ON THE RADIATIVE EQUILIBRIUM OF A STELLAR ATMOSPHERE. XXIII

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ABSTRACT

In this paper the exact H-functions governing the law of diffuse reflection by a semi-infinite plane-parallel atmosphere in accordance with the phase function $\lambda(1 + x \cos \Theta)$ are tabulated. The functions were determined numerically by solving the exact functional equations which they satisfy.

The cases $(1 + x \cos \Theta)$ ($x = -1.0, -0.8, -0.6, -0.4, -0.2, 0.2, 0.4, 0.6, 0.8$, and 1.0) and $\lambda(1 + \cos \Theta)$ ($\lambda = 0.975, 0.950, 0.925, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2$, and 0.1) are covered by the tabulations.

In continuation of Papers XVI[1] and XIX,[2] in which we have tabulated the exact solutions for several problems in the theory of radiative transfer, we provide in this paper tables of H-functions governing the law of diffuse reflection by a semi-infinite plane-parallel atmosphere in accordance with the phase function $\lambda(1 + x \cos \Theta)$.

DIFFUSE REFLECTION IN ACCORDANCE WITH THE PHASE FUNCTION $\lambda(1 + x \cos \Theta)$

The law of diffuse reflection by a semi-infinite plane-parallel atmosphere.—

$$I(\mu, \varphi; \mu_0, \varphi_0) = \tfrac{1}{4}\lambda F\left[H^{(0)}(\mu) H^{(0)}(\mu_0)\left\{1 - c(\mu + \mu_0) - x(1-\lambda)\mu\mu_0\right\} + x(1-\mu^2)^{\frac{1}{2}}(1-\mu_0^2)^{\frac{1}{2}} H^{(1)}(\mu) H^{(1)}(\mu_0)\cos(\varphi - \varphi_0)\right]\frac{\mu_0}{\mu_0 + \mu}.$$

The characteristic functions in terms of which $H^{(0)}(\mu)$ and $H^{(1)}(\mu)$ are defined, are, respectively,

$$\Psi^{(0)}(\mu) = \tfrac{1}{2}\lambda\left[1 + x(1-\lambda)\mu^2\right]$$

and

$$\Psi^{(1)}(\mu) = \tfrac{1}{4}x\lambda(1 - \mu^2).$$

The constant, c, is given by

$$c = x\lambda(1-\lambda)\frac{\alpha_1}{2 - \lambda\alpha_0},$$

where α_0 and α_1 are the moments of order zero and one of $H^{(0)}(\mu)$.

An alternative way of expressing the law of diffuse reflection is

$$I(\mu, \varphi; \mu_0, \varphi_0) = \tfrac{1}{4}\lambda F\left[\psi(\mu)\psi(\mu_0) - x\phi(\mu)\phi(\mu_0) + x(1-\mu^2)^{\frac{1}{2}}(1-\mu_0^2)^{\frac{1}{2}} H^{(1)}(\mu) H^{(1)}(\mu_0)\cos(\varphi - \varphi_0)\right]\frac{\mu_0}{\mu_0 + \mu},$$

where

$$\psi(\mu) = H^{(0)}(\mu)(1 - c\mu), \qquad \phi(\mu) = q\mu H^{(0)}(\mu)$$

and

$$q = \frac{2(1-\lambda)}{2 - \lambda\alpha_0}.$$

[1] *Ap. J.*, **105**, 435, 1947.

[2] *Ibid.*, **106**, 143, 1947.

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TABLE 1

THE FUNCTIONS $H^{(0)}(\mu)$ OBTAINED AS SOLUTIONS OF THE EXACT FUNCTIONAL EQUATIONS THEY SATISFY

(The Case $x = 1$)

μ	$\lambda=0.1$	$\lambda=0.2$	$\lambda=0.3$	$\lambda=0.4$	$\lambda=0.5$	$\lambda=0.6$
0.	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.05	1.0089	1.0183	1.0280	1.0383	1.0492	1.0608
0.10	1.0145	1.0297	1.0459	1.0632	1.0817	1.1020
0.15	1.0188	1.0388	1.0602	1.0832	1.1084	1.1361
0.20	1.0224	1.0463	1.0722	1.1003	1.1311	1.1656
0.25	1.0254	1.0528	1.0825	1.1151	1.1511	1.1918
0.30	1.0280	1.0584	1.0916	1.1281	1.1689	1.2153
0.35	1.0303	1.0634	1.0996	1.1398	1.1850	1.2366
0.40	1.0324	1.0678	1.1069	1.1504	1.1996	1.2562
0.45	1.0343	1.0719	1.1135	1.1600	1.2129	1.2742
0.50	1.0359	1.0755	1.1194	1.1688	1.2252	1.2908
0.55	1.0375	1.0788	1.1249	1.1769	1.2365	1.3063
0.60	1.0389	1.0819	1.1300	1.1844	1.2470	1.3207
0.65	1.0401	1.0847	1.1346	1.1913	1.2568	1.3342
0.70	1.0413	1.0873	1.1389	1.1978	1.2659	1.3468
0.75	1.0424	1.0897	1.1429	1.2038	1.2745	1.3587
0.80	1.0434	1.0919	1.1467	1.2094	1.2825	1.3699
0.85	1.0444	1.0940	1.1502	1.2147	1.2900	1.3805
0.90	1.0453	1.0960	1.1535	1.2196	1.2972	1.3905
0.95	1.0461	1.0978	1.1566	1.2243	1.3039	1.4000
1.00	1.0469	1.0995	1.1595	1.2287	1.3103	1.4090

TABLE 1—*Continued*

μ	$\lambda=0.7$	$\lambda=0.8$	$\lambda=0.9$	$\lambda=0.925$	$\lambda=0.95$	$\lambda=0.975$
0.	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.05	1.0735	1.0876	1.1045	1.1096	1.1153	1.1223
0.10	1.1244	1.1501	1.1819	1.1917	1.2029	1.2169
0.15	1.1673	1.2038	1.2500	1.2645	1.2814	1.3027
0.20	1.2049	1.2516	1.3120	1.3313	1.3539	1.3830
0.25	1.2387	1.2951	1.3695	1.3936	1.4222	1.4593
0.30	1.2693	1.3351	1.4233	1.4523	1.4869	1.5323
0.35	1.2973	1.3722	1.4740	1.5079	1.5487	1.6026
0.40	1.3233	1.4068	1.5220	1.5608	1.6078	1.6706
0.45	1.3473	1.4392	1.5677	1.6114	1.6647	1.7365
0.50	1.3697	1.4697	1.6112	1.6598	1.7195	1.8005
0.55	1.3907	1.4985	1.6528	1.7063	1.7724	1.8627
0.60	1.4103	1.5257	1.6926	1.7510	1.8235	1.9234
0.65	1.4288	1.5515	1.7308	1.7941	1.8730	1.9826
0.70	1.4462	1.5760	1.7674	1.8356	1.9210	2.0403
0.75	1.4627	1.5993	1.8027	1.8757	1.9675	2.0967
0.80	1.4783	1.6216	1.8367	1.9144	2.0127	2.1519
0.85	1.4931	1.6428	1.8694	1.9519	2.0567	2.2058
0.90	1.5072	1.6631	1.9010	1.9882	2.0994	2.2586
0.95	1.5206	1.6825	1.9315	2.0233	2.1409	2.3103
1.00	1.5334	1.7011	1.9610	2.0574	2.1814	2.3609

The constants, q and c, are further related according to

$$q^2 - \frac{c^2}{x} = 1 - \lambda .$$

When $\lambda = 1$, the characteristic function $\Psi^{(0)}(\mu)$ becomes $\frac{1}{2}$, independently of x. The corresponding H-function is therefore the one tabulated in Paper XIX (Table 1, the case $\tilde{\omega}_0 = 1$).

The H-functions tabulated in this paper were computed by the iteration method described in Paper XVI.

TABLE 2

THE MOMENTS α_0 AND α_1 AND THE CONSTANTS q AND c

λ	α_0	α_1	q	c	q^2-c^2
0.1	1.032729	0.519588	0.949003	0.024654	0.899999
.2	1.068832	.541348	.895740	.048491	.799998
.3	1.109034	.565767	.839686	.071260	.699995
.4	1.154378	.593541	.780108	.092605	.599993
.5	1.206366	.625686	.715914	.111984	.499992
.6	1.267352	.663798	.645375	.128520	.399992
.7	1.341368	.710639	.565482	.140649	.299987
.8	1.436535	.771792	.470161	.145147	.199984
.9	1.574492	.862276	.343079	.133123	.099981
.925	1.623024	.894620	.300780	.124451	.074981
.950	1.683484	.935277	.249569	.110873	.049992
0.975	1.767379	0.992380	0.180632	0.087387	0.024992

TABLE 3

THE FUNCTIONS $H^{(1)}(\mu)$ OBTAINED AS SOLUTIONS OF THE EXACT FUNCTIONAL EQUATIONS THEY SATISFY

μ	$x\lambda=1.0$	$x\lambda=0.8$	$x\lambda=0.6$	$x\lambda=0.4$	$x\lambda=0.2$	$x\lambda=-0.2$	$x\lambda=-0.4$	$x\lambda=-0.6$	$x\lambda=-0.8$	$x\lambda=-1.0$
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.05	1.0359	1.0281	1.0206	1.0135	1.0066	0.9936	0.9875	0.9815	0.9758	0.9702
0.10	1.0561	1.0436	1.0319	1.0207	1.0101	0.9903	0.9811	0.9722	0.9637	0.9555
0.15	1.0711	1.0550	1.0400	1.0259	1.0126	0.9880	0.9766	0.9657	0.9553	0.9453
0.20	1.0832	1.0642	1.0466	1.0301	1.0146	0.9862	0.9731	0.9607	0.9488	0.9375
0.25	1.0933	1.0718	1.0520	1.0335	1.0162	0.9847	0.9703	0.9566	0.9436	0.9313
0.30	1.1019	1.0783	1.0565	1.0364	1.0176	0.9835	0.9679	0.9533	0.9393	0.9261
0.35	1.1093	1.0838	1.0604	1.0388	1.0187	0.9825	0.9660	0.9504	0.9358	0.9219
0.40	1.1157	1.0886	1.0638	1.0409	1.0197	0.9816	0.9643	0.9480	0.9327	0.9182
0.45	1.1214	1.0928	1.0667	1.0428	1.0206	0.9808	0.9628	0.9459	0.9300	0.9150
0.50	1.1265	1.0966	1.0694	1.0444	1.0214	0.9801	0.9615	0.9441	0.9277	0.9122
0.55	1.1311	1.1000	1.0718	1.0459	1.0221	0.9795	0.9604	0.9424	0.9256	0.9097
0.60	1.1353	1.1031	1.0739	1.0472	1.0227	0.9789	0.9593	0.9409	0.9237	0.9075
0.65	1.1391	1.1059	1.0759	1.0484	1.0232	0.9784	0.9584	0.9396	0.9220	0.9055
0.70	1.1426	1.1085	1.0776	1.0495	1.0238	0.9780	0.9575	0.9384	0.9205	0.9037
0.75	1.1458	1.1108	1.0792	1.0505	1.0242	0.9776	0.9568	0.9373	0.9191	0.9020
0.80	1.1487	1.1130	1.0807	1.0514	1.0246	0.9772	0.9560	0.9363	0.9179	0.9005
0.85	1.1514	1.1150	1.0821	1.0523	1.0250	0.9769	0.9554	0.9354	0.9167	0.8992
0.90	1.1540	1.1168	1.0834	1.0531	1.0254	0.9765	0.9548	0.9345	0.9156	0.8979
0.95	1.1563	1.1185	1.0846	1.0538	1.0258	0.9762	0.9542	0.9338	0.9146	0.8967
1.00	1.1585	1.1201	1.0857	1.0545	1.0261	0.9760	0.9537	0.9330	0.9137	0.8956

An idea of the accuracy reached in our calculations can be obtained by the comparisons made in Table 4 and 5 between the values of the integrals

$$\int_0^1 H^{(0)}(\mu)\,[1+x(1-\lambda)\,\mu^2]\,d\mu \quad \text{and} \quad \int_0^1 H^{(1)}(\mu)\,(1-\mu^2)\,d\mu\,,$$

evaluated numerically with the aid of the tabulated functions and their exact values, given by the formulae

$$\int_0^1 H^{(0)}(\mu)\,[1+x(1-\lambda)\,\mu^2]\,d\mu = \frac{2}{\lambda}\,[1-\sqrt{(1-\lambda)(1-\tfrac{1}{3}x\lambda)}\,]$$

and

$$\int_0^1 H^{(1)}(\mu)\,(1-\mu^2)\,d\mu = \frac{4}{x\lambda}\,[1-\sqrt{1-\tfrac{1}{3}x\lambda}\,]\,.$$

The values of $q^2 - c^2$ listed in Table 2 may also be compared with $1-\lambda$ in this connection.

TABLE 4

COMPARISON OF THE INTEGRALS $\int_0^1 H^{(0)}(\mu)[1+(1-\lambda)\mu^2]d\mu$ WITH THE AID OF THE TABULATED FUNCTIONS WITH THEIR EXACT VALUES $2[1-\sqrt{(1-\lambda)(1-\frac{1}{3}\lambda)}]\lambda^{-1}$

λ	Iterated	Exact	λ	Iterated	Exact
0.1	1.34523	1.345242	0.7	1.48688	1.486905
.2	1.35900	1.359012	.8	1.54253	1.542573
.3	1.37514	1.375164	.9	1.63422	1.634277
.4	1.39442	1.394449	.925	1.66964	1.669706
.5	1.41799	1.418011	.950	1.71609	1.716122
0.6	1.44770	1.447715	0.975	1.78476	1.784813

TABLE 5

COMPARISON OF THE INTEGRALS $\int_0^1 H^{(1)}(\mu)\,(1-\mu^2)d\mu$ WITH THE AID OF THE TABULATED FUNCTIONS WITH THEIR EXACT VALUES $4[1-\sqrt{1-\frac{1}{3}x\lambda}](x\lambda)^{-1}$

$x\lambda$	Iterated	Exact	$x\lambda$	Iterated	Exact
+1.0	0.73397	0.734014	−0.2	0.65592	0.655910
+0.8	.71822	.718256	−0.4	.64583	.645813
+0.6	.70379	.703819	−0.6	.63632	.636301
+0.4	.69049	.690507	−0.8	.62734	.627314
+0.2	0.67816	0.678164	−1.0	0.61884	0.618802