponding rational map

$$\phi_j(X_1, \dots, X_n) : \mathbb{P}^n \to \mathbb{P}^1.$$

Let $Z_1, \dots, Z_n$ be local coordinates at a point $\pi \in \mathbb{P}^n \setminus \mathbb{C}^n$ and let R be the local ring of π on $\mathbb{P}^n$. Then R is an n-dimensional regular local domain with maximal ideal $M = M(R) = (Z_1, \dots, Z_n)R$ and

$$\phi(X_1, \dots, X_n) = \frac{a_j(Z_1, \dots, Z_n)}{b_j(Z_1, \dots, Z_n)}$$

with

$$a_j = a_j(Z_1, \dots, Z_n) \text{ and } b_j = b_j(Z_1, \dots, Z_n) \text{ in } \mathbb{C}[Z_1, \dots, Z_n].$$

We get a pencil $I_j = (a_j, b_j)R$ in R and we can consider the dicritical set $\mathfrak{D}(R, I_j)$. Let us call the n-tuple $(f_1, \dots, f_n)$ a Jacobian n-tuple if the Jacobian of $f_1, \dots, f_n$ with respect to $X_1, \dots, X_n$ is a nonzero constant, and let us call it an automorphic n-tuple if $\mathbb{C}[f_1, \dots, f_n] = \mathbb{C}[X_1, \dots, X_n]$. The chain rule tells us that every automorphic n-tuple is a Jacobian n-tuple. The Jacobian conjecture predicts that every Jacobian n-tuple is an automorphic n-tuple. It is plausible that if $(f_1, \dots, f_n)$ is a Jacobian n-tuple then the dicritical sets $\mathfrak{D}(R, I_j)_{1 \le j \le n}$ are somehow related to each other and this may help us to prove that $(f_1, \dots, f_n)$ is an automorphic n-tuple.

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