Appendix.

Note on the Theory of Sub-synchronous Maintenance.

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The principal features of interest requiring explanation in regard to the behaviour of the sub-synchronous pendulum are: (1) The actual possibility of the maintenance of the oscillation; (2) the fact that the frequency-ratio has generally an *even* number as the denominator; and (3) the manner in which the amplitude of the maintained oscillation varies in different parts of the range of maintenance. These will now be considered in the light of dynamical theory.

The effect of the periodic field due to the electro-magnet is equivalent to a large increase in the acceleration of gravity over a small part of the arc of swing of the pendulum. The system having only one degree of freedom, its equation of motion may be written in the form

$$\ddot{\theta}+\kappa\dot{\theta}+[n^2-\alpha\theta^2+f(t)\,.\,\mathrm{F}(\theta)]\,\theta=0.$$

In this equation, the damping of the pendulum due to dissipative forces is, as usual, taken to be proportional to the angular velocity. (Strictly speaking, the law of damping would not be the same at all parts of the arc of swing, especially when the pendulum-rod is near the vertical position, owing to the Foucault currents induced in it by the electro-magnet.) The term $-\alpha\theta^2$ which appears in the coefficient of θ is necessary, in view of the large amplitudes of oscillation actually obtained in practice. The term $f(t)\,.\,\mathrm{F}(\theta)$ expresses the effect of the periodic field.

It is sufficient for our present purpose to write $f(t)$ in the form $\beta+\gamma\sin mt$, the higher harmonic components being neglected. With the experimental arrangements actually adopted, $\mathrm{F}(\theta)$ is appreciable only when θ is small, and is practically negligible elsewhere. We may now assume that the pendulum is maintained in a steady oscillation given by

$$\theta=\psi_1\sin(pt+\epsilon_1)+\psi_2\sin(2pt+\epsilon_2)+\psi_3\sin(3pt+\epsilon_3)+,\text{ etc.},$$

and the question to be determined is whether sufficient energy passes from the periodic field to the pendulum in order to sustain the motion.

The loss of energy due to dissipative forces varies as

$$\int\kappa\dot{\theta}^2dt=\tfrac{1}{2}\kappa p^2\psi_1{}^2t.$$

ψ_2, ψ_3, etc., being treated as negligible. The energy which passes from the field into the pendulum is proportional to

$$(\beta+\gamma\sin mt)\,.\,\mathrm{F}(\theta)\,.\,\theta\,.\,\frac{d\theta}{dt}dt=\tfrac{1}{2}p\psi_1{}^2\int(\beta+\gamma\sin mt)\sin 2(pt+\epsilon_1)\,.\,\mathrm{F}(\theta)dt.$$

Changing the origin of time, this may be written in the form

$$\tfrac{1}{2}p\psi_1{}^2\int[\beta+\gamma\sin m(t-\epsilon_1/p)]\sin 2pt\,.\,\mathrm{F}(\theta)dt.$$

To effect the integration, we may, as a first approximation assume that $F(\theta)$ is equal to a constant δ when θ lies between the limits $\pm\phi$, and vanishes elsewhere. The corresponding limits for the variable t are

$$t = \pm\tau + r\pi/p,$$

where $$\tau = 1/p \,.\, \sin^{-1}\phi/\psi_1.$$

From this, it is readily shown that if $m = (2s+1)p$ where s is an integer, the integral evaluated over any number of complete periods is zero. On the other hand, if $m = 2sp$, the integral is finite and increases in proportion to the time. It follows that, on the assumptions made, maintenance is *not* possible when the frequency-ratio is unity divided by an *odd* integer, while if the ratio be unity divided by an even integer, energy may pass from the field to the pendulum in quantity sufficient to maintain its motion.

The other feature requiring explanation is the manner in which the amplitude of the maintained oscillation of the pendulum changes when its free period for small oscillations is altered by moving up the bob. The sequênce of phenomena within any one of the ranges of maintenance, as shown for instance in fig. 1 of the paper, is quite unlike the ordinary type of resonance of a simple vibrator. This is due, in the first place, to the fact that in present case, the amplitude of the maintained oscillation is too large for the ordinary theory of small oscillations to be applicable. Further, the field due to the electro-magnet is appreciable only when the pendulum is nearly in the vertical position. Consequently, when the arc of swing is large, the frequency of the forced oscillation does not differ sensibly from that of the free oscillation. As the bob of the pendulum is moved up, the natural frequency for small oscillations increases, but this is set off by a corresponding increase in the arc of swing, so that the free and forced periods do not differ appreciably at any stage. When the arc of spring is small, however, which is the case near the lower end of the range of maintenance, the constant part β of the field due to the electro-magnet has an appreciable effect, which is equivalent to an increase in the frequency of free oscillation. These considerations fully explain the sequence of phenomena shown in fig. 1 of the paper.

It may be remarked in conclusion, as has indeed been actually observed by Mr. Dey, that the successive ranges within which the bob must lie for maintenance to be possible may possibly overlap in certain cases. The frequency of the maintained oscillation may then assume one or another of the possible series of values according to the actual arc of swing with which the pendulum is started.