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ON COUPLED SECOND-HARMONIC OSCILLATIONS OF THE CONGRUENT DARWIN ELLIPSOIDS

S. CHANDRASEKHAR
University of Chicago
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ABSTRACT

A general theory of coupled second-harmonic oscillations of the congruent Darwin ellipsoids is developed. The class of oscillations studied includes the synchronous oscillations as a special degenerate case.

Subject headings: hydrodynamics — instabilities — rotation

I. INTRODUCTION

In the preceding paper, Monique Tassoul (1975) has correctly pointed out an error in the author's treatment of the problem of the synchronous oscillations of the congruent Darwin ellipsoids which he had formulated (Chandrasekhar 1969*a*; this book will be referred to hereafter as E.F.E.; see also Chandrasekhar 1969*b*). The present paper, which may be considered as a postscript to M. Tassoul's paper, has two objectives: *first*, to generalize the notions underlying synchronous oscillations and provide a basis for a comprehensive discussion of an entire class of second-harmonic oscillations (which includes synchronous oscillations as a special degenerate case); and *second*, to make some cognate observations concerning the attitude one may have toward Darwin's problem as currently formulated and treated.

II. THE EQUATIONS GOVERNING THE SECOND-HARMONIC OSCILLATIONS OF THE CONGRUENT DARWIN ELLIPSOIDS

Under the usual assumptions (E.F.E., pp. 225–228; see also § III below) the congruent Darwin ellipsoids, in equilibrium, consist of two identical ellipsoids (of equal density, mass, and principal axes) facing each other and rotating in circular orbits about their common center of mass with a constant angular velocity Ω. In considering small oscillations about equilibrium, we shall consider the associated Lagrangian displacements in the same frame in which they were originally in equilibrium, namely, in a frame rotating with the same angular velocity Ω as in equilibrium; and we shall further consider the motions of the fluid elements in each of the two ellipsoids in Cartesian frames $[x_1^{(i)}, x_2^{(i)}, x_3^{(i)}; i = 1, 2]$ located at the centers of the respective equilibrium figures, the x_1-axis of one pointing to the (equilibrium) center of the other. And we shall distinguish the quantities pertaining to the two ellipsoids by superscripts $(i) = (1)$ and (2) referring to the "primary" and the "secondary," respectively.

We shall suppose that the two ellipsoids are simultaneously subjected to deformations, in general different, which are described by Lagrangian displacements of the form (E.F.E., § 63, eq. [144])

$$\xi_1^{(i)} = L_{1;0}^{(i)} + L_{1;1}^{(i)}x_1^{(i)} + L_{1;2}^{(i)}x_2^{(i)},$$

$$\xi_2^{(i)} = L_{2;0}^{(i)} + L_{2;1}^{(i)}x_1^{(i)} + L_{2;2}^{(i)}x_2^{(i)},$$

$$\xi_3^{(i)} = L_{3;3}^{(i)}x_3^{(i)} \qquad (i = 1, 2), \tag{1}$$

where a time-dependent factor $e^{\lambda t}$ in the coefficients $L_{j;k}^{(i)}$ has been suppressed. Displacements of this form will preserve the ellipsoidal figures of the components; but their centers will be displaced from their equilibrium positions by the amounts (cf. E.F.E., § 63, eqs. [148]–[152])

$$\frac{1}{M}(V_1^{(i)}, V_2^{(i)}, 0) \qquad (i = 1, 2) \tag{2}$$

(in their respective Cartesian frames); their principal axes will be altered in the ratios

$$\frac{\delta a_j^{(i)}}{a_j} = \frac{5}{2M}\frac{V_{jj}^{(i)}}{a_j^2} \qquad (i = 1, 2)\,; \tag{3}$$

and the ellipsoids, themselves, will be rotated about their x_3-axes by the angles

$$\delta\theta^{(i)} = \frac{5}{M}\frac{V_{12}^{(i)}}{a_1^2 - a_2^2} \qquad (i = 1, 2)\,. \tag{4}$$

In the foregoing equations, $V_k^{(i)}$ and $V_{jk}^{(i)}$ are the first and the second order "virials" as defined in E.F.E. (It will be noticed that we have not distinguished the a_j's by superscripts since they are the same for both ellipsoids; this convention will be adopted in the rest of this paper.)

The equations governing the motions of the fluid elements in the two ellipsoids, in their respective frames, can be written as

$$\rho\frac{du_j^{(i)}}{dt} = -\frac{\partial p^{(i)}}{\partial x_j^{(i)}} + \rho\frac{\partial}{\partial x_j^{(i)}}\{\mathfrak{V}^{(i)} + \mathfrak{V}'(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) - \tfrac{1}{2}Rx_1^{(i)}\Omega^2 + \tfrac{1}{2}\Omega^2[(x_1^{(i)})^2 + (x_2^{(i)})^2]\} + 2\rho\Omega\epsilon_{jl3}u_l^{(i)}\,, \tag{5}$$

where Ω denotes the constant angular velocity of the orbital rotation to be specified (see eq. [17] below) and $\mathfrak{V}'$ is the tidal potential due to the other displaced reoriented ellipsoid of altered principal axes. The expression for the tidal potential $\mathfrak{V}'$ will be expanded in a Taylor series about the equilibrium center (P) of the ellipsoid that is being considered; and in Darwin's problem the series is truncated by fiat (see § III below) at the quadratic terms:

$$\mathfrak{V}'(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) = \mathfrak{V}'(P) + \left(\frac{\partial\mathfrak{V}'}{\partial x_j^{(i)}}\right)_P x_j^{(i)} + \frac{1}{2}\left(\frac{\partial^2\mathfrak{V}'}{\partial x_j^{(i)}\partial x_k^{(i)}}\right)_P x_j^{(i)}x_k^{(i)}\,. \tag{6}$$

It remains to determine the coefficients in this expansion.

a) *The Expansion for the Tidal Potential*

We shall obtain an expansion for $\mathfrak{V}'$ that will be sufficient to allow for reorientations and displacements of the ellipsoids (retaining their ellipsoidal figures) consistent with the Lagrangian displacements assumed in equation (1). Precisely, the geometry of the situation that is being contemplated is illustrated in Figure 1. To be specific, we shall consider the tidal potential of the secondary, "2," on the primary, "1." Then, in the notation of Figure 1,

$$\delta\Theta^{(2)} = \delta\theta^{(2)} + \tfrac{1}{2}\delta\phi^{(2)}\,, \tag{7}$$

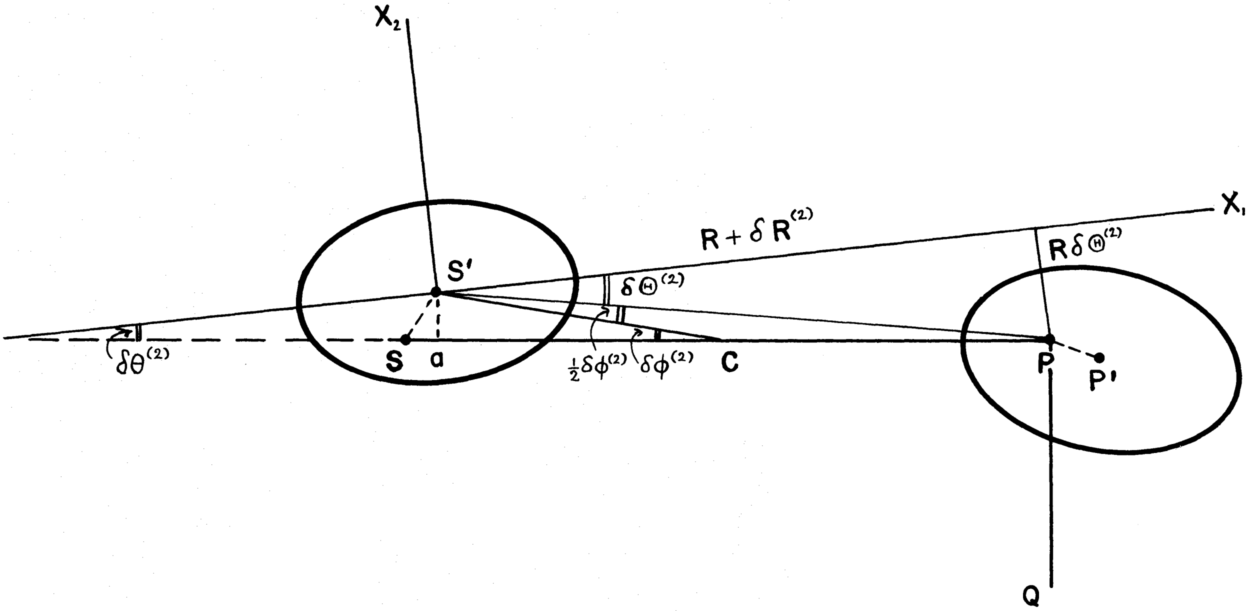

FIG. 1.—The geometry of the system. The system is considered, both when it is in equilibrium and when it is perturbed, in a frame of reference rotating with the uniform constant angular velocity appropriate for equilibrium. The section illustrated is through the orbital plane. In equilibrium, the primary and the secondary, at relative rest, are at P and at S, respectively; their common center of mass is at C, and the distance between them is R. Lagrangian displacements of the kind considered will displace the centers of the primary and the secondary to positions such as P' and S', while rotating them about their vertical axes by certain amounts. Thus, in terms of the displacement considered (eq. [1]) $Sa = L_{1;0}^{(2)}$ and $S'a = L_{2;0}^{(2)}$; and the secondary has been rotated by the angle $\delta\theta^{(2)}$. It is also clear from the illustration that $\delta R^{(2)} = -L_{1;0}^{(2)}$ and $\delta\Theta^{(2)} = \delta\theta^{(2)} + \frac{1}{2}\delta\phi^{(2)}$, where $\delta\phi^{(2)} = 2L_{2;0}^{(2)}/R$.

where

$$\delta\phi^{(2)} = 2\frac{L_{2;0}^{(2)}}{R} = 2\frac{V_2^{(2)}}{RM}\,. \tag{8}$$

Accordingly, in view of equation (4),

$$\delta\Theta^{(2)} = \frac{1}{M}\left(5\frac{V_{12}^{(2)}}{a_1^2 - a_2^2} + \frac{V_2^{(2)}}{R}\right)\,. \tag{9}$$

Also, it is apparent from Figure 1 that

$$\delta R^{(2)} = -L_{1;0}^{(2)} = -\frac{V_1^{(2)}}{M}\,. \tag{10}$$

Now, the various derivatives of the potential of a homogeneous ellipsoid, with principal axes a_j, at an external point $P = (R\cos\delta\Theta, -R\sin\delta\Theta)$ to the first order in $\delta\Theta$, in the (X_1, X_2)-principal plane of the ellipsoid, are given by (E.F.E., § 63, eqs. [154]–[158])

$$\left(\frac{\partial\mathfrak{B}'}{\partial X_1}\right)_P = -2\alpha_1 R\,; \qquad \left(\frac{\partial\mathfrak{B}'}{\partial X_2}\right)_P = 2\alpha_2 R\delta\Theta\,;$$

$$\left(\frac{\partial^2\mathfrak{B}'}{\partial X_1^2}\right)_P = 2(\alpha_2 + \alpha_3)\,; \qquad \left(\frac{\partial^2\mathfrak{B}'}{\partial X_2^2}\right)_P = -2\alpha_2\,; \qquad \left(\frac{\partial^2\mathfrak{B}'}{\partial X_3^2}\right)_P = -2\alpha_3\,;$$

and

$$\left(\frac{\partial^2\mathfrak{B}'}{\partial X_1\partial X_2}\right)_P = -\frac{4a_1a_2a_3R\delta\Theta}{(R^2 + a_2^2 - a_1^2)^{3/2}(R^2 + a_3^2 - a_1^2)^{1/2}} = -\beta_{12}R\delta\Theta \quad \text{(say)}\,. \tag{11}$$

But these formulae do *not* apply to the situation exhibited in Figure 1. Before we can use them, we must incorporate two distinguishing features. They arise from the following circumstances: *first*, the principal axes a_j and the distance R differ from their equilibrium values by amounts specified in equations (3) and (10); and *second*, the orientation of the axes to which equations (11) refer (namely, $S'X_1$ and $S'X_2$) is different from that to which equation (6) refers (namely, PS and PQ).

Allowance for the *first* feature requires that we replace the expressions given in equations (11) by

$$\left(\frac{\partial\mathfrak{B}'}{\partial X_1}\right)_P = -2\alpha_1 R - 2(\delta\alpha_1^{(2)}R + \alpha_1\delta R^{(2)})\,;$$

$$\left(\frac{\partial\mathfrak{B}'}{\partial X_2}\right)_P = 2\alpha_2 R\delta\Theta^{(2)}\,; \qquad \left(\frac{\partial^2\mathfrak{B}'}{\partial X_1\partial X_2}\right)_P = -\beta_{12}R\delta\Theta^{(2)}\,;$$

$$\left(\frac{\partial^2\mathfrak{B}'}{\partial X_1^2}\right)_P = 2(\alpha_2 + \alpha_3) + 2(\delta\alpha_2^{(2)} + \delta\alpha_3^{(2)})\,;$$

$$\left(\frac{\partial^2\mathfrak{B}'}{\partial X_2^2}\right)_P = -2\alpha_2 - 2\delta\alpha_2^{(2)}\,; \qquad \text{and} \qquad \left(\frac{\partial^2\mathfrak{B}'}{\partial X_3^2}\right)_P = -2\alpha_3 - 2\delta\alpha_3^{(2)}\,. \tag{12}$$

The expressions appropriate for $\delta\alpha_j^{(2)}$ readily follow from the equation (cf. eqs. [3] and [10])

$$\delta\alpha_j^{(2)} = \sum_{k=1}^{3}\frac{\partial\alpha_j}{\partial a_k}\delta a_k^{(2)} + \frac{\partial\alpha_j}{\partial R}\delta R^{(2)} = \frac{5}{2M}\sum_{k=1}^{3}\frac{1}{a_k}\frac{\partial\alpha_j}{\partial a_k}V_{kk}^{(2)} - \frac{1}{M}\frac{\partial\alpha_j}{\partial R}V_1^{(2)}\,; \tag{13}$$

and inserting for the partial derivatives of α_j their values, we find:

$$\delta\alpha_1^{(2)} = \frac{1}{2M}[5(Q_{11}V_{11}^{(2)} - \alpha_{12}V_{22}^{(2)} - \alpha_{13}V_{33}^{(2)}) + 2q_1RV_1^{(2)}]\,,$$

$$\delta\alpha_2^{(2)} = \frac{1}{2M}[5(Q_{21}V_{11}^{(2)} - 3\alpha_{22}V_{22}^{(2)} - \alpha_{23}V_{33}^{(2)}) + 2q_2RV_1^{(2)}]\,,$$

$$\delta\alpha_3^{(2)} = \frac{1}{2M}[5(Q_{31}V_{11}^{(2)} - \alpha_{32}V_{22}^{(2)} - 3\alpha_{33}V_{33}^{(2)}) + 2q_3RV_1^{(2)}]\,, \tag{14}$$

where Q_{ij} and q_j have the same meanings as in E.F.E., § 63, equation (186).

Turning to the *second* feature arising from the different orientations of the axes $(S'X_1, S'X_2)$ and (PS, PQ), we must consider the transformation

$$X_1 = -(x_1^{(1)} - R)\cos\delta\theta^{(2)} - x_2^{(1)}\sin\delta\theta^{(2)}\,,$$

and

$$X_2 = +(x_1^{(1)} - R)\sin\delta\theta^{(2)} - x_2^{(1)}\cos\delta\theta^{(2)}\,. \tag{15}$$

In view of this transformation, the required Taylor expansion for $\mathfrak{B}'$ can be written as

$$\begin{aligned}\mathfrak{B}'(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}) = \mathfrak{B}'(P) &- \left(\frac{\partial\mathfrak{B}'}{\partial X_1}\right)_P[x_1^{(1)} + x_2^{(1)}\delta\theta^{(2)}] - \left(\frac{\partial\mathfrak{B}'}{\partial X_2}\right)_P x_2^{(1)}\\ &+ \frac{1}{2}\left(\frac{\partial^2\mathfrak{B}'}{\partial X_1^2}\right)_P[(x_1^{(1)})^2 + 2x_1^{(1)}x_2^{(1)}\delta\theta^{(2)}] + \frac{1}{2}\left(\frac{\partial^2\mathfrak{B}'}{\partial X_2^2}\right)_P[(x_2^{(1)})^2 - 2x_1^{(1)}x_2^{(1)}\delta\theta^{(2)}]\\ &+ \frac{1}{2}\left(\frac{\partial^2\mathfrak{B}'}{\partial X_3^2}\right)_P(x_3^{(1)})^2 + \left(\frac{\partial^2\mathfrak{B}'}{\partial X_1\partial X_2}\right)_P x_1^{(1)}x_2^{(1)}\,,\end{aligned} \tag{16}$$

where for the various partial derivatives of $\mathfrak{B}'$ we must insert the expressions given in equations (12).

With the definitions

$$\Omega^2 = 4\alpha_1\,,\qquad \beta_{11} = 2(2\alpha_1 + \alpha_2 + \alpha_3)\,,\qquad \beta_{22} = 2(2\alpha_1 - \alpha_2)\,,\qquad \beta_{33} = -2\alpha_3\,; \tag{17}$$

$$\delta\beta_1^{(2)} = 2(\delta\alpha_1^{(2)}R + \alpha_1\delta R^{(2)})\,,$$

$$\delta\beta_2^{(2)} = 2R(\alpha_1\delta\theta^{(2)} - \alpha_2\delta\Theta^{(2)})\,,$$

$$\delta\beta_{11}^{(2)} = 2(\delta\alpha_2^{(2)} + \delta\alpha_3^{(2)})\,,\qquad \delta\beta_{22}^{(2)} = -2\delta\alpha_2^{(2)}\,,\qquad \delta\beta_{33}^{(2)} = -2\delta\alpha_3^{(2)}\,,$$

and

$$\delta\beta_{12}^{(2)} = 2(2\alpha_2 + \alpha_3)\delta\theta^{(2)} - \beta_{12}R\delta\Theta^{(2)}\,, \tag{18}$$

we find that the required expression for $\mathfrak{B}'$ can be written in the form

$$\begin{aligned}\mathfrak{B}'(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}) &= \mathfrak{B}'(P) + \delta\beta_1^{(2)}x_1^{(1)} + \delta\beta_2^{(2)}x_2^{(1)}\\ &+ \frac{1}{2}\sum_{k=1}^{3}[\beta_{kk} + \delta\beta_{kk}^{(2)}](x_k^{(1)})^2 + \delta\beta_{12}^{(2)}x_1^{(1)}x_2^{(1)}\,.\end{aligned} \tag{19}$$

With this expression for the tidal potential, the equation of motion governing the fluid elements of the component "1" takes the form

$$\begin{aligned}\rho\frac{du_j^{(1)}}{dt} = -\frac{\partial p^{(1)}}{\partial x_j^{(1)}} + \rho\frac{\partial}{\partial x_j^{(1)}}\Big\{&\mathfrak{B}^{(1)} + \delta\beta_1^{(2)}x_1^{(1)} + \delta\beta_2^{(2)}x_2^{(1)}\\ &+ \frac{1}{2}\sum_{k=1}^{3}[\beta_{kk} + \delta\beta_{kk}^{(2)}](x_k^{(1)})^2 + \delta\beta_{12}^{(2)}x_1^{(1)}x_2^{(1)}\Big\} + 2\rho\Omega\epsilon_{jl3}u_l^{(1)}\,.\end{aligned} \tag{20}$$

The equation of motion governing the fluid elements of the component "2" can be obtained by simply interchanging the superscripts (1) and (2) in equation (20).

When the conditions are stationary, all the $\delta\beta$'s in equation (20) [and in the equation in which the superscripts (1) and (2) are interchanged] vanish; and we shall be led to the same equilibrium figures as are described in E.F.E., § 62.

b) The System of Virial Equations Governing the Oscillations Belonging to the Second Harmonic

From equation (20), the virial equations that govern oscillations of the kind we are presently considering (with a time-dependence $e^{\lambda t}$) can be readily derived in the usual manner. They are in the notation of E.F.E.

$$\begin{aligned}\tfrac{1}{2}\lambda^2 V_{33}^{(1)} &= \delta\mathfrak{W}_{33}^{(1)} + \beta_{33}V_{33}^{(1)} + \delta\beta_{33}^{(2)}I_{33} + \delta\Pi^{(1)}\,,\\ \tfrac{1}{2}\lambda^2 V_{11}^{(1)} - 2\lambda\Omega V_{2;1}^{(1)} &= \delta\mathfrak{W}_{11}^{(1)} + \beta_{11}V_{11}^{(1)} + \delta\beta_{11}^{(2)}I_{11} + \delta\Pi^{(1)}\,,\\ \tfrac{1}{2}\lambda^2 V_{22}^{(1)} + 2\lambda\Omega V_{1;2}^{(1)} &= \delta\mathfrak{W}_{22}^{(1)} + \beta_{22}V_{22}^{(1)} + \delta\beta_{22}^{(2)}I_{22} + \delta\Pi^{(1)}\,,\\ \lambda^2 V_{1;2}^{(1)} - \lambda\Omega V_{22}^{(1)} &= -(2B_{12} - \beta_{11})V_{12}^{(1)} + \delta\beta_{12}^{(2)}I_{22}\,,\\ \lambda^2 V_{2;1}^{(1)} + \lambda\Omega V_{11}^{(1)} &= -(2B_{12} - \beta_{22})V_{12}^{(1)} + \delta\beta_{12}^{(2)}I_{11}\,,\end{aligned} \tag{21}$$

and

$$\lambda^2 V_1^{(1)} - 2\lambda\Omega V_2^{(1)} = \delta\beta_1^{(2)} + \beta_{11} V_1^{(1)} ,$$
$$\lambda^2 V_2^{(1)} + 2\lambda\Omega V_1^{(1)} = \delta\beta_2^{(2)} + \beta_{22} V_2^{(1)} . \tag{22}$$

The $\delta\Pi^{(1)}$'s occurring in equations (21) can be eliminated by standard methods, and we shall be left with (cf. E.F.E., eqs. [176]–[178])

$$(\tfrac{1}{2}\lambda^2 + 3B_{11} - B_{12} - \beta_{11})V_{11}^{(1)} + (-\tfrac{1}{2}\lambda^2 - 3B_{22} + B_{12} + \beta_{22})V_{22}^{(1)} + (B_{13} - B_{23})V_{33}^{(1)} - \delta\beta_{11}^{(2)} I_{11} + \delta\beta_{22}^{(2)} I_{22} = 0 ,$$

$$(\tfrac{1}{2}\lambda^2 + 3B_{11} + B_{12} - 2B_{13} + 2\Omega^2 - \beta_{11})V_{11}^{(1)} + (\tfrac{1}{2}\lambda^2 + 3B_{22} + B_{12} - 2B_{23} + 2\Omega^2 - \beta_{22})V_{22}^{(1)} + (-\lambda^2 - 6B_{33} + B_{13} + B_{23} + 2\beta_{33})V_{33}^{(1)} + 2(\beta_{11} - \beta_{22})\frac{\Omega}{\lambda} V_{12}^{(1)} - \frac{2\Omega}{\lambda}(I_{11} - I_{22})\delta\beta_{12}^{(2)} - \delta\beta_{11}^{(2)} I_{11} - \delta\beta_{22}^{(2)} I_{22} + 2\delta\beta_{33}^{(2)} I_{33} = 0 ,$$

and

$$\lambda\Omega V_{11}^{(1)} - \lambda\Omega V_{22}^{(1)} + (\lambda^2 + 4B_{12} - \beta_{11} - \beta_{22})V_{12}^{(1)} - \delta\beta_{12}^{(2)}(I_{11} + I_{22}) = 0 . \tag{23}$$

Equations (22) and (23) must be supplemented by the solenoidal condition

$$\sum_{k=1}^{3} \frac{V_{kk}^{(1)}}{a_k^2} = 0 \cdot \tag{24}$$

Equations (22)–(24), together with the equations obtained from them by interchanging the superscripts (1) and (2), provide a complete set of 12 homogeneous equations for the 12 virials $V_1^{(i)}$, $V_2^{(i)}$, $V_{11}^{(i)}$, $V_{22}^{(i)}$, $V_{33}^{(i)}$, and $V_{12}^{(i)}$ (i = 1 and 2); and the determinant of this set of equations will provide the characteristic equation for determining all the characteristic frequencies belonging to the modes of oscillation of the type considered.

By setting the V_j's and the V_{jk}'s pertaining to the two components as equal, we shall obtain from equations (22)–(24) two identical uncoupled sets of equations, six each; and the determinant of either set will provide the characteristic frequencies belonging to the "synchronous" modes of oscillation. I have verified by direct numerical calculation that the characteristic frequencies of the synchronous modes of oscillations so obtained are in agreement with those determined by M. Tassoul (1975).

Similarly, by setting the V_j's and the V_{jk}'s pertaining to the two components as equal in magnitude but opposite in sign, we shall again obtain two identical, uncoupled sets of equations, six each; and the determinant of either set will determine the characteristic frequencies of oscillation during which the two components are *exactly out of phase*. The enumeration of the characteristic frequencies belonging to these out-of-phase oscillations, as well as those belonging to the general coupled system of all 12 equations, may bear on the following problem considered in another—albeit erroneous—context (Chandrasekhar 1970).

Suppose that we perturb one of the components of the binary, keeping the other initially rigid. Then the component that is perturbed may be assumed to be set in one of the "natural" modes of oscillation determined in an earlier paper (Chandrasekhar 1964). The tidal potential of this oscillating component will vary periodically over the other. As a result, the other component will be set into forced *oscillation*. Since the natural frequencies of oscillation of the two components, being congruent, are equal, it is clear that, under the circumstances envisaged, we shall have a case of *resonant* forced oscillations. The amplitudes of such resonant forced oscillations will, in the first instance, increase linearly with time. The energy in the oscillation of the first component will thus begin to be drained to the second; and this transfer of energy may entail the excitation of one of the general coupled modes of oscillation considered in this paper—presumably, the out-of-phase oscillations.[1]

[1] The problem of such resonant forced oscillations has been considered (Chandrasekhar 1970). It may be of interest to revise the analysis of that paper in accordance with the equations derived in this paper. The formal solution given in that paper (eqs. [15]–[18]) is valid; but the expression for the forcing term (given in eq. [12]) may need revision.

III. SOME COGNATE OBSERVATIONS

Finally, some remarks concerning the attitude one may adopt toward Darwin's problem may be appropriate.[2] If one wishes to consider, in a mathematically and a physically consistent manner, the effect of the mutual tidal distortions of the components of a binary system on their orbital motion and on their figures, then the assumption of *exact* ellipsoidal figures cannot be made since it breaks down already beyond their first order departures from sphericity. The reason is that in the expansion of the tidal potential in powers of a_1/R, the terms inclusive of order $(a_1/R)^5$ do not depend even on the lowest order departures from sphericity of the tidally distorting body.[3] And the allowance for the terms of orders $(a_1/R)^4$ and $(a_1/R)^5$, which have angular dependences given by spherical harmonics of orders 3 and 4, distort the figures by amounts which are of orders lower than the departures of exact ellipsoidal figures from sphericity. The importance of including the terms in spherical harmonics of orders 3 and 4 has been emphasized in some early investigations in which the development of "furrows" on the boundaries of distorted polytropes has been described (cf. Chandrasekhar 1933*b*, Figs. 1 and 2). It would appear then that the chief interest in the Darwin ellipsoids arises from the fact that by truncating, by fiat, the Taylor expansion of the tidal potential after the quadratic terms, we have an exactly soluble problem in the mathematical theory of the ellipsoidal figures of equilibrium.

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[2] With respect to the understandings of Darwin and Jeans regarding the problem of the stability of the Roche and the Darwin ellipsoids, the reader may care to read their views quoted in some of the author's earlier papers: Chandrasekhar 1963 (p. 1182, § I); 1964 (p. 599, n. 2; p. 601, in the last paragraph of § II, and p. 618, § *c*). Their views, particularly on the stability of the Roche ellipsoid, may be contrasted with what is presently known (E.F.E., § 59).

[3] This is a well-known fact; it is quoted, for example, in Chandrasekhar (1933*a*, eqs. [1]–[5]).

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S. CHANDRASEKHAR: Laboratory for Astrophysics and Space Research, 933 E. 56th Street, Chicago, IL 60637