ON THE RADIATIVE EQUILIBRIUM OF A STELLAR ATMOSPHERE. IV

C. U. Cesco, S. Chandrasekhar, and J. Sahade
Yerkes Observatory
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ABSTRACT

In this paper the solution to the problem of line formation given in an earlier paper is obtained in its numerical form in the first three approximations. Tables for computing the residual intensity are given.

1. *Introduction.*—In an earlier paper[1] the "standard case" in the theory of the formation of stellar absorption lines has been considered and the solution obtained in the form

$$I_i = I(\mu_i) = b_\nu \left\{ \sum_{a=1}^{n} \frac{L_a e^{-k_a t}}{1+\mu_i k_a} + \mu_i + \frac{a_{\nu_0}}{b_\nu} + t \right\} \quad (i = \pm 1, \ldots, \pm n), \quad (1)$$

where the μ_i's represent the Gaussian division of the interval $(-1, +1)$, the k_a's the n positive roots of the equation

$$1 = (1-\lambda) \sum_{j=1}^{n} \frac{a_j}{1-\mu_j^2 k^2}, \quad (2)$$

and the L_a's ($a = 1, \ldots, n$) the n constants of integration to be determined by the boundary conditions

$$\sum_{a=1}^{n} \frac{L_a}{1-\mu_i k_a} + \frac{a_{\nu_0}}{b_\nu} = \mu_i \qquad (i = 1, \ldots, n). \quad (3)$$

Furthermore, in solution (1), t denotes the total optical depth, λ is a constant related to the ratio of the line to the continuous absorption coefficient (cf. II, eq. [87]) and which depends on the frequency in the line considered, and a_{ν_0} and b_ν are two constants which represent the coefficients in a Taylor expansion of the Planck intensity B_{ν_0}, valid near $t = 0$:

$$B_{\nu_0} = a_{\nu_0} + b_\nu t. \quad (4)$$

In this paper we propose to study this solution in greater detail.

2. *The formula for the residual intensity.*—We shall first derive the formula for the residual intensity. This is defined by

$$r = \frac{F_\nu(0)}{F_c(0)}, \quad (5)$$

where $F_\nu(0)$ denotes the emergent flux at the frequency considered and $F_c(0)$ the intensity of the background in the continuous sprectrum. Since

$$F = 2\int_{-1}^{+1} I\mu d\mu, \quad (6)$$

[1] S. Chandrasekhar, *Ap. J.*, **100**, 76, 1944. Referred to hereafter as "II."

we can write, in the approximation in which solution (1) is valid,

$$F = 2 \sum_{-n}^{n} I_j \mu_j a_j . \tag{7}$$

Substituting for the I_j's according to equation (1), we obtain

$$F = \tfrac{4}{3} b_\nu \left\{ 1 - \frac{3\lambda}{1-\lambda} \sum_{a=1}^{n} \frac{L_a}{k_a} e^{-k_a t} \right\} . \tag{8}$$

Accordingly, for the emergent flux we have

$$F_\nu (0) = \tfrac{4}{3} b_\nu \left\{ 1 - \frac{3\lambda}{1-\lambda} \sum_{a=1}^{n} \frac{L_a}{k_a} \right\} . \tag{9}$$

To obtain the emergent flux F_c (0) corresponding to the background continuum, we have to pass to the limit $\lambda = 1$ and $\eta = 0$ in the foregoing equation. But the result can be obtained more readily by starting with the equation of transfer appropriate for this case; for the resulting equation then admits of an elemental integration, and we obtain

$$F_c (0) = a_{\nu_0} + \tfrac{2}{3} b_{\nu_0} , \tag{10}$$

where it should be noted that the quantity b_{ν_0} [$= b_\nu$ $(1+\eta)$] is now defined in terms of the expansion

$$B_{\nu_0} = a_{\nu_0} + b_{\nu_0} \tau , \tag{11}$$

τ denoting the optical depth for the continuous opacity only. Combining equations (9) and (10), we have

$$r = \frac{b_\nu}{b_{\nu_0}} \frac{1 - \dfrac{3\lambda}{1-\lambda} \displaystyle\sum_{a=1}^{n} \frac{L_a}{k_a}}{\dfrac{1}{2} + \dfrac{3}{4} \dfrac{a_{\nu_0}}{b_{\nu_0}}} . \tag{12}$$

Since equation (4) is to be regarded as a Taylor expansion valid near $t = 0$, we can readily find an explicit expression for b_ν / a_{ν_0}. Clearly,

$$\frac{b_\nu}{a_{\nu_0}} = \left(\frac{1}{B_{\nu_0}} \frac{dB_{\nu_0}}{dt} \right)_{T=T_0} \quad \text{and} \quad \frac{b_{\nu_0}}{a_{\nu_0}} = \left(\frac{1}{B_{\nu_0}} \frac{dB_{\nu_0}}{d\tau} \right)_{T=T_0} , \tag{13}$$

where T_0 denotes the boundary temperature. Using Milne's well-known relation

$$T^4 = T_0^4 (1 + \tfrac{3}{2} \tau) , \tag{14}$$

we derive

$$\frac{b_\nu}{a_{\nu_0}} = \frac{3}{8} u \frac{e^u}{e^u - 1} \left(\frac{d\tau}{dt} \right) = \frac{3}{8} u \frac{e^u}{e^u - 1} \frac{1}{1+\eta} , \tag{15}$$

where

$$u = \frac{h\nu_0}{kT} . \tag{16}$$

Equation (12) can now be re-written as

$$r=\frac{1}{1+\eta}\,\frac{1-\dfrac{3\lambda}{1-\lambda}\displaystyle\sum_{a=1}^{n}\frac{L_a}{k_a}}{\dfrac{1}{2}+2\,\dfrac{1}{u}\,\dfrac{e^u-1}{e^u}}\,; \tag{17}$$

or, writing

$$\mathcal{L}_a=\lambda L_a\,;\qquad x=\frac{1}{u}\,\frac{e^u-1}{e^u}\,, \tag{18}$$

we have

$$r=\frac{1}{1+\eta}\left\{\frac{1-\dfrac{3}{1-\lambda}\displaystyle\sum_{a=1}^{n}\frac{\mathcal{L}_a}{k_a}}{\frac{1}{2}+2x}\right\}. \tag{19}$$

The $\mathcal{L}_a$'s are now determined by the conditions

$$\sum_{a=1}^{n}\frac{\mathcal{L}_a}{1-\mu_i k_a}=\lambda\mu_i-\tfrac{8}{3}\lambda(1+\eta)\,x\quad(i=1,\,\ldots\,,\,n)\,. \tag{20}$$

Now equations (2) and (20) allow us to derive values of k_a and $\mathcal{L}_a$ for all values of $\lambda < 1$. To obtain the limiting form of the solution for $\lambda = 1$, we proceed as follows: It can be readily shown that, as $\lambda \to 1$, the characteristic roots k_i tend to

$$\frac{1}{\mu_i}\left[1-\tfrac{1}{2}(1-\lambda)\,a_i\right]\qquad(\lambda\to 1)\,. \tag{21}$$

From this we derive

$$\mathcal{L}_i\to\tfrac{1}{2}\lambda a_i\left(\mu_i-\tfrac{8}{3}\left[1+\eta\right]x\right)(1-\lambda)\qquad(\lambda\to 1)\,, \tag{22}$$

and the formula for r becomes

$$r=\frac{\frac{1}{2}+4x\displaystyle\sum_{i=1}^{n}a_i\mu_i}{\frac{1}{2}+2x}\,. \tag{23}$$

Since $\sum_{i=1}^{n} a_i\mu_i$ tends to $\frac{1}{2}$ as $n\to\infty$, $r\to 1$ as the order of approximation increases. Hence the departure of $\sum_{i=1}^{n} a_i\mu_i$ from $\frac{1}{2}$ is an index of the accuracy of our approximation.

3. *Numerical results.*—The numerical forms for the first three approximations to the solution have been worked out for various values of λ and x and for the most important case: $\epsilon = 0$ or $\lambda = (1+\eta)^{-1}$.

But before we tabulate these solutions, we may note here the equations that determine the characteristic roots in the first three approximations. They are:

$$k^2=3\lambda\,, \tag{24}$$

$$\mu_1^2\mu_2^2k^4-\left\{\tfrac{1}{3}+\lambda\left[\mu_1^2+\mu_2^2-\tfrac{1}{3}\right]\right\}k^2+\lambda=0\,, \tag{25}$$

and

$$\left.\begin{aligned}\mu_1^2\mu_2^2\mu_3^2k^6-\left\{\lambda\left[\mu_1^2\mu_2^2+\mu_1^2\mu_3^2+\mu_2^2\mu_3^2\right]+(1-\lambda)\left[\tfrac{1}{3}(\mu_1^2+\mu_2^2+\mu_3^2)-\tfrac{1}{5}\right]\right\}k^4&\\+\left\{\lambda\left[\mu_1^2+\mu_2^2+\mu_3^2-\tfrac{1}{3}\right]+\tfrac{1}{3}\right\}k^2-\lambda=0\,.&\end{aligned}\right\} \tag{26}$$

In Table 1, the resulting roots k_a for various values of λ are tabulated.

In Tables 2, 3, and 4 we have tabulated the residual intensity as derived after the evaluation of the necessary constants.

TABLE 1

VALUES OF k

λ	First Approximation (k_1)	Second Approximation		Third Approximation		
		k_1	k_2	k_1	k_2	k_3
0.05	0.387298	2.01257	0.379497	3.24943	0.379484	1.23255
0.10	0.547723	2.05519	0.525560	3.29669	0.525430	1.24087
0.20	0.774597	2.14577	0.711876	3.39312	0.710456	1.26096
0.30	0.948683	2.24183	0.834508	3.49150	0.829085	1.28609
0.40	1.09545	2.34138	0.922637	3.59119	0.909563	1.31607
0.50	1.22474	2.44271	0.988751	3.69160	0.964672	1.34962
0.60	1.34164	2.54448	1.03980	3.79227	1.00260	1.38474
0.70	1.44914	2.64575	1.08012	3.89279	1.02909	1.41957
0.80	1.54919	2.74589	1.11259	3.99286	1.04801	1.45283
0.90	1.64317	2.84450	1.13917	4.09225	1.06192	1.48385
0.95	1.68819	2.89315	1.15071	4.14163	1.06752	1.49843
1.00	1.73205	2.94134	1.16126	4.19078	1.07242	1.51238

TABLE 2

VALUES OF r GIVEN BY THE FIRST APPROXIMATION

λ	x												
	0.1	0.125	0.15	1/6	0.175	0.20	0.225	0.25	0.275	0.3	0.325	1/3	0.35
0.05	0.1790	0.1952	0.2093	0.2178	0.2218	0.2330	0.2429	0.2519	0.2600	0.2674	0.2741	0.2762	0.2802
0.10	0.2671	0.2862	0.3030	0.3131	0.3178	0.3310	0.3428	0.3534	0.3630	0.3717	0.3797	0.3822	0.3870
0.20	0.4013	0.4221	0.4404	0.4513	0.4564	0.4707	0.4835	0.4950	0.5054	0.5149	0.5235	0.5262	0.5315
0.30	0.5104	0.5309	0.5488	0.5595	0.5646	0.5786	0.5912	0.6025	0.6127	0.6220	0.6305	0.6332	0.6383
0.40	0.6057	0.6250	0.6418	0.6519	0.6567	0.6699	0.6817	0.6924	0.7020	0.7108	0.7188	0.7213	0.7261
0.50	0.6917	0.7094	0.7248	0.7341	0.7385	0.7506	0.7614	0.7712	0.7800	0.7880	0.7954	0.7977	0.8021
0.60	0.7710	0.7868	0.8006	0.8089	0.8128	0.8237	0.8334	0.8421	0.8500	0.8572	0.8638	0.8658	0.8698
0.70	0.8450	0.8588	0.8709	0.8782	0.8816	0.8910	0.8995	0.9071	0.9140	0.9203	0.9260	0.9278	0.9313
0.80	0.9148	0.9265	0.9368	0.9429	0.9458	0.9538	0.9610	0.9675	0.9733	0.9786	0.9835	0.9850	0.9880
0.90	0.9810	0.9906	0.9989	1.0039	1.0063	1.0129	1.0187	1.0240	1.0288	1.0331	1.0371	1.0383	1.0407
0.95	1.0130	1.0214	1.0288	1.0333	1.0354	1.0412	1.0464	1.0510	1.0553	1.0591	1.0626	1.0637	1.0658
1.00	1.0442	1.0516	1.0580	1.0619	1.0637	1.0688	1.0733	1.0774	1.0810	1.0844	1.0874	1.0884	1.0902

TABLE 3

VALUES OF r GIVEN BY THE SECOND APPROXIMATION

λ	x												
	0.1	0.125	0.15	1/6	0.175	0.20	0.225	0.25	0.275	0.3	0.325	1/3	0.35
0.05	0.1729	0.1884	0.2021	0.2103	0.2141	0.2248	0.2344	0.2430	0.2508	0.2578	0.2643	0.2663	0.2702
0.10	0.2559	0.2741	0.2899	0.2995	0.3040	0.3164	0.3275	0.3376	0.3466	0.3549	0.3624	0.3648	0.3693
0.20	0.3822	0.4014	0.4182	0.4282	0.4330	0.4461	0.4579	0.4685	0.4780	0.4868	0.4947	0.4972	0.5020
0.30	0.4855	0.5038	0.5198	0.5294	0.5339	0.5465	0.5577	0.5678	0.5770	0.5853	0.5929	0.5953	0.5998
0.40	0.5766	0.5933	0.6078	0.6166	0.6207	0.6321	0.6423	0.6515	0.6598	0.6674	0.6743	0.6765	0.6806
0.50	0.6598	0.6744	0.6872	0.6948	0.6984	0.7084	0.7174	0.7255	0.7328	0.7394	0.7454	0.7473	0.7510
0.60	0.7374	0.7497	0.7604	0.7669	0.7699	0.7784	0.7859	0.7927	0.7988	0.8044	0.8095	0.8111	0.8142
0.70	0.8107	0.8205	0.8291	0.8343	0.8367	0.8435	0.8495	0.8549	0.8599	0.8643	0.8684	0.8697	0.8722
0.80	0.8805	0.8878	0.8942	0.8980	0.8998	0.9048	0.9093	0.9133	0.9170	0.9203	0.9233	0.9242	0.9261
0.90	0.9475	0.9522	0.9563	0.9587	0.9599	0.9631	0.9660	0.9686	0.9709	0.9730	0.9750	0.9756	0.9768
0.95	0.9801	0.9835	0.9864	0.9882	0.9890	0.9913	0.9934	0.9952	0.9969	0.9984	0.9998	1.0003	1.0011
1.00	1.0122	1.0142	1.0160	1.0170	1.0175	1.0189	1.0202	1.0213	1.0223	1.0232	1.0240	1.0243	1.0248

It may be of interest to compare these solutions with that derived by the more familiar Milne-Eddington approximation of setting $K = \frac{1}{3} J$. We have[2] in this approximation:

$$F = \frac{4}{3} b_{\nu_0} \frac{\sqrt{\lambda} + \frac{8}{\sqrt{3}} x}{\frac{1}{\sqrt{\lambda}} + \frac{2}{\sqrt{3}}} . \tag{27}$$

In Table 5 the comparison between this solution and the ones obtained in this paper is made.

TABLE 4

VALUES OF r GIVEN BY THE THIRD APPROXIMATION

λ	x: 0.1	1/6	0.25	1/3	0.35
0.05	0.1722	....	0.2419	....	0.2690
0.10	0.2544	....	0.3355	....	0.3670
0.20	0.3794	0.4248	0.4645	0.4928	0.4976
0.30	0.4815	....	0.5622	....	0.5936
0.40	0.5717	....	0.6446	....	0.6729
0.50	0.6542	....	0.7174	....	0.7419
0.60	0.7313	....	0.7836	....	0.8040
0.70	0.8042	....	0.8451	....	0.8610
0.80	0.8739	....	0.9028	....	0.9141
0.90	0.9409	....	0.9576	....	0.9641
0.95	0.9736	....	0.9841	....	0.9881
1.00	1.0057	....	1.0100	....	1.0116

TABLE 5

COMPARISON WITH EDDINGTON'S APPROXIMATION
(Standard Case: λ=0.20)

x	Eddington's Approximation	First Approximation	Second Approximation	Third Approximation
0.100	0.3575	0.3746	0.3568	0.3541
1/6	.4786	.5014	.4758	.4719
0.250	.6299	.6600	.6246	.6193
1/3	.7813	.8186	.7734	.7666
0.350	0.8115	0.8503	0.8032	0.7961

[2] Eddington, *M.N.*, **89**, 620, 1929.