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THE POST-NEWTONIAN EFFECTS OF GENERAL RELATIVITY ON THE EQUILIBRIUM OF UNIFORMLY ROTATING BODIES. V. THE DEFORMED FIGURES OF THE MACLAURIN SPHEROIDS (*Continued*)

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ABSTRACT

The theory of the (deformed) post-Newtonian Maclaurin spheroid developed in an earlier paper is specialized suitably to make a comparison with the Newtonian spheroid having the same angular momentum and baryon number.

I. INTRODUCTION

In an earlier paper (Chandrasekhar 1967; this paper will be referred to hereafter as Paper II) the deformed figures of the Maclaurin spheroid, in the first post-Newtonian approximation to the equations of general relativity, were determined. Precisely, configurations with a specified uniform density ρ, rotating with a constant angular velocity Ω, were considered; and it was shown how the figures of equilibrium can be obtained, consistently with the equations of post-Newtonian hydrodynamics (Chandrasekhar 1965), by subjecting the Newtonian spheroids, with the same density ρ, to a Lagrangian displacement of the form

$$\boldsymbol{\xi} = \frac{\pi G \rho a_1^2}{c^2} (S_1 \boldsymbol{\xi}^{(1)} + S_2 \boldsymbol{\xi}^{(2)}) , \tag{1}$$

where a_1 denotes the semimajor axis of the Newtonian spheroid,

$$\boldsymbol{\xi}^{(1)} = (x_1, x_2, -2z) , \qquad \boldsymbol{\xi}^{(2)} = \frac{1}{a_1^2} (\varpi^2 x_1, \varpi^2 x_2, -4\varpi^2 z) , \tag{2}$$

and S_1 and S_2 are certain constants to be determined. (Note that since the chosen displacement $\boldsymbol{\xi}$ is divergence free, the coordinate volumes of the Newtonian and the post-Newtonian configurations have been assumed to be the same.)

On carrying through the analysis, it was found that while the constant S_2 was uniquely determined (Paper II, eq. [99]), the constant S_1 was left undetermined. This indeterminacy in the solution was correctly interpreted by the statement that "the free choice we have in selecting the amplitude of the displacement $\boldsymbol{\xi}^{(1)}$ corresponds to the free choice we have in selecting a particular Maclaurin spheroid (among neighboring ones) for comparison with the post-Newtonian configuration." But it was wrongly concluded that "the availability of this choice has clearly no physical content." As Bardeen (1971) has pointed out, there *is* a physically meaningful comparison that can be made, namely, when *the Newtonian and the post-Newtonian configurations have, for a given density ρ, the same baryon number and the same angular momentum.*

The renormalization of the solution given in Paper II, to be in accord with Bardeen's specification, can be readily accomplished. As we shall see, the additional calculations that are needed to accomplish this renormalization are, indeed, very few.

In his paper, Bardeen has already carried out the necessary renormalization by solving the entire problem ab initio by a different method. Nevertheless, it is useful to demonstrate that the two essentially different methods of solution—the writer's and

Bardeen's—lead to identical results, particularly as, in Bardeen's method, one starts with the exact equations of general relativity in the context of the problem and proceeds from there to the post-Newtonian approximation. The agreement of the solutions provides therefore a confirmation of the equations and methods of post-Newtonian hydrodynamics as developed by the writer.

II. THE ANGULAR MOMENTUM OF THE POST-NEWTONIAN CONFIGURATION

Under stationary conditions, the linear momentum in the post-Newtonian approximation is given by (cf. Chandrasekhar and Nutku 1969, eq. [74])

$$\pi_\alpha = \rho v_\alpha + \frac{1}{c^2}\rho\left[v_\alpha\left(v^2 + 6U + \Pi + \frac{p}{\rho}\right) - 4U_\alpha\right], \tag{3}$$

where the various quantities have their standard meanings in this theory. Accordingly, under the circumstances of the problem presently considered, the angular momentum of an element of fluid, about the axis of rotation, is given by

$$L_3 = x_1\pi_2 - x_2\pi_1 = \rho\Omega\varpi^2\left[1 + \frac{1}{c^2}\left(\Omega^2\varpi^2 + 6U + \frac{p}{\rho} - 4\mathfrak{D}\right)\right]. \tag{4}$$

Explicit expressions for the various quantities occurring in equation (2) are given in Paper II, equations (15), (16), (17), (53), and (55).

The angular momentum, $\mathfrak{M}$, of the entire mass is obtained by integrating L_3 over the volume occupied by the fluid. Thus,

$$\mathfrak{M} = \int_{\text{post N}} \rho\Omega\varpi^2 dx + \frac{1}{c^2}\Omega\int_{\text{Mc}} \rho\varpi^2\left(\Omega^2\varpi^2 + 6U + \frac{p}{\rho} - 4\mathfrak{D}\right)\Big]dx\,, \tag{5}$$

where the first integral must be evaluated, correctly to $O(c^{-2})$, over the deformed figure of the post-Newtonian configuration, while it will suffice to evaluate the second integral over the volume of the undeformed Maclaurin spheroid.

Considering first the contribution to $\mathfrak{M}$ by the second integral on the right-hand side of equation (5), we find with the aid of the expressions given in Paper II (eqs. [15]–[17]) that

$$\begin{aligned}
&\frac{1}{c^2}\Omega\int_{\text{Mc}} \rho\varpi^2\left(\Omega^2\varpi^2 + 6U + \frac{p}{\rho} - 4\mathfrak{D}\right)dx \\
&= \frac{\pi G\rho}{c^2}\rho\Omega\int_{\text{Mc}}\left[(8a_1^2A_1 + 7a_3^2A_3)\varpi^2 - \left(4B_{11} + 3\frac{a_3^2}{a_1^2}A_3\right)\varpi^4 \right. \\
&\qquad \left. + (4a_1^2A_{13} - 7A_3)\varpi^2z^2\right]dx \\
&= \frac{\pi G\rho a_1^2}{c^2}(\Omega I_{11})_{\text{Mc}}Q \text{ (say)}\,,
\end{aligned} \tag{6}$$

where

$$Q = 2\left[\left(8A_1 + 7\frac{a_3^2}{a_1^2}A_3\right) - \frac{4}{7}\left(4B_{11} + 3\frac{a_3^2}{a_1^2}A_3\right) + \frac{1}{7}\left(4a_3^2A_{13} - 7\frac{a_3^2}{a_1^2}A_3\right)\right]. \tag{7}$$

Considering next the contribution to $\mathfrak{M}$ by the first integral on the right-hand side of equation (5), we can write

$$\int_{\text{post N}} \rho\Omega\varpi^2 dx = (2\Omega I_{11})_{\text{Mc}} + 2\delta\Omega I_{11} + 2\Omega V_{11}\,. \tag{8}$$

The first term on the right-hand side of equation (8) represents the angular momentum of a Newtonian Maclaurin spheroid having the same density and coordinate volume as the post-Newtonian configuration; and the remaining two terms represent the contributions to $\mathfrak{M}$ arising from the fact that the angular velocity and the moment of inertia of the post-Newtonian configuration differ from those of the Newtonian configuration by the amounts $\delta\Omega$ and $V_{11}(=\delta I_{11})$, respectively. Expressions for $\delta\Omega$ and V_{11} are given in Paper II, equations[1] (54) and (55). Combining the contributions arising from these terms in $\delta\Omega$ and V_{11} with that already given in equation (6), we obtain

$$\mathfrak{M} = (2\Omega I_{11})_{\mathrm{Mc}} + (\delta\mathfrak{M})_{\mathrm{coord.\ vol.}} \,, \tag{9}$$

where

$$(\delta\mathfrak{M})_{\mathrm{coord.\ vol.}} = \frac{\pi G\rho a_1^2}{c^2}(\Omega I_{11})_{\mathrm{Mc}}\left\{\left(Q + \frac{E_{13}}{a_1^2\Omega^2}\right) + \frac{4}{7}\left[1 - \frac{2}{\Omega^2}(a_1^2 A_{11} - 2a_3^2 A_{13})\right](7S_1 + 4S_2)\right\} . \tag{10}$$

In the foregoing equations we have distinguished $(\delta\mathfrak{M})$ by a subscript "coord. vol." to emphasize that this is the difference in the angular momenta of the post-Newtonian and the Newtonian configurations at constant coordinate volume.

III. ADJUSTMENT TO EQUAL BARYON NUMBER

As we have already remarked, by the choice of the Lagrangian displacement (1) that is divergence free, we have arranged that the coordinate volumes of the Newtonian and the post-Newtonian configurations are the same. But the *proper volume* of the post-Newtonian configuration, correct to $O(c^{-2})$, is given by (cf. Chandrasekhar 1969, eq. [44])

$$\int_{\mathrm{Mc}} u^0\sqrt{-g}\, d\boldsymbol{x} = \int_{\mathrm{Mc}}\left[1 + \frac{1}{c^2}(\tfrac{1}{2}v^2 + 3U)\right]d\boldsymbol{x} \,; \tag{11}$$

and this differs from the volume of the Maclaurin spheroid. As Bardeen has pointed out, it is physically more meaningful to arrange that the two configurations have equal baryon numbers for the same density ρ; and this stipulation requires that the proper volume of the post-Newtonian configurations, as determined by equation (11), agrees with the volume, $\frac{4}{3}\pi a_1^2 a_3$, of the Maclaurin spheroid to $O(c^{-2})$. To satisfy this last requirement, we must subject the post-Newtonian configuration[2] to a uniform expansion of the form

$$\boldsymbol{\xi}^{(0)} = \frac{\pi G\rho a_1^2}{c^2} S_0\boldsymbol{x} \text{ (say)} \,, \tag{12}$$

and determine the constant S_0 by the condition

$$\mathrm{div}\, \boldsymbol{\xi}^{(0)} = \frac{\pi G\rho a_1^2}{c^2}(3S_0) = -\frac{1}{(\frac{4}{3}\pi a_1^2 a_3)c^2}\int_{\mathrm{Mc}}(\tfrac{1}{2}\Omega^2\varpi^2 + 3U)d\boldsymbol{x} \,. \tag{13}$$

[1] In these equations we can set $S_3 = 0$ without any loss of generality (see the remarks preceding eq. [80] in Paper II).

[2] This subjection of the post-Newtonian configuration to a uniform expansion will not affect in any way the analysis carried out in terms of the displacement (1) since we can alternatively subject the comparison Maclaurin spheroid to an equal uniform contraction (keeping the density ρ and the post-Newtonian configuration unchanged) and avail ourselves, once again, of the freedom we have in selecting "the particular Maclaurin spheroid (among neighboring ones) for comparison with the post-Newtonian configuration."

Evaluating this last integral with the aid of the known expressions for Ω^2 and U, we find

$$S_0 = -\frac{1}{3}\left(5.2A_1 + 2\frac{a_3^2}{a_1^2}A_3\right). \tag{14}$$

Now comparing the angular momentum of the post-Newtonian configuration, after this expansion, with that of the Newtonian spheroid, we obtain (cf. eqs. [9] and [10])

$$(\delta\mathfrak{M})_{\text{proper vol.}} = (\delta\mathfrak{M})_{\text{coord. vol.}} + \frac{\pi G\rho a_1^2}{c^2}(2\Omega I_{11})_{\text{Mc}}(5S_0)$$

$$= \frac{\pi G\rho a_1^2}{c^2}(\Omega I_{11})_{\text{Mc}}\left\{-\frac{10}{3}\left(5.2A_1 + 2\frac{a_3^2}{a_1^2}A_3\right) + Q + \frac{E_{13}}{a_1^2\Omega^2}\right.$$

$$\left. + \frac{4}{7}\left[1 - \frac{2}{\Omega^2}(a_1^2A_{11} - 2a_3^2A_{13})\right](7S_1 + 4S_2)\right\}. \tag{15}$$

The requirement that the angular momenta of the post-Newtonian and the Newtonian configurations be the same leads to the equation

$$7S_1 + 4S_2 = \frac{\frac{10}{3}(5.2A_1 + 2a_3^2A_3/a_1^2) - Q - E_{13}/a_1^2\Omega^2}{\frac{4}{7}[1 - 2(a_1^2A_{11} - 2a_3^2A_{13})/\Omega^2]}. \tag{16}$$

With S_2 already determined (Paper II, eq. [99]), equation (16) determines S_1; and the corresponding value of $\delta\Omega$ follows from Paper II, equation (55) (with S_3 set equal to zero). The results of the calculations are summarized in Table 1; it includes the values of S_2 already listed in Paper II. It is found that these results are in complete numerical agreement with those given by Bardeen.[3]

IV. THE BINDING ENERGY

The "binding energy" of a configuration, as commonly defined, is no more than the integral over its volume of what has been called the "conserved energy" in the papers on post-Newtonian hydrodynamics (Chandrasekhar 1965, 1969; Chandrasekhar and Nutku 1969; Chandrasekhar and Esposito 1970). We shall now evaluate this quantity along the sequence of the post-Newtonian Maclaurin spheroids.

The expression for the conserved energy in the first post-Newtonian approximation is (Chandrasekhar and Nutku 1969, eq. [67])

$$\mathfrak{E} = \rho(\tfrac{1}{2}v^2 - \tfrac{1}{2}U + \Pi)$$

$$+ \frac{1}{c^2}\rho\left[\tfrac{5}{8}v^4 + \tfrac{5}{2}v^2U - \tfrac{5}{2}U^2 + 2U\Pi + v^2\left(\Pi + \frac{p}{\rho}\right) - \tfrac{1}{2}v_\mu P_\mu\right]. \tag{17}$$

Under the circumstances of the problem presently considered, $P_\mu = 4U_\mu$; and the terms in the internal energy Π can also be ignored. Accordingly, the binding energy of the post-

[3] The relations between the constants as defined here and as defined by Bardeen are

$$\left(\frac{S_0}{\beta}\right)_{\text{B}} = \frac{3}{8}\frac{a_1}{a_3}(S_0)_{\text{C}}; \quad \left(\frac{S_4}{\beta}\right)_{\text{B}} = \frac{3}{7}\frac{a_1}{a_3}(S_2)_{\text{C}};$$

$$\left(\frac{S_2}{\beta}\right)_{\text{B}} = -3\frac{a_1}{a_3}[\tfrac{1}{7}(S_2)_{\text{C}} + \tfrac{1}{4}(S_1)_{\text{C}}]; \quad \left(\frac{\Delta\ln\Omega}{\beta}\right)_{\text{B}} = \frac{3}{8}\frac{a_1}{a_3}\left(\frac{\delta\Omega}{\Omega}\right)_{\text{C}}.$$

TABLE 1

THE VALUES OF S_0, S_1, S_2, AND $\delta\Omega/\Omega$ WHICH DESCRIBE THE POST-NEWTONIAN CONFIGURATIONS DERIVED FROM MACLAURIN SPHEROIDS OF VARIOUS ECCENTRICITIES

(e_J = 0.8126700; e_{max} = 0.9299555; e^* = 0.985226)

e	S_0	S_1	S_2	$c^2\delta\Omega/(\Omega\pi G\rho a_1^2)$	e	S_0	S_1	S_2	$c^2\delta\Omega/(\Omega\pi G\rho a_1^2)$
0	−1.60000	0	0	0	0.92	−0.82604	−0.04481	+ 0.08505	+0.32196
0.20	−1.57975	−0.00497	0.00003	0.20905	e_{max}	−0.78545	−0.04631	+ 0.09977	+0.30391
0.35	−1.53565	−0.01455	0.00032	0.24584	0.94	−0.73987	−0.04970	+ 0.11983	+0.28156
0.40	−1.51446	−0.01858	0.00057	0.26182	0.95	−0.68858	−0.05691	+ 0.14885	+0.25401
0.45	−1.48948	−0.02289	0.00098	0.27929	0.96	−0.62940	−0.07315	+ 0.19652	+0.21945
0.50	−1.46024	−0.02737	0.00160	0.29793	0.97	−0.55892	−0.11739	+ 0.29662	+0.17516
0.55	−1.42610	−0.03186	0.00254	0.31731	0.975	−0.51760	−0.17614	+ 0.41222	+0.14804
0.60	−1.38623	−0.03622	0.00394	0.33682	0.980	−0.47046	−0.35278	+ 0.73506	+0.11650
0.65	−1.33947	−0.04020	0.00606	0.35562	0.981	−0.46014	−0.43901	+ 0.88881	+0.10956
0.70	−1.28419	−0.04354	0.00928	0.37242	0.982	−0.44947	−0.57896	+ 1.13659	+0.10239
0.75	−1.21797	−0.04588	0.01428	0.38525	0.983	−0.43843	−0.84503	+ 1.60509	+0.09496
0.80	−1.13696	−0.04682	0.02236	0.39079	0.984	−0.42696	−1.5458	+ 2.83442	+0.08727
e_J	−1.11339	−0.04679	0.02517	0.39042	0.985	−0.41504	−8.4634	+14.943	+0.07930
0.82	−1.09908	−0.04672	0.02699	0.38977	e^*	−0.41228	$\pm\infty$	$\pm\infty$	+0.07746
0.84	−1.05720	−0.04632	0.03284	0.38607	0.986	−0.40261	+2.48319	− 4.21064	+0.07104
0.85	−1.03450	−0.04601	0.03636	0.38298	0.987	−0.38963	+1.09181	− 1.77288	+0.06246
0.86	−1.01045	−0.04565	0.04039	0.37890	0.988	−0.37601	+0.70323	− 1.09009	+0.05356
0.87	−0.98491	−0.04525	0.04502	0.37367	0.989	−0.36168	+0.52027	− 0.76721	+0.04430
0.88	−0.95767	−0.04485	0.05042	0.36711	0.990	−0.34654	+0.41356	− 0.57790	+0.03468
0.89	−0.92849	−0.04448	0.05677	0.35898	0.995	−0.25244	+0.19918	− 0.19554	−0.01912
0.90	−0.89710	−0.04424	0.06435	0.34900	0.999	−0.11764	+0.10091	− 0.05716	−0.05779
0.91	−0.86310	−0.04427	0.07357	0.33682					

Newtonian Maclaurin spheroid is given by

$$\int \mathfrak{E} dx = \int_{\text{post N}} \rho(\tfrac{1}{2}v^2 - \tfrac{1}{2}U)dx + \frac{1}{c}\int_{\text{Mc}} \rho(\tfrac{5}{8}v^4 + \tfrac{5}{2}v^2U - \tfrac{5}{2}U^2 + v^2p/\rho - 2v_\mu U_\mu)dx\,, \tag{18}$$

where the first integral must be evaluated correctly to $O(c^{-2})$ over the deformed figure of post-Newtonian configuration, while it will suffice to evaluate the second integral over the undeformed Maclaurin spheroid.

Considering first the contribution to the binding energy by the second integral on the right-hand side of equation (18), we find with the aid of the expressions given in Paper II (eqs. [15]–[17]) that

$$\frac{1}{c^2}\int_{\text{Mc}} \rho\{\Omega^2(x_1^2 + x_2^2)[\tfrac{5}{8}\Omega^2(x_1^2 + x_2^2) + \tfrac{5}{2}U + p/\rho - 2\mathfrak{D}] - \tfrac{5}{2}U^2\}dx$$
$$= \frac{1}{c^2}(\pi G\rho)^2(\tfrac{4}{15}\pi a_1^2 a_3\rho) \times \tfrac{1}{56}[40a_1^4\Omega^4 - 160a_1^4A_1\Omega^2 - 160a_1^4A_1^2$$
$$- 60a_3^4A_3^2 - 420I^2 - 8a_1^2a_3^2\Omega^2 - 80a_1^2a_3^2A_1A_3 + 280\Omega^2a_1^2I$$
$$+ 32\Omega^2(4a_1^6A_{11} + a_1^4a_3^2A_{13} - 7a_1^4A_{11})]\,, \tag{19}$$

where Ω is measured in the unit $(\pi G\rho)^{1/2}$ and

$$I = 2a_1^2A_1 + a_3^2A_3\,. \tag{20}$$

Considering next the contribution to the binding energy by the first integral on the right-hand side of equation (18), we can write

$$\int_{\text{post N}} \rho(\tfrac{1}{2}v^2 - \tfrac{1}{2}U^2)dx = (\Omega^2 I_{11} + \mathfrak{W})_{\text{Mc}} + \delta\Omega^2 I_{11} + \Omega^2 V_{11} + \delta\mathfrak{W}$$
$$= (\Omega^2 I_{11} + \mathfrak{W})_{\text{Mc}} + \delta\Omega^2 I_{11} + 2\Omega^2 V_{11}\,. \tag{21}$$

The first term on the right-hand side of equation (21) represents the binding energy of the Newtonian Maclaurin spheroid having the same density and coordinate volume as the post-Newtonian configuration; and the remaining terms represent the contribution arising from the fact that the angular velocity, the moment of inertia, and the potential energy of the post-Newtonian configuration differ from those of the Newtonian configuration by the amounts $\delta\Omega$, $\delta I_{11}(=V_{11})$, and $\delta\mathfrak{W} = \Omega^2 V_{11}$ (as may be readily verified). Combining the contributions arising from these terms in $\delta\Omega$ and V_{11} with that given in equation (19), we have

$$\int \mathfrak{E} dx = E_0 + (\Delta E)_{\text{coord. vol.}}\,, \tag{22}$$

where

$$E_0 = (\Omega^2 I_{11} + \mathfrak{W})_{\text{Mc}} = \pi G\rho(\tfrac{4}{15}\pi\rho a_1^2 a_3)(\Omega^2 a_1^2 - 2I) \tag{23}$$

and

$$(\Delta E)_{\text{coord vol.}} = \frac{1}{c^2}(\pi G\rho)^2(\tfrac{4}{15}\pi a_1^2 a_3\rho)$$

$$\times \Big\{a_1^2 E_{13} + \tfrac{4}{7}\Omega^2 a_1^4\Big[1 - \frac{2}{\Omega^2}(a_1^2 A_{11} - 2a_3^2 A_{13})\Big](7S_1 + 4S_2)$$

$$+ \tfrac{1}{56}[40a_1^4\Omega^4 - 160a_1^4 A_1\Omega^2 - 160a_1^4 A_1^2 - 60a_3^4 A_3^2 - 420I^2 - 8a_1^2 a_3^2\Omega^2$$

$$- 80a_1^2 a_3^2 A_1 A_3 + 280\Omega^2 I a_1^2 + 32\Omega^2(4a_1^6 A_{11} + a_1^4 a_3^2 A_{13} - 7a_1^4 A_1)]\Big\}\,. \tag{24}$$

In equation (24) we have distinguished (ΔE) by a subscript "coord. vol." to emphasize that this is the difference in the binding energy of the post-Newtonian and the New-

TABLE 2

THE BINDING ENERGY ALONG THE POST-NEWTONIAN SEQUENCE OF MACLAURIN SPHEROIDS

e	E_0	$c^2(\Delta E)/(E_0\pi G\rho a_1^2)$	e	E_0	$c^2(\Delta E)/(E_0\pi G\rho a_1^2)$
0.	−4.0000	+0.09524	0.920	−1.5422	−0.03269
0.20	−3.9243	+0.09362	e_{max}	−1.4394	−0.04053
0.35	−3.7616	+0.08961	0.940	−1.3272	−0.04902
0.40	−3.6843	+0.08745	0.950	−1.2050	−0.05803
0.45	−3.5941	+0.08473	0.960	−1.06957	−0.06751
0.50	−3.4896	+0.08131	0.970	−0.91597	−0.07713
0.55	−3.3692	+0.07700	0.975	−0.82983	−0.08171
0.60	−3.2307	+0.07155	0.980	−0.73512	−0.08578
0.65	−3.0712	+0.06464	0.981	−0.71490	−0.08649
0.70	−2.8869	+0.05581	9.982	−0.69419	−0.08716
0.75	−2.6722	+0.04439	0.983	−0.67294	−0.08776
0.80	−2.4187	+0.02940	0.984	−0.65111	−0.08829
e_J	−2.3469	+0.02487	0.985	−0.62865	−0.08874
0.82	−2.3037	+0.02208	e^*	−0.62348	−0.08883
0.84	−2.1791	+0.01382	0.986	−0.60550	−0.08909
0.85	−2.1128	+0.00928	0.987	−0.58158	−0.08933
0.86	−2.0434	+0.00444	0.988	−0.55682	−0.08943
0.87	−1.9707	−0.00073	0.989	−0.53111	−0.08936
0.88	−1.8943	−0.00626	0.990	−0.50433	−0.08909
0.89	−1.8138	−0.01218	0.995	−0.34677	−0.08219
0.90	−1.7288	−0.01854	0.999	−0.14780	−0.05242
0.91	−1.6385	−0.02536			

tonian configurations at constant volume. The adjustment to equal proper volume is easily made: the considerations of § III lead to the result

$$(\Delta E)_{\text{proper vol.}} = (\Delta E)_{\text{coord. vol.}} + \frac{1}{c^2}(\pi G\rho)^2(\tfrac{4}{15}\pi a_1^2 a_3 \rho) a_1^2(\Omega^2 a_1^2 - 2I)(5S_0) , \quad (25)$$

where S_0 has the value given in equation (15).

In Table 2 we list the values of $(\Delta E)_{\text{proper vol.}}/E_0$ obtained with the aid of the foregoing formula.[4]

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[4] There are some small but (unexplained) systematic differences between the values of the binding energy given in Table 2 and those tabulated by Bardeen (1971).