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ON A CRITERION FOR THE OCCURRENCE OF A DEDEKIND-LIKE POINT OF BIFURCATION ALONG A SEQUENCE OF AXISYMMETRIC SYSTEMS. II. NEWTONIAN THEORY FOR DIFFERENTIALLY ROTATING CONFIGURATIONS

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ABSTRACT

The equations in the Newtonian theory, which govern quasi-stationary nonaxisymmetric deformations of axisymmetric configurations in nonuniform rotation, are derived; and the condition for the existence of such deformations with a ϕ-dependence of the form $e^{im\phi}$ (where m is an integer greater than or equal to 1) is expressed in terms of a variational principle. The condition for the case $m = 2$ applies for the occurrence of a Dedekind-like point of bifurcation.

In an appendix the variational principle governing the axisymmetric modes of oscillation of differentially rotating systems is reformulated in a manner that avoids the solution of a second-order partial differential equation.
Subject heading: rotation

I. INTRODUCTION

The recent studies of Ostriker and his associates (Bodenheimer and Ostriker 1973; Ostriker and Bodenheimer 1973; and the references listed in these papers) have shown that sequences of differentially rotating compressible masses—their "zero-viscosity" sequences—exhibit remarkable similarities with the classical Maclaurin sequence of uniformly rotating incompressible masses. For example, these studies strongly suggest that sequences of differentially rotating configurations will exhibit points of bifurcation (i.e., points of neutral stability) where $|\mathfrak{T}/\mathfrak{W}| \simeq 0.14$, where $\mathfrak{T}$ denotes the kinetic energy and $\mathfrak{W}$ the gravitational potential energy of the configuration. Ostriker and Bodenheimer locate the points of bifurcation along their sequences by determining the characteristic frequencies of the relevant nonaxisymmetric modes of oscillation in accordance with the formulae of Ostriker and Tassoul (1969; see also Tassoul and Ostriker 1968); these formulae were derived from the tensor-virial equations in the manner of Chandrasekhar and Lebovitz (1962, 1963) in their treatment of uniformly rotating masses. However, it would appear that the application of the tensor-virial equations is subject to greater uncertainties in the context of differentially rotating systems than it is in the context of uniformly rotating systems. In any event, it would be useful to establish a criterion for the occurrence of a point of neutral stability that is exact and which is susceptible, at the same time, to an algorism that will locate the point with increasing precision. But it is important to observe, first, that the definition of a "neutral point" for rotating systems is subject to an ambiguity: it arises from the freedom we have in the choice of a coordinate frame in which we wish to specify the characteristic frequencies belonging to the various normal modes. Thus, in the case of the Maclaurin sequence, its *entire* stable part can be considered as a curve of neutral stability in *four* different ways corresponding to the four Riemann sequences which branch off from *each* point (cf. Chandrasekhar 1969, § 36; this book will be referred to hereafter as *E.F.E.*). Nevertheless, in the context of uniformly rotating systems two frames of reference are naturally distinguished: the frame rotating with the uniform angular velocity Ω of the configuration, and the inertial frame. Along the Maclaurin sequence, the neutral points in the two frames occur at *exactly* the same point; but the modes to which they belong are different: the neutral mode in the rotating frame deforms the Maclaurin spheroid into a Jacobi ellipsoid while the neutral mode in the inertial frame deforms it into a Dedekind ellipsoid (cf. *E.F.E.*, p. 94, the remarks following eq. [106]). It is also known that the onset of *secular instability* at the common point can be by either mode depending on the nature of the dissipative process that may be operative; thus gravitational radiation-reaction makes the Maclaurin spheroid unstable by the Dedekind mode while normal viscous dissipation makes it similarly unstable by the Jacobi mode (cf. Chandrasekhar 1970).

When a configuration is rotating nonuniformly—with Ω as a function of ϖ (the distance from the rotation axis), for example—we cannot distinguish in any unambiguous way a particular rotating frame as unique. But no such ambiguity beclouds the definition of the inertial frame. Accordingly, the definition of a Dedekind-like point of bifurcation is not subject to the ambiguity which is inherent in the definition of a Jacobi-like point of bifurcation.

In view of the foregoing remarks, we shall restrict ourselves in this paper to the location of a Dedekind-like point of bifurcation along a sequence of differentially rotating configurations. For this purpose, we shall establish a criterion which will specify the circumstances under which an initial configuration in equilibrium can be subjected to a nontrivial quasi-stationary nonaxisymmetric deformation without violating any of the governing equations. The present paper provides, therefore, the Newtonian version of the relativistic theory developed in the preceding paper (Chandrasekhar and Friedman 1973*b*; this paper will be referred to hereafter as Paper I).

II. THE EQUATIONS GOVERNING STATIONARY NON-AXISYMMETRIC SYSTEMS

In a system of cylindrical-polar coordinates, (ϖ, z, ϕ), the equations of hydrodynamics governing a perfect fluid, in a stationary state and in gravitational equilibrium, are

$$\frac{Dv^{\varpi}}{ds} = \frac{V^2}{\varpi} + \frac{\partial \mathfrak{B}}{\partial \varpi} - \frac{1}{\rho}\frac{\partial p}{\partial \varpi} \quad \text{and} \quad \frac{Dv^z}{ds} = \frac{\partial \mathfrak{B}}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{1}$$ [1]

where v^{ϖ}, v^z, and $V(=\varpi\Omega)$ are the linear velocities along the three principal directions, $\mathfrak{B}$ is the Newtonian gravitational potential, ρ is the density, p is the pressure, and

$$\frac{D}{ds} = \Omega\frac{\partial}{\partial \phi} + v^{\varpi}\frac{\partial}{\partial \varpi} + v^z\frac{\partial}{\partial z}. \tag{2}$$

Besides, we have the equation of continuity

$$\frac{\partial}{\partial \phi}(\rho V) + \frac{\partial}{\partial \varpi}(\rho\varpi v^{\varpi}) + \frac{\partial}{\partial z}(\rho\varpi v^z) = 0, \tag{3}$$

and Poisson's equation

$$(\varpi\mathfrak{B}_{,\varpi})_{,\varpi} + (\varpi\mathfrak{B})_{,zz} + 4\pi G\varpi\rho = -\frac{1}{\varpi}\frac{\partial^2\mathfrak{B}}{\partial \phi^2}. \tag{4}$$

[1] The ϕ-component of the equation is not written out here; its role in the present analysis is replaced by the Bernoulli integral (see eq. [5] below; also footnote 2).

And if the entropy of a fluid element is conserved during its motion, we have the *Bernoulli integral*

$$\frac{D}{ds}\left(\tfrac{1}{2}|v|^2 - \mathfrak{V} + \frac{p}{\rho} + \Pi\right) = 0\,, \tag{5}$$

where Π denotes the thermodynamic internal energy (per unit mass).

a) The Equations Governing Equilibrium

As our initial equilibrium configuration (which we shall presently subject to quasi-stationary deformations) we shall consider one which is stationary and axisymmetric and in which only rotational motions (specified by Ω) are present:

$$V = \varpi\Omega\,, \qquad v^{\varpi} = v^z = 0\,. \tag{6}$$

We shall further suppose that

$$\Omega \equiv \Omega(\varpi)\,. \tag{7}$$

The equations governing the equilibrium of such configurations are

$$(\mathfrak{V} + \tfrac{1}{2}V^2)_{,\alpha} - \frac{1}{\rho}p_{,\alpha} = \tfrac{1}{2}\varpi^2\Omega^2{}_{,\alpha} \qquad (\alpha = \varpi, z)\,, \tag{8}$$

and

$$(\varpi\mathfrak{V}_{,\varpi})_{,\varpi} + (\varpi\mathfrak{V})_{,zz} = -4\pi G\varpi\rho\,. \tag{9}$$

It can be verified that for distributions of pressure and density compatible with equations (7) and (8)

$$\frac{\partial(p, \rho)}{\partial(\varpi, z)} = 0 \qquad \text{or} \qquad p \equiv p(\rho)\,. \tag{10}$$

(This result is indeed obvious; but it seems to have been explicitly stated first by Wavre 1932.)

III. THE EQUATIONS GOVERNING INFINITESIMAL NONAXISYMMETRIC DEFORMATIONS

We shall suppose that a configuration initially axisymmetric, stationary, and governed by equations (6)–(9) is subjected to an infinitesimal nonaxisymmetric deformation in a quasi-stationary manner (i.e., infinitely slowly). We shall describe such a deformation in terms of a Lagrangian displacement $\xi^\alpha(\varpi, z; \phi)$ related to the velocity v^α (strictly, the Eulerian change δv^α in the velocity) by

$$v^\alpha = \Omega\xi^\alpha{}_{,\phi}\,; \tag{11}$$

and further suppose that

$$\xi^\alpha(\varpi, z; \phi) = \xi^\alpha(\varpi, z)e^{im\phi}\,, \tag{12}$$

where m is an integer greater than or equal to 1.

As usual, we shall distinguish between an Eulerian and a Lagrangian change that a quantity experiences by virtue of the deformation. These changes can be expressed as the result of an Eulerian (δ) or a Lagrangian (Δ) operator acting on it; and the relation between the two operators is

$$\Delta = \delta + \xi^{\alpha} \frac{\partial}{\partial x^{\alpha}} \,. \tag{13}$$

The equations governing the deformation can be obtained by linearizing equations (1)–(5) about the equilibrium solution. Thus, the linearization of the Bernoulli integral (5) gives

$$V \Delta V - \Delta \mathfrak{V} + \frac{\gamma p}{\rho^2} \Delta \rho = 0 \,, \tag{14}$$

where use has been made of the relations

$$\frac{\Delta p}{p} = \gamma \frac{\Delta \rho}{\rho} \quad \text{and} \quad p \Delta \left(\frac{1}{\rho} \right) = -\Delta \Pi \,, \tag{15}$$

which ensure that each element of the fluid conserves its entropy during its motion. In equation (15), γ denotes the adiabatic exponent as commonly defined. Alternative forms of equation (14) which will be found useful are

$$V \delta V = -\frac{1}{\rho} \delta p + \delta \mathfrak{V} - 2 \xi^{\varpi} \varpi \Omega^2 - \tfrac{1}{2} \xi^{\varpi} \varpi^2 \Omega^2{}_{,\varpi} \tag{16}$$

and

$$\frac{\delta V}{V} = -\beta^2 \frac{\Delta \rho}{\rho} + \frac{\Delta \mathfrak{V}}{V^2} - \xi^{\varpi} \frac{\Omega_{,\varpi}}{\Omega} - \frac{\xi^{\varpi}}{\varpi} \,, \tag{17}$$

where

$$\beta = \frac{\text{velocity of sound}}{\text{transverse linear velocity}} = \frac{1}{V} \left(\frac{\gamma p}{\rho} \right)^{1/2} . \tag{18}$$

Similarly, the linearization of the equation of continuity (3) gives

$$\Delta(\rho V) = -\rho V \xi^{\alpha}{}_{,\alpha} \,; \tag{19}$$

or, alternatively

$$\frac{\Delta \rho}{\rho} = -\frac{\delta V}{V} - \xi^{\alpha}{}_{,\alpha} - \frac{\xi^{\varpi}}{\varpi} - \xi^{\varpi} \frac{\Omega_{,\varpi}}{\Omega} \,. \tag{20}$$

Solving equations (17) and (20) for $\Delta \rho / \rho$ and $\delta V / V$, we obtain

$$\frac{\Delta \rho}{\rho} = \frac{1}{\beta^2 - 1} \left(\frac{\Delta \mathfrak{V}}{V^2} + \xi^{\alpha}{}_{,\alpha} \right) \tag{21}$$

and

$$\frac{\delta V}{V} + \xi^{\varpi} \frac{\Omega_{,\varpi}}{\Omega} + \frac{\xi^{\varpi}}{\varpi} = -\frac{1}{\beta^2 - 1} \left(\frac{\Delta \mathfrak{V}}{V^2} + \beta^2 \xi^{\alpha}{}_{,\alpha} \right) . \tag{22}$$

We may conclude from equation (21) that

$$\frac{\Delta\mathfrak{V}}{V^2} + \xi^\alpha{}_{,\alpha} = 0 \qquad \text{when} \qquad \beta = 1\,, \tag{23}$$

i.e., when the transverse linear velocity becomes sonic. That the *sonic line* should appear explicitly in this manner in the present analysis has a clear physical origin; but it requires a more careful consideration than we shall attempt at this time.

Turning next to the hydrodynamical equations (1), we obtain their linearized versions

$$-m^2\Omega^2\xi^\varpi = \frac{\delta V^2}{\varpi} + \delta\mathfrak{V}_{,\varpi} + \frac{\delta\rho}{\rho^2}\,p_{,\varpi} - \frac{1}{\rho}\,\delta p_{,\varpi} \tag{24}$$

and

$$-m^2\Omega^2\xi^z = \delta\mathfrak{V}_{,z} + \frac{\delta\rho}{\rho^2}\,p_{,z} - \frac{1}{\rho}\,\delta p_{,z}\,. \tag{25}$$ [2]

By making use of equations (8) and (15), we can bring the foregoing equations to the form

$$-m^2\rho\Omega^2\xi^\alpha = -\left(\frac{\gamma p}{\rho}\,\Delta\rho\right)_{,\alpha} + \rho[\Delta(\mathfrak{V} + \tfrac{1}{2}V^2)]_{,\alpha} + \frac{\Delta\rho}{\rho}\,p_{,\alpha} - \rho\varpi^2(\Omega\Delta\Omega)_{,\alpha} - \rho\varpi\xi^\varpi\Omega^2{}_{,\alpha}\,. \tag{26}$$

We observe the remarkable similarity of equation (26) to the standard form of the pulsation equation governing the axisymmetric oscillations of these same configurations; and it plays the same role in the present context.

And finally, we have the linearized version of Poisson's equation:

$$(\varpi\delta\mathfrak{V}_{,\varpi})_{,\varpi} + (\varpi\delta\mathfrak{V})_{,zz} + 4\pi G\varpi\delta\rho = \frac{m^2}{\varpi}\,\delta\mathfrak{V}\,. \tag{27}$$

Equations (26) and (27) provide three *dynamical equations* for the problem while equations (15), (17), and (20) are to be considered as providing *initial-value equations.* These equations must be supplemented by the *boundary conditions*

$$\xi^\alpha \quad \text{and} \quad \delta\mathfrak{V} \quad \text{are everywhere finite}\,,$$

$$\Delta p = 0 \quad \text{and} \quad n^\alpha\delta\mathfrak{V}_{,\alpha}|^{+}_{-} = -4\pi G\rho n_\alpha\xi^\alpha\,,$$

$$\text{on the boundary where} \quad p = 0\,,$$

and

$$\delta\mathfrak{V} \prec r^{-3} \qquad \text{as} \qquad r \to \infty\,, \tag{28}$$

where n_α is the unit outward normal to the boundary. The equations and boundary condition which we have enumerated constitute a characteristic value problem for m^2. We should not, of course, expect that for an arbitrary initial configuration an

[2] In addition to equations (24) and (25), we might also have written an equation governing the Lagrangian displacement ξ^ϕ in the ϕ-direction. In the present treatment, equation (17), derived from the Bernoulli integral, replaces that equation to which, in fact, it can be shown to be equivalent.

allowed characteristic value for m will be an integer (≥ 1). *We are, however, interested precisely in the circumstances which will permit an integral characteristic value* (*greater than* 1). In particular, if $m = 2$ is an allowed characteristic value, we should conclude that the initial configuration is susceptible to a Dedekind-like point of bifurcation.

IV. THE VARIATIONAL PRINCIPLE

First, we define a *trial displacement* $\bar{\xi}^\alpha$ and a *trial change* $\bar{\delta}\mathfrak{V}$ by the sole requirement that they satisfy the same boundary conditions as are demanded of the proper solutions of the characteristic-value problem; and we *determine* the associated barred variations $\bar{\Delta}\rho$, $\bar{\delta}V$, etc., in terms of the chosen $\bar{\xi}^\alpha$ and $\bar{\delta}\mathfrak{V}$ with the aid of the initial-value equations (15), (17), and (20).

We next multiply the pulsation equation (26) by $\varpi\bar{\xi}^\alpha$, sum over α (i.e., ϖ and z), and integrate over all ϖ and z (i.e., over $0 \leq \varpi < \infty$ and $-\infty < z < +\infty$). After a sequence of reductions involving several integrations by parts and substitutions from the initial-value equations (which the barred and the unbarred quantities must satisfy) and the dynamical equations (27)[3] governing $\delta\mathfrak{V}$, we obtain

$$m^2 \iint \left(\rho\varpi\Omega^2\xi^\alpha\bar{\xi}^\alpha - \frac{1}{4\pi G\varpi}\,\delta\mathfrak{V}\bar{\delta}\mathfrak{V}\right)d\varpi dz$$

$$= \iint \Bigg[\varpi\gamma p\,\frac{\Delta\rho\bar{\Delta}\rho}{\rho^2} + \frac{\varpi}{\rho}\,\xi^\alpha p_{,\alpha}\bar{\xi}^\beta p_{,\beta} - \varpi p_{,\alpha}\left(\xi^\alpha\,\frac{\bar{\Delta}\rho}{\rho} + \bar{\xi}^\alpha\,\frac{\Delta\rho}{\rho}\right)$$

$$- \rho\varpi(\delta V + \varpi\xi^\varpi\Omega_{,\varpi})(\bar{\delta}V + \varpi\bar{\xi}^\varpi\Omega_{,\varpi}) - \rho\varpi^2\Omega^2{}_{,\varpi}\xi^\varpi\bar{\xi}^\varpi$$

$$- 2\rho\varpi\Omega(\xi^\varpi\bar{\delta}V + \bar{\xi}^\varpi\delta V) - \varpi(\delta\mathfrak{V}\bar{\delta}\rho + \bar{\delta}\mathfrak{V}\delta\rho)$$

$$+ \frac{\varpi}{4\pi G}(\delta\mathfrak{V}_{,\varpi}\bar{\delta}\mathfrak{V}_{,\varpi} + \delta\mathfrak{V}_{,z}\bar{\delta}\mathfrak{V}_{,z})\Bigg]d\varpi dz$$

$$+ \int_S [n_\alpha\xi^\alpha n_\beta\bar{\xi}^\beta|\text{grad}\,p| - \rho n_\alpha(\xi^\alpha\bar{\delta}\mathfrak{V} + \bar{\xi}^\alpha\delta\mathfrak{V})]ds\,, \tag{29}$$

where n_α is a unit vector normal to the boundary of the configuration on which p vanishes and the integral over s is a line integral extended over a meridional section (for $\varpi \geq 0$) of the boundary. (If ρ should vanish on the boundary, the line integral makes no contribution since in this case grad p will also vanish on the boundary.) It should also be noted that in equation (29), the integration over the terms in $\delta\mathfrak{V}$ must be extended over the *entire* admissible ranges of ϖ and z; but the integrations over the remaining terms need be extended only over the regions occupied by the fluid since these terms include p or ρ as a factor.

We observe that equation (29) is manifestly symmetric in the barred and unbarred quantities. It can be shown that on this account, equation (29) provides the basis for a variational determination of m^2 in the following sense.

We *formally* identify the barred and the unbarred quantities in equation (29) to obtain

[3] It *cannot* be assumed that $\delta\mathfrak{V}$ satisfies this equation.

$$\bar{m}^2 \iint \left\{\rho\varpi\Omega^2[(\xi^\varpi)^2 + (\xi^z)^2] - \frac{\delta\mathfrak{V}^2}{4\pi G\varpi}\right\} d\varpi dz$$
$$= \iint \left\{\rho\varpi \frac{\gamma p}{\rho}\left(\frac{\Delta\rho}{\rho}\right)^2 + \frac{\varpi}{\rho}\xi^\alpha p_{,\alpha}\xi^\beta p_{,\beta} - 2\varpi\frac{\Delta\rho}{\rho}\xi^\alpha p_{,\alpha} - 2\varpi\delta\mathfrak{V}d\rho\right.$$
$$- \rho\varpi V^2\left(\frac{\delta V}{V} + \xi^\varpi\frac{\Omega_{,\varpi}}{\Omega}\right)^2 - 4\rho V^2\xi^\varpi\frac{\delta V}{V} - 2\rho V^2(\xi^\varpi)^2\frac{\Omega_{,\varpi}}{\Omega}$$
$$\left. + \frac{\varpi}{4\pi G}[(\delta\mathfrak{V}_{,\varpi})^2 + (\delta\mathfrak{V}_{,z})^2]\right\} d\varpi dz + \int_S [(n_\alpha\xi^\alpha)^2|\mathrm{grad}\, p| - 2\rho\delta\mathfrak{V}n_\alpha\xi^\alpha]ds\,, \tag{30}$$

where we have replaced m by $\bar{m}$ to emphasize that we are presently requiring not that $\bar{m}$ be an integer but only that it be a characteristic value of the problem which we have formulated.

We now consider equation (30) as a *formula* for $\bar{m}^2$ in which ξ^α and $\delta\mathfrak{V}$ are a trial displacement and a trial change (in the sense we have defined these quantities) and $\Delta\rho$ and δV have the values given by the initial-value equations (in terms of ξ^α and $\delta\mathfrak{V}$). Suppose we now evaluate $\bar{m}^2$ successively with the aid of two sets of trial functions $(\xi^\alpha, \delta\mathfrak{V})$ and $(\xi^\alpha + \frac{1}{2}\delta\xi^\alpha, \delta\mathfrak{V} + \frac{1}{2}\delta^2\mathfrak{V})$. The effect $(\delta\bar{m}^2)$, on $\bar{m}^2$ given by equation (30), of (arbitrary) increments $(\frac{1}{2}\delta\xi^\alpha, \frac{1}{2}\delta^2\mathfrak{V})$ in the selected trial functions can be written down directly from equation (30) by subjecting it to the desired variation. If we now demand that $\delta\bar{m}^2$ vanish identically for an arbitrarily selected $\delta\xi^\alpha$ and $\delta^2\mathfrak{V}$ (consistent only with the boundary conditions of the problem), then it can be shown (by essentially retracing the steps that led to eq. [29] starting from the pulsation equation) that the chosen ξ^α and $\delta\mathfrak{V}$ satisfy the *dynamical equations* (26) and (27).

By making use of equations (21) and (22), we can reduce equation (30) to the more convenient form

$$\bar{m}^2 \iint \left\{\rho\varpi\Omega^2[(\xi^\varpi)^2 + (\xi^z)^2] - \frac{\delta\mathfrak{V}^2}{4\pi G\varpi}\right\} d\varpi dz$$
$$= \iint \left\{\rho\varpi V^2\left[\frac{1}{1-\beta^2}\left(\frac{\Delta\mathfrak{V}}{V^2} + \xi^\alpha{}_{,\alpha}\right)^2 - \left(\xi^\alpha{}_{,\alpha} - \frac{\xi^\varpi}{\varpi}\right)^2\right]\right.$$
$$+ 2\rho\varpi\Omega^2(\xi^\varpi)^2\left(2 + \varpi\frac{\Omega_{,\varpi}}{\Omega}\right) + \frac{\varpi}{\rho}\xi^\alpha p_{,\alpha}\xi^\beta p_{,\beta}$$
$$\left. + 2\varpi\delta\mathfrak{V}\xi^\alpha\rho_{,\alpha} + \frac{\varpi}{4\pi G}[(\delta\mathfrak{V}_{,\varpi})^2 + (\delta\mathfrak{V}_{,z})^2]\right\} d\varpi dz$$
$$+ \int_S [(n_\alpha\xi^\alpha)^2|\mathrm{grad}\, p| - 2\rho\delta\mathfrak{V}n_\alpha\xi^\alpha]ds\,. \tag{31}$$

a) *An Alternative Form of Equation* (*31*)

In equation (31), the terms in $\delta\mathfrak{V}$ require to be integrated over the infinite ranges of ϖ and z. We shall now describe a procedure which will partially eliminate the need for the evaluation of such infinite integrals.

Let $\tilde{\xi}(\varpi, z; \phi)$ be a three-dimensional displacement with the components

$$\tilde{\xi}^\varpi = \xi^\varpi \cos m\phi\,, \qquad \tilde{\xi}^z = \xi^z \cos m\phi\,, \qquad \text{and} \qquad \tilde{\xi}^\phi = \xi^\phi \sin m\phi\,, \tag{32}$$

where ξ^ϖ and ξ^z have the same meanings that they have had hitherto. The definition of $\tilde{\xi}$ thus "enlarges" our earlier definition of ξ^α to include a ϕ-component as well.

The three-dimensional displacement $\tilde{\xi}$ will generate a density-like function $\delta\tilde{\rho}(\varpi, z; \phi)$ given by

$$\delta\tilde{\rho} = -\text{div}\,(\rho\tilde{\xi})\,, \tag{33}$$

or

$$\delta\tilde{\rho} = \delta\hat{\rho}(\varpi, z)\cos m\phi\,, \tag{34}$$

where

$$\delta\hat{\rho} = -\frac{1}{\varpi}(\varpi\rho\xi^\varpi)_{,\varpi} - (\rho\xi^z)_{,z} - \frac{m\rho}{\varpi}\xi^\phi\,. \tag{35}$$

The displacement $\tilde{\xi}$ will also generate a $\delta\tilde{\mathfrak{V}}$ given by

$$\delta\tilde{\mathfrak{V}} = \delta\mathfrak{V}\cos m\phi\,, \tag{36}$$

where $\delta\mathfrak{V}$ is determined as a solution of the equation

$$\frac{1}{\varpi}(\varpi\delta\mathfrak{V}_{,\varpi})_{,\varpi} + (\delta\mathfrak{V})_{,zz} - \frac{m^2}{\varpi^2}\delta\mathfrak{V} = -4\pi G\delta\hat{\rho}\,, \tag{37}$$

satisfying the condition

$$n_\alpha\delta\mathfrak{V}_{,\alpha}|_-^+ = -4\pi G\rho n_\alpha\xi^\alpha\,. \tag{38}$$

With $\delta\mathfrak{V}$ defined in the foregoing manner, it can be readily verified that

$$\frac{1}{4\pi G}\int_{-\infty}^{+\infty}\int_0^\infty \varpi[(\delta\mathfrak{V}_{,\varpi})^2 + (\delta\mathfrak{V}_{,z})^2]d\varpi dz$$
$$= \iint \varpi\delta\hat{\rho}\delta\mathfrak{V}d\varpi dz - \frac{m^2}{4\pi G}\int_{-\infty}^{+\infty}\int_0^\infty \frac{\delta\mathfrak{V}^2}{\varpi}d\varpi dz + \int_S \rho\delta\mathfrak{V}n_\alpha\xi^\alpha ds\,. \tag{39}$$

Inserting this last result in equation (31), we obtain

$$\bar{m}^2\iint \rho\varpi\Omega^2(\xi^z)^2 d\varpi dz + (\bar{m}^2 - 4)\iint \rho\varpi\Omega^2(\xi^\varpi)^2 d\varpi dz + \frac{m^2 - \bar{m}^2}{4\pi G}\iint \frac{\delta\mathfrak{V}^2}{\varpi}d\varpi dz$$
$$= \iint\left\{\rho\varpi V^2\left[\frac{1}{1-\beta^2}\left(\frac{\Delta\mathfrak{V}}{V^2} + \xi^\alpha{}_{,\alpha}\right)^2 - \left(\xi^\alpha{}_{,\alpha} - \frac{\xi^\varpi}{\varpi}\right)^2\right]\right.$$
$$\left. + 2\rho V^2(\xi^\varpi)^2\frac{\Omega_{,\varpi}}{\Omega} + \frac{\varpi}{\rho}\xi^\alpha p_{,\alpha}\xi^\beta\rho_{,\beta} + 2\varpi\delta\mathfrak{V}\xi^\alpha\rho_{,\alpha} + \varpi\delta\mathfrak{V}\delta\hat{\rho}\right\}d\varpi dz$$
$$+ \int_S [(n_\alpha\xi^\alpha)^2|\text{grad}\,p| - \rho\delta\mathfrak{V}n_\alpha\xi^\alpha]ds\,, \tag{40}$$

where it will be observed that no infinite integrals occur on the right-hand side of the equation.

One cautionary remark concerning the use of equation (40) as a variational basis for evaluating $\bar{m}^2$ should be noted: it concerns the apparent singularity of the integrand (on the right-hand side) at $\beta = 1$. As we have noted earlier, we should require of a true proper solution that it satisfies the condition expressed in equation (23). If a selected trial function does not satisfy equation (23), we may include as a part of the variational assumption that the singularity at $\beta = 1$ is to be avoided by taking the principal value of the integral.

V. AN ILLUSTRATIVE EXAMPLE

It is known that along the Maclaurin sequence, the Dedekind bifurcation occurs by the displacement

$$\xi^1 = x^1 , \qquad \xi^2 = -x^2 , \qquad \text{and} \qquad \xi^3 = 0 . \tag{41}$$

In the notation of § IV*a*, this displacement corresponds to

$$\xi^\varpi = \varpi , \qquad \xi^z = 0 , \qquad \text{and} \qquad \xi^\phi = -\varpi . \tag{42}$$

The variations $\delta\hat{\rho}$ and $\delta\mathfrak{V}$ induced by this displacement are

$$\delta\hat{\rho} = -\varpi\frac{\partial\rho}{\partial\varpi} \qquad \text{and} \qquad \delta\mathfrak{V} = -\varpi\frac{\partial\mathfrak{D}}{\partial\varpi} , \tag{43}$$

where $\mathfrak{D}$ is the solution of the equation

$$(\varpi^3\mathfrak{D}_{,\varpi})_{,\varpi} + (\varpi^3\mathfrak{D}_{,z})_{,z} = -4\pi G\rho\varpi^3 . \tag{44}$$ [4]

The corresponding Lagrangian change in $\mathfrak{V}$ is given by

$$\Delta\mathfrak{V} = \varpi(\mathfrak{V} - \mathfrak{D})_{,\varpi} . \tag{45}$$

Inserting from equations (43) and (45) in equation (40) we obtain the following approximate criterion—approximate since it is derived from an assumed trial displacement—for the occurrence of a Dedekind-like point of bifurcation along a sequence of differentially rotating configurations:

$$\iint\left\{\frac{\rho\varpi V^2}{1-\beta^2}\left[\frac{\varpi}{V^2}(\mathfrak{V}-\mathfrak{D})_{,\varpi}+1\right]^2 + 2\rho\varpi^2V^2\frac{\Omega_{,\varpi}}{\Omega} + \frac{\varpi^3}{\rho}p_{,\varpi}\rho_{,\varpi} - \varpi^3\frac{\partial\rho}{\partial\varpi}\frac{\partial\mathfrak{D}}{\partial\varpi}\right\}d\varpi dz = 0 , \tag{46}$$

where we have not included the line integral over the meridional section of the boundary on the assumption that ρ vanishes on the boundary.

It can be verified that the condition (46) (including the line integral) is identically satisfied at the known point of bifurcation along the Maclaurin sequence.[5]

[4] The solution for $\delta\mathfrak{V}$ follows from the equation

$$\delta\mathfrak{V} = -\left(\frac{\partial\mathfrak{D}_1}{\partial x^1} - \frac{\partial\mathfrak{D}_2}{\partial x^2}\right) ,$$

where $\mathfrak{D}_i$ is the Newtonian potential induced by the "density distribution" ρx^i (cf. *E.F.E.*, pp. 24 and 107). For axisymmetric distributions we are presently considering $\mathfrak{D}_i = x^i\mathfrak{D}$ where $\mathfrak{D}$ is defined by equation (44).

[5] In fact, at the point of bifurcation $\Delta\mathfrak{V} + V^2 \equiv 0$.

VI. CONCLUDING REMARKS

The present paper exploits in the Newtonian framework the correspondence between time-dependent axisymmetric systems and time-independent nonaxisymmetric systems that was noticed in Paper I in the framework of general relativity. The correspondence is manifest in general relativity when one compares the metrics

$$ds^2 = -e^{2\nu}(dt)^2 + e^{2\psi}(d\phi - q_2 dx^2 - q_3 dx^3 - \omega dt)^2 + e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2 , \tag{47}$$

and

$$ds^2 = -e^{2\mathfrak{n}}(dt - \mathfrak{q}_2 dx^2 - \mathfrak{q}_3 dx^3 - \mathfrak{w} d\phi)^2 + e^{2\mathfrak{p}}(d\phi)^2 + e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2 , \tag{48}$$

which are appropriate for the two cases; and it is exemplified by the analogous roles played by the equations expressing the conservation of angular momentum per baryon, $(\epsilon + p)u_1/N$, and the conservation of "energy" per baryon, $(\epsilon + p)u_0/N$, in the two cases. As we have seen, in the Newtonian theory, the analogous roles are played by the angular momentum integral and the Bernoulli integral: a fact which one might not have suspected. The reason why the correspondence, which is so manifest in general relativity, is obscured in the Newtonian theory is that the terms which vanish in the Newtonian limit in the axisymmetric case are the dominant ones in the stationary case. Nevertheless, the correspondence which "really exists" also in the Newtonian theory remains to be analyzed and exploited.

In the more immediate context of this paper, while the methods developed are capable of wide generalizations, the most pressing practical problem is to test the accuracy of the Ostriker-Tassoul criterion by making use of the variational principle derived in this paper.

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APPENDIX

In formulating a variational principle for the characteristic frequencies of axisymmetric oscillations of rotating systems in general relativity, it was found (Chandrasekhar and Friedman 1972, 1973*a*) that its use, at no stage, required the solution of a second-order partial differential equation: the most that is required is the solution of a quasi-linear differential equation. This is contrary to one's experience in the Newtonian treatment of this same problem (Lynden-Bell and Ostriker 1967; Lebovitz 1970): in this treatment one is required to solve an equation of Poisson's type as an initial-value equation. We shall now show how by a slight reformulation of the problem the latter requirement can be avoided in the Newtonian theory as well.

The basic equations that govern the axisymmetric modes of oscillation of a differentially rotating configuration, with $\Omega \equiv \Omega(\varpi)$, can be written in the forms (Lebovitz 1970, eqs. [20], [30], and [35]–[37])

$$-\rho\sigma^2\xi^\varpi = \left[\frac{\gamma p}{\varpi}(\varpi\xi^\alpha)_{,\alpha}\right]_{,\varpi} + (\xi^\alpha p_{,\alpha})_{,\varpi} + \frac{\delta\rho}{\rho}p_{,\varpi} - \frac{\rho}{\varpi^3}(\varpi^4\Omega^2)_{,\varpi}\xi^\varpi + \rho\delta\mathfrak{V}_{,\varpi}$$

$$= (\rho \mathfrak{Q}\boldsymbol{\xi})^\varpi + \rho\delta\mathfrak{V}_{,\varpi} \text{ (say)} \tag{A1}$$

and

$$-\rho\sigma^2\xi^z = \left[\frac{\gamma p}{\varpi}(\varpi\xi^\alpha)_{,\alpha}\right]_{,z} + (\xi^\alpha p_{,\alpha})_{,z} + \frac{\delta\rho}{\rho}p_{,z} + \rho\delta\mathfrak{V}_{,z}$$
$$= (\rho \boldsymbol{Q}\boldsymbol{\xi})^z + \rho\delta\mathfrak{V}_{,z}\,, \tag{A2}$$

where $\delta\mathfrak{V}$ is to be determined as a solution of the equation,

$$\nabla^2\delta\mathfrak{V} = \frac{1}{\varpi}(\varpi\delta\mathfrak{V}_{,\varpi})_{,\varpi} + (\delta\mathfrak{V})_{,zz} = \frac{4\pi G}{\varpi}(\rho\varpi\xi^\alpha)_{,\alpha} \quad \text{in} \quad V$$
$$= 0 \quad \text{in} \quad V^c\,, \tag{A3}$$

which satisfies the boundary condition

$$n_\alpha\delta\mathfrak{V}_{,\alpha}\Big|_-^+ = -4\pi G\rho n_\alpha\xi^\alpha \quad \text{on} \quad \partial V\,, \tag{A4}$$

where V denotes the region occupied by the fluid, V^c the complementary region exterior to the fluid, and ∂V the boundary of V (on which p and Δp vanish); also n_α is the unit outward-normal to ∂V.

With the definition of the inner product

$$(\boldsymbol{\xi}, \boldsymbol{\eta}) = \int_V \rho\xi^\alpha\eta^\alpha\varpi d\varpi dz\,, \tag{A5}$$

it can be shown by the methods of Lynden-Bell and Ostriker and of Lebovitz, that the operator $\boldsymbol{Q}$ defined in equations (A1) and (A2) is *self-adjoint*:

$$(\boldsymbol{\eta}, \boldsymbol{Q}\boldsymbol{\xi}) = (\boldsymbol{Q}\boldsymbol{\eta}, \boldsymbol{\xi})\,. \tag{A6}$$

It is this symmetry of $\boldsymbol{Q}$ that enables a variational formulation of the characteristic-value problem underlying equations (A1) and (A2). But in that formulation equation (A3) is considered as an initial-value equation for $\delta\mathfrak{V}$ and, as such, requiring solution. The necessity to solve equation (A3) can be avoided by the following reformulation.

Consider the *formula*

$$-\sigma^2(\boldsymbol{\xi}, \boldsymbol{\xi}) = (\boldsymbol{\xi}, \boldsymbol{Q}\boldsymbol{\xi}) - \frac{1}{4\pi G}\int_{-\infty}^{+\infty}\int_0^\infty \varpi[(\delta\mathfrak{V}_{,\varpi})^2 + (\delta\mathfrak{V}_{,z})^2]d\varpi dz$$
$$+ 2\int_V\int \rho\xi^\alpha\delta\mathfrak{V}_{,\alpha}\varpi d\varpi dz\,. \tag{A7}$$

It can be verified that the effect $(\delta\sigma^2)$ on σ^2, given by equation (A7), of an arbitrary variation $\delta\xi^\alpha$ in ξ^α and $\delta^2\mathfrak{V}$ in $\delta\mathfrak{V}$ (compatible only with the boundary conditions on these quantities) is given by

$$-\delta\sigma^2 = \frac{2}{(\boldsymbol{\xi}, \boldsymbol{\xi})}\Bigg\{(\delta\boldsymbol{\xi}, \boldsymbol{Q}\boldsymbol{\xi} + \text{grad}\,\delta\mathfrak{V} + \sigma^2\boldsymbol{\xi})$$
$$+ \int_V\int \delta^2\mathfrak{V}\left[\frac{1}{4\pi G}\nabla^2\delta\mathfrak{V} - \frac{1}{\varpi}(\rho\varpi\xi^\alpha)_{,\alpha}\right]\varpi d\varpi dz$$
$$+ \int_{V^c}\int \frac{\delta^2\mathfrak{V}}{4\pi G}\nabla^2\delta\mathfrak{V}\varpi d\varpi dz + \int_{\partial V}\delta^2\mathfrak{V}\left[\rho n_\alpha\xi^\alpha + \frac{1}{4\pi G}n_\alpha\delta\mathfrak{V}_{,\alpha}\Big|_-^+\right]ds\Bigg\}\,. \tag{A8}$$

From this equation it follows that if we demand that $\delta\sigma^2$ vanishes identically for all arbitrary variations $\delta\xi^\alpha$ and $\delta^2\mathfrak{V}$, then the original equations (A1)–(A4) follow directly. Accordingly, *formula (A7) provides a variational basis for determining* σ^2 *which avoids the solution of the second-order equation (A3).*

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