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ON A CRITERION FOR THE ONSET OF DYNAMICAL INSTABILITY BY A NONAXISYMMETRIC MODE OF OSCILLATION ALONG A SEQUENCE OF DIFFERENTIALLY ROTATING CONFIGURATIONS

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ABSTRACT

In this paper, a variational base is derived for locating the point of onset of dynamical instability along a sequence of differentially rotating configurations. The underlying idea is to seek the conditions for the existence of quasi-stationary, nonaxisymmetric modes of deformation in frames of reference rotating with various preassigned angular velocities.

Subject headings: hydrodynamics — instabilities — rotation

I. INTRODUCTION

In a recent paper (Chandrasekhar and Lebovitz 1973; this paper will be referred to hereafter as Paper II) a variational base for locating a Dedekind-like point of bifurcation along a sequence of differentially rotating configurations was derived. The method consists in determining whether a configuration allows, in an inertial frame, a quasi-stationary, nonaxisymmetric deformation with a dependence on the azimuthal angle ϕ of the form $e^{im\phi}$, where m is an integer greater than or equal to 1. It was, however, emphasized that the notion of a *neutral point* (where an instability of some sort could set in) for a rotating system is ambiguous: the ambiguity arises from the freedom we have in the choice of the coordinate frame in which we may seek the neutral mode. On this account, the possibility of a neutral deformation need not necessarily be a signal that some instability will set in. Accordingly, it would be useful if a criterion for the onset of *dynamical instability* can be given, since *its* onset cannot depend on the choice of frame. In this paper, we shall show how by a simple generalization of the analysis of Paper II, we can establish a criterion for the onset of dynamical instability by a nonaxisymmetric mode of overstable oscillations.

II. THE BASIC IDEA

Suppose that in the inertial frame, the configuration allows a nonaxisymmetric mode of oscillation described by a Lagrangian displacement of the form

$$\xi^{\alpha}{}_{\text{inertial}}(\varpi, z; \phi, t) = \xi^{\alpha}(\varpi, z)e^{i(\sigma t + m\phi)} \qquad (\alpha = \varpi, z)\,, \tag{1}$$

where (ϖ, z, ϕ) define a system of cylindrical polar coordinates, m is an integer greater than or equal to one, and σ is a characteristic frequency. We now inquire[1] how this deformation will appear to an observer at rest in a frame rotating with a uniform angular velocity Ω_0.

Let ϕ_0 denote the azimuthal angle in the observer's rotating frame. It is related to the angle ϕ in the inertial frame by

$$\phi_0 = \phi - \Omega_0 t\,. \tag{2}$$

Accordingly, the displacement will appear to the observer in the rotating frame as

$$\xi^{\alpha}{}_{\text{rotating frame}} = \xi^{\alpha}(\varpi, z)\exp\left[i(\sigma + m\Omega_0)t + im\phi_0\right]. \tag{3}$$

From equation (3) it follows: *so long as σ is real and the mode of oscillation is a stable one, there exists an Ω_0 $(= -\sigma/m)$ such that, for an observer, at rest in a frame rotating uniformly with this angular velocity Ω_0, the mode will appear as one associated with a neutral deformation; and the signal for the onset of dynamical instability is when two such values of Ω_0 coincide to become complex conjugates.*

From the foregoing, it is clear that what we have to seek is the condition for the occurrence of a neutral mode of deformation with a ϕ-dependence $e^{im\phi}$ in a frame of reference rotating with some initially specified angular

[1] The argument which follows is essentially that described in a different context in Chandrasekhar (1969, § 36); this book will be referred to hereafter as *E.F.E.*

velocity Ω_0. From such determinations of Ω_0 along a sequence of configurations, we can locate the point at which dynamical instability will set in: *it is the point where two allowed values of* Ω_0 *coincide.*

III. THE EQUATIONS GOVERNING STATIONARY NONAXISYMMETRIC SYSTEMS IN A ROTATING FRAME

In a frame of reference rotating with a uniform angular velocity Ω_0, the hydrodynamic equations governing a perfect fluid in a stationary state, and in gravitational equilibrium are

$$\frac{Dv^{\varpi}}{ds} = \frac{V^2}{\varpi} + \frac{\partial \mathfrak{V}}{\partial \varpi} - \frac{1}{\rho}\frac{\partial p}{\partial \varpi} + {\Omega_0}^2\varpi + 2V\Omega_0 \tag{4}$$

and

$$\frac{Dv^z}{ds} = \frac{\partial \mathfrak{V}}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{5}$$

where v^{ϖ}, v^z, and $V\,(=\varpi\Omega)$ are the linear velocities along the three principal directions, $\mathfrak{V}$ is the Newtonian gravitational potential, ρ is the density, p is the pressure, and

$$\frac{D}{ds} = \Omega\frac{\partial}{\partial \phi} + v^{\varpi}\frac{\partial}{\partial \varpi} + v^z\frac{\partial}{\partial z}\,. \tag{6}$$

Besides, we have the equation of continuity and Poisson's equation; these have the same forms as in the inertial frame. But the Bernoulli integral has a different form; we now have

$$\frac{D}{ds}\left(\tfrac{1}{2}|v|^2 - \mathfrak{V} - \tfrac{1}{2}{\Omega_0}^2\varpi^2 + \frac{p}{\rho} + \Pi\right) = 0\,, \tag{7}$$

where Π denotes the thermodynamic internal energy (per unit mass).

a) *The Equations Governing Equilibrium*

As our initial equilibrium configuration (which we shall presently subject to quasi-stationary deformations) we shall consider one which is stationary and axisymmetric and in which only rotational motions (specified by Ω) are present:

$$V = \varpi\Omega\,, \qquad v^{\varpi} = v^z = 0\,. \tag{8}$$

We shall further suppose that

$$\Omega \equiv \Omega(\varpi)\,. \tag{9}$$

It should be noticed that Ω is the angular velocity as viewed in the rotating frame. The angular velocity Ω_i, in the inertial frame, is given by

$$\Omega_i = \Omega + \Omega_0\,. \tag{10}$$

Corresponding to Ω_0 and Ω_i, we have the linear velocities

$$V_0 = \varpi\Omega_0 \quad \text{and} \quad V_i = \varpi\Omega_i = \varpi(\Omega + \Omega_0) = V + V_0\,. \tag{11}$$

From equations (4) and (5), we readily verify that the equation governing equilibrium is

$$\mathfrak{V}_{,\alpha} - \frac{1}{\rho}p_{,\alpha} = -\delta_{\alpha}{}^{\varpi}{\Omega_i}^2\varpi \qquad (\alpha = \varpi, z)\,; \tag{12}$$

equation (12) is, in fact, the same as the equation in the inertial frame (cf. Paper II, eq. [8]). And again, by virtue of equation (9),

$$p \equiv p(\rho)\,. \tag{13}$$

IV. THE EQUATIONS GOVERNING INFINITESIMAL QUASI-STATIONARY, NONAXISYMMETRIC DEFORMATIONS

We shall suppose that a configuration, initially axisymmetric, stationary, and governed by equations (8), (12), and (13) is subjected to an infinitesimal nonaxisymmetric deformation in a quasi-stationary manner, *in the rotating* *frame considered.* As in Paper II, we shall describe the deformation in terms of a Lagrangian displacement $\xi^\alpha(\varpi, z; \phi)$ related to v^α by

$$v^\alpha = \Omega \xi^\alpha{}_{,\phi}\,; \tag{14}$$

and further suppose that

$$\xi^\alpha(\varpi, z; \phi) = \xi^\alpha(\varpi, z) e^{im\phi}\,, \tag{15}$$

where m is an integer greater than or equal to 1. And again, we shall distinguish between the Eulerian and the Lagrangian changes defined in terms of the operators δ and Δ.

The linearized version of the Bernoulli integral (7) now gives

$$V\Delta V - \Delta\mathfrak{V} - \Omega_0{}^2 \varpi \xi^\varpi + \frac{\gamma p}{\rho^2} \Delta\rho = 0\,. \tag{16}$$

Alternative forms of equation (16), which we shall find useful, are (cf. Paper II, eqs. [16] and [17])

$$V\delta V = -\frac{1}{\rho}\,\delta p + \delta\mathfrak{V} - 2\varpi\xi^\varpi \Omega\Omega_i - \varpi^2 \xi^\varpi \Omega\Omega_{,\varpi} \tag{17}$$

and

$$\frac{\delta V}{V} = -\beta^2 \frac{\Delta\rho}{\rho} + \frac{\Delta\mathfrak{V}}{V^2} + \frac{\xi^\varpi}{\varpi}\,\frac{\Omega_0{}^2 - \Omega^2}{\Omega^2} - \xi^\varpi \frac{\Omega_{,\varpi}}{\Omega}\,, \tag{18}$$

where

$$\beta = \left(\frac{\gamma p}{\rho}\right)^{1/2} \frac{1}{V}\,. \tag{19}$$

The linearized version of the equation of continuity has the same form as in the inertial frame; we have

$$\Delta(\rho V) = -\rho V \xi^\alpha{}_{,\alpha} \qquad \text{or} \qquad \delta(\rho V) = -(\rho V \xi^\alpha)_{,\alpha}\,. \tag{20}$$

An alternative form of this equation is

$$\frac{\Delta\rho}{\rho} = -\frac{\delta V}{V} - \xi^\alpha{}_{,\alpha} - \frac{\xi^\varpi}{\varpi} - \xi^\varpi \frac{\Omega_{,\varpi}}{\Omega}\,. \tag{21}$$

Combining equations (18) and (21), we obtain (cf. Paper II, eq. [21])

$$\frac{\Delta\rho}{\rho} = \frac{1}{\beta^2 - 1}\left(\frac{\Delta\mathfrak{V}}{V^2} + \xi^\alpha{}_{,\alpha} + \frac{\xi^\varpi}{\varpi}\,\frac{\Omega_0{}^2}{\Omega^2}\right). \tag{22}$$

Linearizing the hydrodynamical equations, we obtain, as in Paper II, the pulsation equation,

$$-m^2 \rho\Omega^2 \xi^\alpha = -\left(\frac{\gamma p}{\rho}\,\Delta\rho\right)_{,\alpha} + \rho[\Delta(\mathfrak{V} + \tfrac{1}{2}V_i{}^2)]_{,\alpha} + \frac{\Delta\rho}{\rho}\,p_{,\alpha} - \rho\varpi^2(\Omega_i\Delta\Omega)_{,\alpha} - \rho\varpi\xi^\varpi \Omega_i{}^2{}_{,\alpha}\,. \tag{23}$$

This equation is, in fact, identical with the equation in the inertial frame (cf. Paper II, eq. [26]).

And finally, we have the linearized version of Poisson's equation:

$$(\varpi\delta\mathfrak{V}_{,\varpi})_{,\varpi} + (\varpi\delta\mathfrak{V})_{,zz} + 4\pi G\varpi\delta\rho = \frac{m^2}{\varpi}\,\delta\mathfrak{V}\,. \tag{24}$$

Equations (23) and (24) provide three *dynamical equations,* while equations (18) and (22) are to be considered as providing *initial-value equations.* These equations must be supplemented by the boundary conditions

$$\xi^\alpha \quad \text{and} \quad \delta\mathfrak{V} \text{ are everywhere finite}\,,$$

$$\Delta p = \frac{\gamma p}{\rho}\,\Delta\rho = 0 \quad \text{and} \quad n^\alpha \delta\mathfrak{V}_{,\alpha}\Big|_-^+ = -4\pi G\rho n_\alpha \xi^\alpha \text{ on the boundary where } p = 0\,,$$

and

$$\delta\mathfrak{V} \prec r^{-3} \quad \text{as} \quad r \to \infty\,, \tag{25}$$

where n^α is the unit outward normal to the boundary.

The equations and boundary conditions which we have enumerated constitute a characteristic-value problem for m^2; and we shall treat it in the same manner as in Paper II.

V. THE VARIATIONAL PRINCIPLE

Defining a *trial displacement* $\bar{\xi}^\alpha$ and a *trial change* $\delta\bar{\mathfrak{V}}$ as in Paper II (§ 4) and proceeding along identical steps, we now obtain (in place of Paper II, eq. [29])

$$m^2 \iint \left(\rho\varpi\Omega^2\xi^\alpha\bar{\xi}^\alpha - \frac{\delta\mathfrak{V}\delta\bar{\mathfrak{V}}}{4\pi G\varpi}\right)d\varpi dz$$
$$= \iint \Bigg[\varpi\gamma p \frac{\Delta\rho\bar{\Delta}\rho}{\rho^2} + \frac{\varpi}{\rho}\xi^\alpha p_{,\alpha}\bar{\xi}^\beta \rho_{,\beta} - \varpi p_{,\alpha}\left(\xi^\alpha\frac{\bar{\Delta}\rho}{\rho} + \bar{\xi}^\alpha\frac{\Delta\rho}{\rho}\right) - \rho\varpi(\delta V + \varpi\xi^\varpi\Omega_{,\varpi})(\delta\bar{V} + \varpi\bar{\xi}^\varpi\Omega_{,\varpi})$$
$$- \rho\varpi^2\xi^\varpi\bar{\xi}^\varpi\Omega_i{}^2{}_{,\varpi} - 2\rho\varpi\Omega_i(\xi^\varpi\delta\bar{V} + \bar{\xi}^\varpi\delta V) - \varpi(\delta\mathfrak{V}\delta\bar{\rho} + \delta\bar{\mathfrak{V}}\delta\rho)$$
$$+ \frac{\varpi}{4\pi G}(\delta\mathfrak{V}_{,\varpi}\delta\bar{\mathfrak{V}}_{,\varpi} + \delta\mathfrak{V}_{,z}\delta\bar{\mathfrak{V}}_{,z})\Bigg]d\varpi dz$$
$$+ \int_S [n_\alpha\xi^\alpha n_\beta\bar{\xi}^\beta|\text{grad } p| - \rho n_\alpha(\xi^\alpha\delta\bar{\mathfrak{V}} + \bar{\xi}^\alpha\delta\mathfrak{V})]ds\,. \tag{26}$$

The manifest symmetry of this equation in the barred and the unbarred quantities, clearly implies that we can formulate a variational base for solving the characteristic-value problem specified by equations (18), (22), (23), (24), and (25). Instead of Paper I, equation (31), we now have

$$\bar{m}^2 \iint \left\{\rho\varpi\Omega^2[(\xi^\varpi)^2 + (\xi^z)^2] - \frac{\delta\mathfrak{V}^2}{4\pi G\varpi}\right\}d\varpi dz$$
$$= \iint \Bigg\{\rho\varpi V^2\left[\frac{1}{1-\beta^2}\left(\frac{\Delta\mathfrak{V}}{V^2} + \xi^\alpha{}_{,\alpha} + \frac{\xi^\varpi}{\varpi}\frac{\Omega_0{}^2}{\Omega^2}\right)^2 - \left(\xi^\alpha{}_{,\alpha} - \frac{\xi^\varpi}{\varpi}\right)^2\right] + 2\rho\Omega^2(\xi^\varpi)^2\left(2 + \frac{\varpi\Omega_{,\varpi}}{\Omega}\right) + \frac{\varpi}{\rho}\xi^\alpha p_{,\alpha}\xi^\beta\rho_{,\beta}$$
$$+ 2\varpi\delta\mathfrak{V}\xi^\alpha\rho_{,\alpha} + 2\rho\varpi\Omega\Omega_0\xi^\varpi\left(2\xi^\varpi + 2\varpi\xi^\alpha{}_{,\alpha} + \varpi\xi^\varpi\frac{\Omega_{,\varpi}}{\Omega}\right) + \frac{\varpi}{4\pi G}[(\delta\mathfrak{V}_{,\varpi})^2 + (\delta\mathfrak{V}_{,z})^2]\Bigg\}d\varpi dz$$
$$+ \int_S [(n_\alpha\xi^\alpha)^2|\text{grad } p| - 2\rho\delta\mathfrak{V}n_\alpha\xi^\alpha]ds\,. \tag{27}$$

And again "enlarging" the definition of ξ^α to include a ϕ-component (as in Paper II, § IV*a*), and defining a density

$$\delta\hat{\rho} = -\frac{1}{\varpi}(\varpi\rho\xi^\varpi)_{,\varpi} - (\rho\xi^z)_{,z} - \frac{m\rho}{\varpi}\xi^\phi\,, \tag{28}$$

we can reduce equation (27) to the form (cf. Paper II, eq. [40])

$$\bar{m}^2\iint \rho\varpi\Omega^2(\xi^z)^2 d\varpi dz + (\bar{m}^2 - 4)\iint \rho\varpi\Omega^2(\xi^\varpi)^2 d\varpi dz + \frac{m^2 - \bar{m}^2}{4\pi G}\iint \frac{\delta\mathfrak{V}^2}{\varpi}d\varpi dz$$
$$= \iint \Bigg\{\rho\varpi V^2\left[\frac{1}{1-\beta^2}\left(\frac{\Delta\mathfrak{V}}{V^2} + \xi^\alpha{}_{,\alpha} + \frac{\xi^\varpi}{\varpi}\frac{\Omega_0{}^2}{\Omega^2}\right)^2 - \left(\xi^\alpha{}_{,\alpha} - \frac{\xi^\varpi}{\varpi}\right)^2\right] + 2\rho\varpi^2(\xi^\varpi)^2\Omega_i\Omega_{,\varpi} + \frac{\varpi}{\rho}\xi^\alpha p_{,\alpha}\xi^\beta\rho_{,\beta}$$
$$+ 2\varpi\delta\mathfrak{V}\xi^\alpha\rho_{,\alpha} + \varpi\delta\mathfrak{V}\delta\hat{\rho} + 4\rho\Omega_0\Omega\varpi\xi^\varpi(\xi^\varpi + \varpi\xi^\alpha{}_{,\alpha})\Bigg\}d\varpi dz$$
$$+ \int_S [(n_\alpha\xi^\alpha)^2|\text{grad } p| - \rho\delta\mathfrak{V}n_\alpha\xi^\alpha]ds\,, \tag{29}$$

where it should be remembered that $\Omega = (\Omega_i - \Omega_0)$ is the angular velocity in the rotating frame.

Equation (29) is more conveniently considered as an equation for Ω_0 for some assigned integral value for $\bar{m}$ rather than as an equation for determining where, along a sequence of configurations, $\bar{m}$ takes an integral value for some assigned Ω_0.

VI. AN ILLUSTRATIVE EXAMPLE

For the trial function

$$\xi^{\varpi} = \varpi\,, \qquad \xi^{z} = 0\,, \qquad \text{and} \qquad \xi^{\phi} = -\varpi \tag{30}$$

appropriate for $m = 2$, considered in Paper II, § V, equation (29) reduces to

$$\iint \left\{\frac{\rho\varpi V^2}{1-\beta^2}\left[\frac{\varpi}{V^2}(\mathfrak{B}-\mathfrak{D})_{,\varpi} + 1 + \frac{\Omega_0^{\,2}}{\Omega^2}\right]^2 + 2\rho\varpi^3\Omega\Omega_{,\varpi} + \frac{\varpi^3}{\rho}p_{,\varpi}\rho_{,\varpi} - \varpi^3\rho_{,\varpi}\mathfrak{D}_{,\varpi} + 2\rho\varpi^3\Omega\Omega_0\left(4 + \varpi\frac{\Omega_{,\varpi}}{\Omega}\right)\right\}d\varpi dz + \int_S [(n_\alpha\xi^\alpha)^2|\text{grad}\,p| - \rho\delta\mathfrak{B}n_\alpha\xi^\alpha]ds = 0\,, \tag{31}$$

where $\mathfrak{D}$ is the solution of the equation

$$(\varpi^3\mathfrak{D}_{,\varpi})_{,\varpi} + (\varpi^3\mathfrak{D}_{,z})_{,z} = -4\pi G\rho\varpi^3\,. \tag{32}$$

a) Verification of Equation (31) *for the Maclaurin Sequence*

We shall now verify that equation (31) correctly predicts, for each member of the Maclaurin sequence, the two values of Ω_0 for which quasi-stationary deformations of the form considered exist; and further that the criterion for the onset of dynamical instability is when the two values of Ω_0 coincide.

Along the Maclaurin sequence

$$\mathfrak{B} = \pi G\rho(I - A_1\varpi^2 - A_3z^2) \qquad (I = 2A_1a_1^{\,2} + A_3a_3^{\,2})$$

and

$$\mathfrak{D} = \pi G\rho a_1^{\,2}(A_1 - A_{11}\varpi^2 - A_{13}z^2)\,, \tag{33}$$

where the index-symbols A_1, A_{11}, B_{11}, etc., have their standard meanings (cf. *E.F.E.*, § 21).

In the present context, we should require

$$\frac{\varpi}{V^2}(\mathfrak{B}-\mathfrak{D})_{,\varpi} + 1 + \frac{\Omega_0^{\,2}}{\Omega^2} = 0\,, \tag{34}$$

since this quantity is proportional to $\Delta\rho$. By measuring angular velocities in the unit $(\pi G\rho)^{1/2}$, we obtain from equation (34) the relation

$$\Omega^2 + \Omega_0^{\,2} - 2B_{11} = 0\,. \tag{35}$$

If Ω_{Mc}, in the same unit, denotes the angular velocity of rotation of the Maclaurin spheroid (in the inertial frame), equation (35) can be rewritten in the form

$$(\Omega_{\text{Mc}} - \Omega_0)^2 + \Omega_0^{\,2} - 2B_{11} = 0\,, \tag{36}$$

or

$$\Omega_0^{\,2} - \Omega_0\Omega_{\text{Mc}} - (B_{11} - \tfrac{1}{2}\Omega_{\text{Mc}}^{\,2}) = 0\,. \tag{37}$$

The surviving part of equation (31) is

$$\iint 8\rho\varpi^3\Omega\Omega_0 d\varpi dz + \int_S [(n_\alpha\xi^\alpha)^2|\text{grad}\,p| - \rho\delta\mathfrak{B}n_\alpha\xi^\alpha]ds = 0\,. \tag{38}$$

The surface integral in equation (38), transformed back into a volume integral, is

$$-\iint \{\rho[\varpi^3(\mathfrak{B} + \tfrac{1}{2}\Omega_{\text{Mc}}^{\,2}\varpi^2)_{,\varpi}]_{,\varpi} + \rho(\varpi\xi^{\varpi}\delta\mathfrak{B})_{,\varpi}\}d\varpi dz = \pi G\rho\iint 8\rho\varpi^3(B_{11} - \tfrac{1}{2}\Omega_{\text{Mc}}^{\,2})d\varpi dz\,, \tag{39}$$

where, in the reductions, we have made use of the relation (cf. Paper II, eq. [43])

$$\delta\mathfrak{B} = -\varpi\frac{\partial\mathfrak{D}}{\partial\varpi} = 2\pi G\rho a_1^{\,2}A_{11}\varpi^2\,. \tag{40}$$

Equation (38) accordingly requires

$$\Omega\Omega_0 + B_{11} - \tfrac{1}{2}\Omega_{Mc}^2 = 0\,; \tag{41}$$

and this condition is clearly equivalent to (37).

Solving equation (37) for Ω_0, we obtain

$$\Omega_0 = \tfrac{1}{2}[\Omega_{Mc} \pm (4B_{11} - \Omega_{Mc}^2)^{1/2}]\,. \tag{42}$$

This last expression for Ω_0 is in agreement with the known result (cf. *E.F.E.*, p. 94, eq. [106]). The two values of Ω_0 given by equation (42) coincide when

$$\Omega_{Mc}^2 = 4B_{11}\,; \tag{43}$$

and this is indeed the point where dynamical instability sets in along the Maclaurin sequence.

VII. CONCLUDING REMARK

In view of the doubts that have sometimes been expressed concerning the criteria, for the stability of differentially rotating stars, that have been derived by the method of the tensor virial, it would be useful to have those results confirmed by making use of the variational principle derived in this paper.

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