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ON THE EVOLUTION OF THE MAIN-SEQUENCE STARS

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ABSTRACT

The evolution of the stars on the main sequence consequent to the gradual burning of the hydrogen in the central regions is examined. It is shown that, as a result of the decrease in the hydrogen content in these regions, the convective core (normally present in a star) eventually gives place to an isothermal core. It is further shown that there is an upper limit (~ 10 per cent) to the fraction of the total mass of hydrogen which can thus be exhausted. Some further remarks on what is to be expected beyond this point are also made.

1. *General considerations.*—The problem of stellar evolution is intimately connected with that of energy production in the stars. Both Bethe and Weizsäcker² showed that the source of the energy radiated by the main-sequence stars is the transformation of hydrogen into helium through the so-called “carbon cycle.” On the basis of the Bethe-Weizsäcker theory, G. Gamow³ outlined a picture of stellar evolution.

The Gamow theory is based on three fundamental assumptions: (*a*) the stars evolve gradually through a sequence of equilibrium configurations; (*b*) the successive equilibrium configurations are homologous; and (*c*) the nuclear reaction continues to take place until the entire hydrogen in the star is exhausted.

Such a picture of stellar evolution presents certain difficulties. The assumption that the successive equilibrium configurations are homologous cannot be expected to be rigorously valid; for the nuclear reaction reduces the hydrogen content in the neighborhood of the center of the star, and therefore the molecular weight in this region becomes increasingly larger than that of the rest of the stellar material, unless we suppose that a diffusion process rapidly mixes the whole of the stellar mass. The only region in which the mixing can be supposed to take place is the Cowling convective core; but stellar configurations in which the ratios of the molecular weights of the convective core and envelope are different are not homologous, contrary to assumption (*b*).

Another difficulty of the Gamow theory is the uncertainty of the amount of hydrogen that can be burned. It is important to determine this quantity accurately, since its value affects essentially the final luminosity and effective temperature given by the homology formulae. Again it does not appear probable that the entire hydrogen content could be

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² H. Bethe, *Phys. Rev.*, **55**, 434, 1939; and C. F. von Weizsäcker, *Phys. Zs.*, **39**, 633, 1938.

³ *Phys. Rev.*, **53**, 59, 1938; **55**, 718, 1939; *Nature*, **144**, 575, 1939.

001 3 1942

exhausted, since it would imply a thorough mixing of the stellar material, in order that the entire content could reach the center, where the nuclear reactions principally take place. Sometimes it is assumed that only 14.5 per cent of the entire hydrogen content (which is the fraction of the total mass contained inside the Cowling convective core) participate in the carbon cycle. This hypothesis, though apparently more plausible, should be further amended, for it would be valid only if the turbulence in the convective core mixed the material rapidly enough to avoid the formation of an isothermal region at the center which would tend to stop convection. However, even if there is no formation of an isothermal core, the fraction of the stellar mass contained in the convective region would be expected to diminish as the molecular weight increases relatively to that of the rest of the stellar material. We shall obtain the precise amount of this diminution; but it is clear that only a small fraction of the stellar hydrogen could be burned if only the hydrogen in the convective core was available for the nuclear reaction. We should not conclude, however, that only that small amount of hydrogen could be burned, since there is the possibility of the formation of an isothermal core after the exhaustion of hydrogen has stopped the convection in the central regions. Such a possibility results, as we shall show, from the existence of equilibrium configurations formed by isothermal cores surrounded by point-source envelopes, the mass of the isothermal cores being larger than that of the limiting convective core of vanishing hydrogen content.

In the isothermal core-radiative envelope models the nuclear reaction takes place at the interface of core and envelope. The fraction of the mass contained in the isothermal core cannot exceed a fixed value, so that the nuclear reaction will finally cease when the mass in the core reaches its maximum value. A possibility which should not be overlooked is that, during the transition to gravitational energy production, larger cores could be formed that would not be equilibrium configurations; however, the lifetime of such configurations is presumably small, so that in a first approximation we can neglect such possibilities.

2. *Stellar models formed by convective cores and radiative envelopes with different molecular weights.*—The construction of these models can be done either by the method used by T. G. Cowling⁴ or by that proposed by Chandrasekhar.⁵

At the interface the values of the temperature, pressure, and mass of the core should be identical:

$$P(r_i)_{\text{core}} = P(r_i)_e; \quad T(r_i)_{\text{core}} = T(r_i)_e; \quad M(r_i)_{\text{core}} = M(r_i)_e, \quad (1)$$

where P , M , and T denote the total pressure, the mass within the radius r , and the temperature, respectively. The index i indicates that the values refer to the interface and the index e that the quantities correspond to the envelope solution of the equilibrium equations.

Conditions (1) are not sufficient; it is further necessary that the effective polytropic index of the envelope be 1.5 at the interface

$$\left(\frac{d \ln P}{d \ln T}\right)_e = \left(\frac{d \ln P}{d \ln T}\right)_{\text{core}}. \quad (2)$$

It is convenient to introduce the homology invariant quantities U and V for the core as follows:

$$U = \frac{4\pi\rho r^3}{M(r)}; \quad V = \frac{2}{5} \frac{GM(r)\rho}{rP}. \quad (3)$$

Conditions (1) can now be written in terms of U_i and V_i as

$$U_i = \frac{4\pi\rho_e(r_i)r_i^3\mu_c}{M_e(r_i)\mu_e}; \quad V_i = \frac{2}{5} \frac{GM_e(r_i)\rho_e(r_i)\mu_c}{r_iP_e(r_i)\mu_e}, \quad (4)$$

⁴ *M.N.*, 91, 92, 1931.

⁵ *An Introduction to the Study of Stellar Structure*, p. 352, Chicago, 1939.

where μ_c and μ_e denote the molecular weights of the core and envelope, respectively. The appearance of the ratio of the molecular weights in formulae (4) is due to the discontinuity of the density at the interface. If radiation pressure is negligible, the ratio of the values of the density on both sides of the interface is simply the ratio of the corresponding molecular weights. Indeed, in that case the total pressure may be identified with the gas pressure; and so

$$P = \frac{k}{\mu H} \rho T. \tag{5}$$

Since both P and T are continuous at the interface, equation (5) implies the continuity of ρ/μ .

For given values of the total mass M , luminosity L , radius R , and molecular weights μ_c and μ_e it is generally possible to find ∞^1 solutions satisfying conditions (4), each of these solutions corresponding to a value of the opacity constant κ_0 that appears in Kramer's law,

$$\kappa = \kappa_0 \rho T^{-3.5}. \tag{6}$$

Condition (2) eliminates the arbitrariness of κ_0 .

In order to determine the effect of the increase of the ratio μ_c/μ_e on the size of the convective core, we have applied the method described to a star with the solar values M_\odot , L_\odot , R_\odot , $\mu_e = 1$, and $\mu_c/\mu_e = 2$, neglecting radiation pressure. We found that such a con-

TABLE 1

$\frac{\mu_c}{\mu_e}$	ξ	η	S	$\frac{\rho_c}{\rho}$	$\frac{L_0}{[\frac{2}{5}S]^{1/2}} \times 10^{-25}$	$(1-\beta_c) \left(\frac{M_\odot}{M}\right)^2 \mu_e^{-4}$	$-\omega$
1.....	0.169	0.145	2.250	37	0.573	0.0053	0.554
2.....	0.083	0.076	1.853	179	1.140	0.0190	0.803

vective core could be approximately fitted to an envelope described by the Inger Nielsen solution with $\log \kappa_0 = 24.992$. The characteristics of the Cowling model and those of the limiting convective core with $\mu_c/\mu_e = 2$ are given in Table 1. In this table we have tabulated the quantities

$$\xi = \frac{r_i}{R}; \quad \eta = \frac{M_i}{M}; \quad S = \frac{T_c}{T_i} \frac{\eta}{\xi V_i}; \quad \omega = - \left[\frac{6}{7} \frac{\eta^2}{\xi V_i} \left(\frac{4}{5} U_i + V_i - 1 \right) + \frac{5}{8} J \right], \tag{7}$$

where T_c is the central temperature and J an integral that has to be evaluated numerically, as will be explained later. All the tabulated quantities are homology invariants, and it is possible to express in terms of them the radius R , the luminosity L , and the gravitational energy Ω of the star as follows:

$$R = \frac{2}{5} \frac{\mu_c H}{k} \frac{GM}{T_c} S \left(\frac{\mu_c}{\mu_e} \right), \tag{8}$$

$$L = \frac{L_0 \left(\frac{\mu_c}{\mu_e} \right)}{\left[\frac{2}{5} S \left(\frac{\mu_c}{\mu_e} \right) \right]^{1/2}} \left(\frac{k}{GH} \right)^{1/2} \frac{1}{\kappa_0} M^5 T_c^{0.5} \mu_e^{7.5} \mu_c^{-0.5}, \tag{9}$$

$$\Omega = \omega \frac{GM^2}{R}, \tag{10}$$

where L_0 is a numerical constant depending only on the ratio of the molecular weights. Before proceeding further we shall indicate how formulae (8), (9) and (10) arise.

3. According to the second of equations (4), we have

$$r_i = \frac{2}{5} \frac{GM \eta \rho_{\text{core}}(r_i)}{V_i P_i},$$

and so

$$R = \frac{2}{5} GM \frac{\eta}{\xi V_i} \frac{\rho_{\text{core}}(r_i)}{P_i};$$

but in the core we have

$$\frac{\rho}{P} = \frac{\mu_c H}{kT}.$$

Hence,

$$R = \frac{2}{5} \frac{\mu_c H}{k} \frac{GM}{T_c} \frac{\eta}{\xi V_i T_i} = \frac{2}{5} \frac{\mu_c H}{k} \frac{GM}{T_c} S \left(\frac{\mu_c}{\mu_e} \right),$$

which is our formula (8).

For any homologous family of configurations, derived on the basis of Kramer's law of opacity, the luminosity is given by the formula

$$L = L_0 \frac{M^{5.5}}{\kappa_c R^{0.5}} \mu_e^{7.5}, \quad (11)$$

L_0 being a characteristic constant of the family. In the present case L_0 can depend only on μ_c/μ_e . Applying equation (11) to the star of the family with the solar values M_\odot , L_\odot , R_\odot , and $\mu_e = 1$ and using solar units, we get for L_0 the formula

$$L_0 = (\kappa_0)_{\text{int}}. \quad (9a)$$

Introducing into equation (11) expression (8) for R , we obtain formula (9).

4. We shall now evaluate the potential energy Ω of the composite models. Quite generally

$$\Omega = - \int_0^R \frac{GM(r) dM(r)}{r}, \quad (12)$$

or

$$\Omega = \frac{1}{2} \int_0^R W dM(r). \quad (13)$$

W being the gravitational potential.

Now, for a composite model, Ω is the sum of the potential energies of the core and envelope, i.e.,

$$\Omega = \Omega_{\text{core}} + \Omega_e, \quad (14)$$

$$\Omega_{\text{core}} = \frac{1}{2} \int_0^{r_i} W dM(r), \quad \Omega_e = \frac{1}{2} \int_{r_i}^R W dM(r).$$

The core is a gaseous sphere of polytropic index 1.5; hence⁶

$$\frac{5}{2} \left(\frac{P}{\rho} - \frac{P_i}{\rho_i} \right) = W_i - W = - \frac{GM_i}{r_i} - W; \quad (15)$$

and therefore

$$\Omega_{\text{core}} = - \frac{5}{4} \int_{\text{core}} P d\tau + \frac{5}{4} \frac{P_i M_i}{\rho_i} - \frac{1}{2} \frac{GM_i^2}{r_i},$$

⁶ Chandrasekhar, *op. cit.*, p. 100.

where $d\tau$ is an element of volume. Using a formula due to E. A. Milne,⁷ we can express Ω_c as

$$\Omega_c = -3 \int_{\text{core}} P d\tau + 4\pi P_i r_i^3. \quad (16)$$

Introducing this value of $\int P d\tau$ in the expression for Ω_c , we get

$$\Omega_{\text{core}} = \frac{5}{12} \Omega_{\text{core}} - \frac{5\pi}{3} P_i r_i^3 + \frac{5}{4} \frac{P_i M_i}{\rho_i} - \frac{1}{2} \frac{GM_i^2}{r_i}$$

or

$$\Omega_{\text{core}} = -\frac{6}{7} \left[\frac{GM_i^2}{r_i} + \frac{5}{2} P_i \left(\frac{4\pi}{3} r_i^3 - \frac{M_i}{\rho_i} \right) \right]. \quad (17)$$

But

$$\frac{GM_i^2}{r_i} = \frac{GM^2}{R} \frac{\eta^2}{\xi}. \quad (17a)$$

On the other hand, according to the conditions (4) we have

$$P_i r_i^3 = \frac{2}{5} \frac{GM \eta P_{\text{core}}(r_i)}{V_i} \quad r_i^2 = \frac{2}{5} \frac{GM^2 U_i \eta^2}{4\pi R \xi V_i} \quad (17b)$$

and

$$\frac{P_i}{\rho_i} = \frac{2}{5} \frac{GM}{R} \frac{\eta}{\xi V_i}. \quad (17c)$$

Taking into account these last three formulae, we get for Ω_{core}

$$\Omega_{\text{core}} = -\frac{6}{7} \frac{\eta^2}{\xi} \left[1 + \frac{1}{3} \frac{U_i}{V_i} - \frac{1}{V_i} \right] \frac{GM^2}{R}. \quad (18)$$

Turning next to the part Ω_e of Ω , we start with the formula (cf. eq. [16])

$$\Omega_c = -3 \int_{\text{env}} P d\tau - 4\pi P_i r_i^3. \quad (16a)$$

The integral in equation (16a) can be put in the form

$$\int_{\text{env}} P d\tau = \frac{2}{5} J \frac{GM^2}{R}, \quad (19)$$

where J is a homology invariant quantity that has to be evaluated numerically. Taking into account equations (17b and (19) we get for Ω_e

$$\Omega_e = - \left[\frac{2}{5} J + \frac{2}{5} \frac{U_i \eta^2}{\xi V_i} \right] \frac{GM^2}{R}. \quad (20)$$

Thus, combining equations (18) and (20), we finally obtain

$$\Omega = \Omega_e = \Omega_{\text{core}} = - \left[\frac{6}{7} \frac{\eta^2}{\xi V_i} \left(\frac{4}{5} U_i + V_i - 1 \right) + \frac{2}{5} J \right] \frac{GM^2}{R}, \quad (21)$$

which is the last of the formulae (4). We can express equation (21) more simply as

$$\Omega = \omega \frac{GM^2}{R}. \quad (10)$$

It is this quantity ω which is tabulated in Table 1.

⁷ *M.N.*, **89**, 739, 1929; **96**, 179, 1936.

It is now easy to derive the formulae for the internal energy H and the total energy E as follows:

$$H = \int_0^R c_v T dM(r) = \left[(c_v)_{\text{core}} \frac{\mu_c H}{k} \int_0^{r_i} \frac{P}{\rho} dM + (c_v)_e \frac{\mu_e H}{k} \int_{r_i}^R \frac{P}{\rho} dM \right],$$

or

$$H = \left[\left(\frac{c_v}{c_p - c_v} \right)_{\text{core}} \int_{\text{core}} P d\tau + \left(\frac{c_v}{c_p - c_v} \right)_e \int_{\text{env}} P d\tau \right] \\ = \left[\frac{1}{\gamma_{\text{core}} - 1} \int_{\text{core}} P d\tau + \frac{1}{\gamma_e - 1} \int_{\text{env}} P d\tau \right].$$

Taking into account equations (16) and (16a), we get

$$H = -\frac{1}{3} \left[\frac{\Omega_{\text{core}}}{\gamma_{\text{core}} - 1} + \frac{\Omega_e}{\gamma_e - 1} - 4\pi P_i r_i^3 \left(\frac{1}{\gamma_{\text{core}} - 1} - \frac{1}{\gamma_e - 1} \right) \right].$$

But $\gamma_{\text{core}} = \gamma_{\text{env}} = \gamma$; and so

$$H = -\frac{\Omega}{3(\gamma - 1)}. \quad (22)$$

For the total energy E we have

$$E = \Omega + H = \Omega - \frac{\Omega}{3(\gamma - 1)} = \frac{3\gamma - 4}{3(\gamma - 1)} \Omega. \quad (23)$$

Formulae (22) and (23) hold for any model in which γ is constant through the whole stellar mass. The preceding argumentation is, however, necessary to prove the validity of the Ritter-Perry formula (22) for composite configurations with different values of μ in the different regions.

5. *Stellar models with isothermal cores and radiative envelopes with different molecular weights.*—The method of constructing stellar models formed by an isothermal core surrounded by a radiative envelope was discussed by L. R. Henrich and S. Chandrasekhar⁸ for the case in which the values of μ are the same in both regions. We will now examine the more general case of different molecular weights.

It is necessary to fit an E solution corresponding to an isothermal core with molecular weight μ_c to a radiative envelope with molecular weight μ_e . At the interface the values of the pressure, temperature, and mass of the core given by both solutions should be identical:

$$P(r_i)_{\text{core}} = P(r_i)_e; \quad T(r_i)_{\text{core}} = T(r_i)_e; \quad M(r_i)_{\text{core}} = M(r_i)_e. \quad (24)$$

The density has a discontinuity at the interface. Neglecting radiation pressure, we get, as in section 2, for the ratio of the densities on both sides of the interface the value of the ratio of the respective molecular weights. The conditions (24) are the only ones to be fulfilled, and so we get a family of ∞^1 configurations for any given set of values of M , L , and R .

It is convenient to introduce the homology invariant functions u and v to describe the isothermal core,

$$u = 4\pi \frac{r^3 \rho_{\text{core}}}{M(r)}, \quad v = \frac{\mu_c H}{k} \frac{GM(r)}{rT(r)}. \quad (25)$$

The equations of fit in the new variables are

$$u_i = 4\pi \left[\frac{r^3 \rho(r_i)_e}{M(r_i)_e} \right] \frac{\mu_c}{\mu_e}, \quad v_i = \left[\frac{\mu_e H}{k} \frac{GM(r_i)_e}{r_i T(r_i)_e} \right] \frac{\mu_c}{\mu_e}. \quad (26)$$

⁸ *Ap. J.*, 94, 525, 1941.

The quantities q and ν ,

$$q = \frac{r_i}{R} \quad \text{and} \quad \nu = \frac{M(r_i)}{M}, \quad (27)$$

are homology invariants, and each of them can be used to label the different configurations corresponding to the same stellar mass and the same central temperature.

From the second of equations (26) we readily obtain the following formulae for the radius R :

$$R(q) = Q(q) \frac{\mu_c H}{k} \frac{GM}{T_c}, \quad (28)$$

where

$$Q(q) = \frac{\nu}{v_i q}. \quad (29)$$

For the present case the luminosity formula takes the form

$$L = L_0(q, \mu_c/\mu_e) \frac{1}{\kappa_0} \frac{M^{5.5}}{R^{0.5}} \mu_e^{7.5}. \quad (30)$$

Equation (30) is analogous to equation (9) and can be derived in the same way. The quantity L_0 depends on both q and μ_c/μ_e . Our numerical integrations were done with the solar values M_\odot , L_\odot , R_\odot , and $\mu_e = 2.2$; hence we get for L_0

$$L_0 = \frac{(\kappa_0)_{\text{int}}}{(2.2)^{7.5}}. \quad (31)$$

Introducing the expression (28) of R in equation (30), we find

$$L = \frac{L_0}{\sqrt{Q}} \left(\frac{k}{GH} \right)^{1/2} \frac{1}{\kappa} M^5 T_c^{0.5} \mu_e^{7.5} \mu_c^{-0.5}. \quad (32)$$

The variation of the luminosity at constant central temperature is determined by the factor L_0/\sqrt{Q} .

The thickness Δr_i of the energy-producing shell is related to the luminosity by the equation

$$L = 4\pi r_i^2 \rho(r_i) \epsilon \Delta r_i \epsilon_0, \quad (33)$$

where ϵ_0 is the energy production per gram per second.

Introducing in equation (33) the expression for L , we get

$$\frac{\epsilon_0 \Delta r_i}{R} = \frac{q L_0}{Q^{1/2} \mu_i \nu} \frac{1}{\kappa_0} \left(\frac{k}{GH} \right)^{1/2} M^4 T_c^{0.5} \mu_e^{6.5} \mu_c^{0.5}, \quad (34)$$

or

$$\frac{\Delta r_i}{R} \propto \frac{q L_0}{Q^{1/2} \mu_i \nu}. \quad (34a)$$

The potential energy Ω is the sum of two terms Ω_{core} and Ω_e . The quantity Ω_c is given by formula (16), since in the derivation of this formula the nature of the core does not matter. Introducing in equation (16) the value of P given by the gas equation and taking into account that the temperature is constant throughout the core, we obtain for Ω_c

$$\Omega_c = -3 \frac{k T_c}{\mu_c H} \int_{\text{cor}} \rho d\tau + 4\pi P_i r_i^3. \quad (35)$$

Similarly, Ω_e is given by equation (16a), and therefore

$$\Omega = \Omega_e + \Omega_c = -3 \frac{kT_c}{\mu_c H} \nu M - 3 \int_{\text{env}} P d\tau. \quad (36)$$

Taking into account formulae (19) and (28), we can simplify the expression of

$$\Omega = -3 (Q\nu + \frac{2}{3}J) \frac{GM^2}{R}. \quad (37)$$

The results of the numerical integration are given in Table 2 and illustrated graphically in Figures 1, 2, 3, 4, 5, and 6. The points represented by circles refer to the models with isothermal cores, the squares refer to models with convective cores.

TABLE 2*

q	ν	Q	$-\omega$	$\frac{L_0}{\sqrt{Q}} \times 10^{-25}$	$\frac{\rho_c}{\bar{\rho}} \times 10^{-2}$	$C \frac{\Delta r_i}{R}$	$(1-\beta_i) \left(\frac{M_\odot}{M}\right)^2 \mu_e^{-4}$
0.045	0.079	0.747	0.936	1.759	369	1.680	0.0004
.046	.071	.709	.896	1.610	628	1.750	.0005
.049	.093	.761	.988	1.956	124	1.520	.0005
.058	.101	.713	.976	2.020	32.1	1.306	.0005
.063	.094	.661	.918	1.869	14.5	1.176	.0004
.067	.089	.625	.879	1.714	9.21	1.116	.0004
.071	.072	.571	.813	1.425	4.49	1.024	.0004
0.073	0.065	0.555	0.786	1.318	3.46	1.000	0.0004

* C is a constant factor; $(1-\beta)$ is the ratio of radiation pressure to gas pressure; ρ_c is the central density; and $\bar{\rho}$ the mean density.

6. From Table 1 we derive the main features of the models with convective cores. As the ratio of molecular weights μ_c/μ_e increases from 1 to 2;

a) the fraction of the stellar mass in convective equilibrium decreases from 0.145 to 0.076.

b) the fraction of the radius occupied by the convective core decreases from 0.169 to 0.083.

c) the radius of the star increases by a factor ~ 1.65 , assuming that T_c remains constant.

d) the potential energy and the total energy decrease, the heat content, H , increases, and there is, therefore, liberation of gravitational energy when the star is still burning hydrogen.

e) The luminosity increases by a factor ~ 1.41 , assuming again the constancy of T_c .

f) The effective temperature decreases by a factor ~ 1.18 . The decrease of T_{eff} is due to the large increase of the radius. Actually, the situation is different, for the sharp reduction of hydrogen content in the core raises the central temperature and hence counteracts the tendency of the radius to grow.

For the models with isothermal cores we have the following properties:

a) There are no equilibrium configurations with cores containing less than 0.065 or more than 0.101 of the stellar mass. The lower limit is due to the appearance of convective instability at the interface, while the upper one is due to the impossibility of fitting a core to an envelope. The upper limit is a decreasing function of μ_c/μ_e , since it is ~ 0.35 for the case of equal molecular weights, as was shown by Henrich and Chandrasekhar.⁷

b) Starting from its minimum value, ν increases rapidly as q grows, reaches its absolute maximum, and starts spiraling around a certain value.

c) The radius of the model starts increasing as ν grows, reaches an absolute maximum $1.7 R_{\text{Cow}}$, and afterward starts to spiral.

d) The potential energy of gravitation Ω and the total nonnuclear energy E start decreasing as ν increases, reach an absolute minimum, and then finally start increasing and spiraling. In other words, the gravitational binding increases until it reaches a maximum and afterward spirals. Since the nuclear binding is the same for two models with the same ν , it results that there can be two or more configurations with the same amount of burned hydrogen and with different total bindings.

e) The luminosity starts increasing as ν grows until an absolute maximum $\sim 2.5 L_{\text{Cow}}$ is reached and then spirals. The total change in the luminosity from the Cowling model to the model with the maximum isothermal core is only 1 mag., instead of 5 mag. as in the Gamow theory.

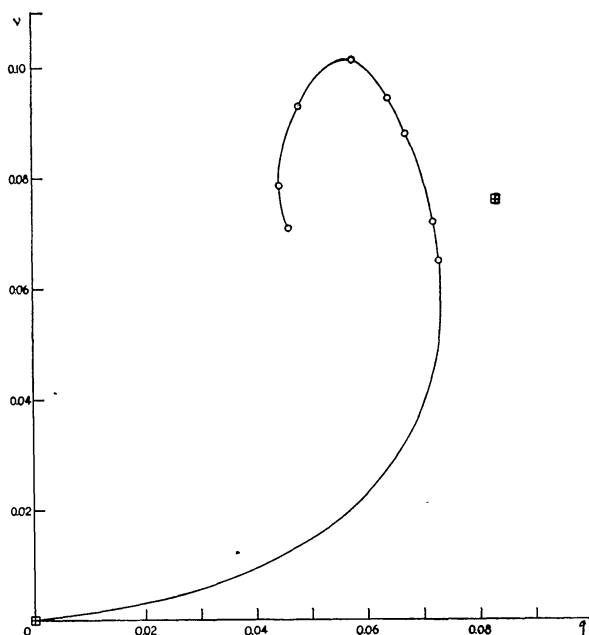


FIG. 1

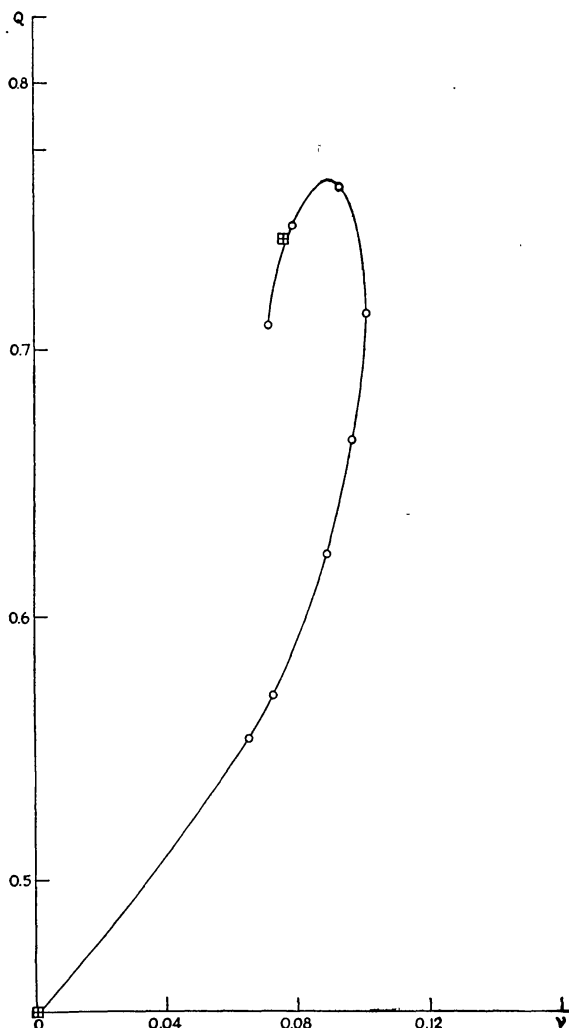


FIG. 2

f) The effective temperature remains practically constant as L grows, then decreases and starts spiraling.

g) The thickness of the energy-generating shell varies very slightly.

h) The central density increases continually along the spiral in Figure 1. The ratio of the central density of the isothermal core with maximum ν to that of the Cowling model is ~ 22 .

7. *Applications to stellar evolution.*—Now we can give a more detailed picture of the evolution of the main-sequence stars. The hydrogen combustion in the center may have either of the following effects: formation of an isothermal core at the center or shrinkage of the convective core. Which of the effects takes place depends on the rapidity of mix-

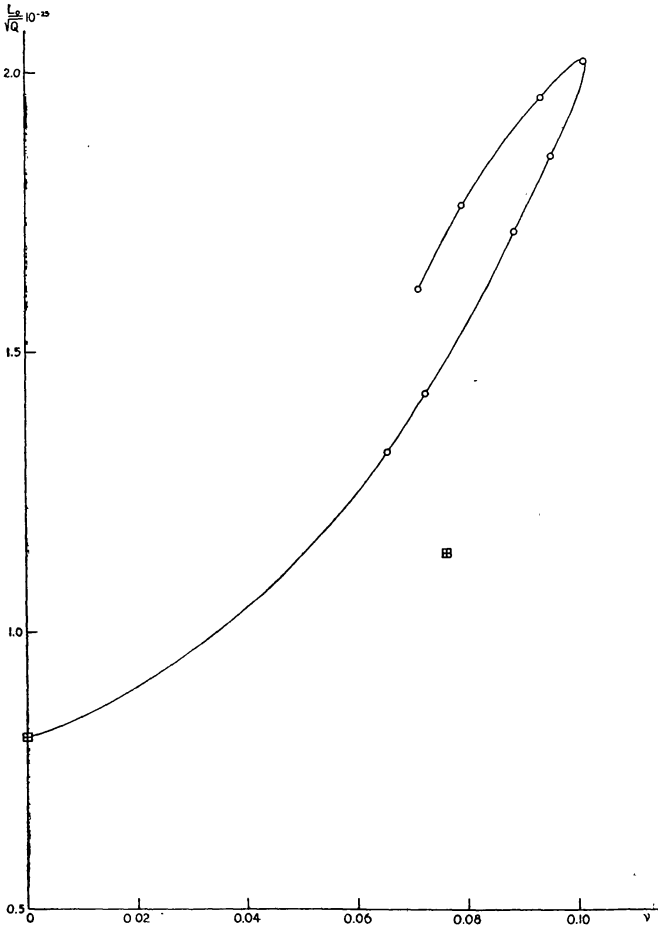


FIG. 3

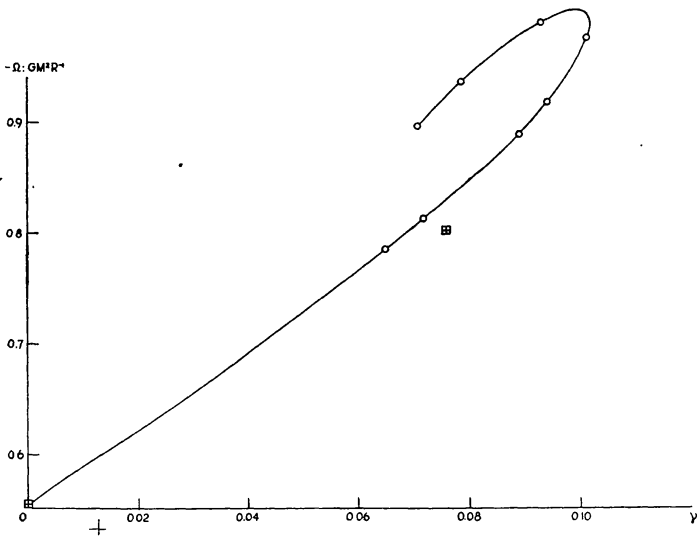


FIG. 4

ing. The superadiabatic gradient being larger for very bright stars, the mixing will proceed faster and the convective core will shrink; in stars of low luminosity an isothermal core may be formed.

The formation of an isothermal core surrounded by a convective shell and a radiative envelope seems, at first sight, to present difficulties, because at the interface of the convective and isothermal regions the polytropic indices would be different: ∞ and 1.5. On the other hand, since the energy-producing layers would lie at the boundary of the isothermal core, it is reasonable to suppose that there should be convective instability at the outer boundary of those layers and stability at the inner one. It should be remembered, in this connection, that the nonexistence of sources of energy in the isothermal core would make convection impossible, since there would be no driving mechanism. Consequently, we need not require the

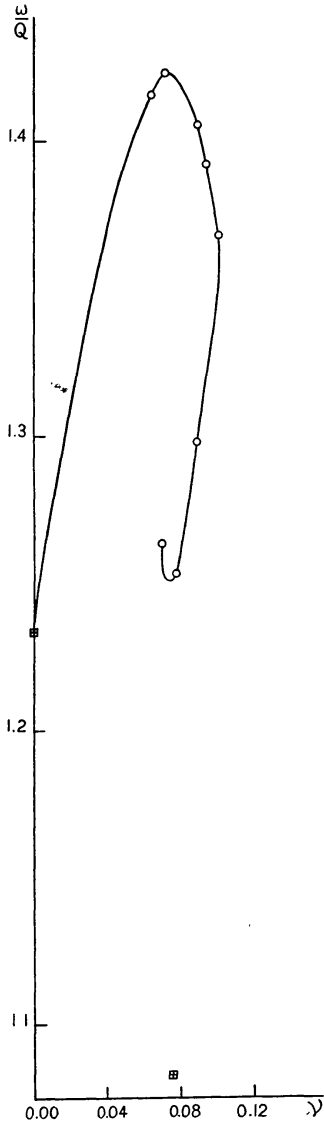


FIG. 5

convective and isothermal regions the polytropic indices would be different: ∞ and 1.5. On the other hand, since the energy-producing layers would lie at the boundary of the isothermal core, it is reasonable to suppose that there should be convective instability at the outer boundary of those layers and stability at the inner one. It should be remembered, in this connection, that the nonexistence of sources of energy in the isothermal core would make convection impossible, since there would be no driving mechanism. Consequently, we need not require the

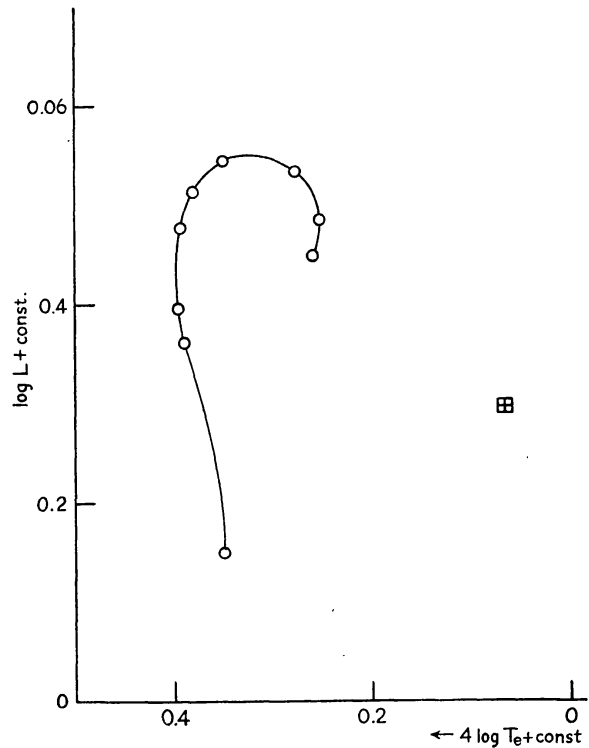


FIG. 6

equality of polytropic indices at the interface of the convective and isothermal regions. In any case it may be expected that, even if there are no equilibrium configurations of the type considered, there may still exist quasi-equilibrium configurations of that type.

The growth of an isothermal core at the center would slow down and finally stop convection when the fraction of the stellar mass inside it exceeded 0.065. The stellar model would then go over into the sequence formed by isothermal cores surrounded by radiative envelopes.

In the case where there is rapid mixing the convective core would start shrinking and

the luminosity would increase somewhat, while the radius would expand rapidly. The star would not, however, expand so much as the results of Table 1 might suggest, for the scarcity of hydrogen would make the central temperature rise. As the central temperature rises, so will the temperature at the outer points, and the energy production will spread to outer regions. The convective currents must stop before the whole hydrogen is burned, and that will be taken care of by the spreading of the energy production and consequent leveling of the temperature gradient. Finally, the star will readjust itself to an isothermal core-radiative envelope model. The later stages of the evolution would be the same as in the case of slow mixing.

It is to be noted that the liberation of the gravitational energy has already started during the time when the nuclear reaction is going on and not merely after the reaction is over, as it is usually supposed. This liberation of gravitational energy is in turn responsible for the increase of the heat content of the star.

So far we have discussed the evolution of a star only during the relatively early stages of the exhaustion of hydrogen in its central regions. The question now arises as to what can be said concerning the evolution during the later stages, i.e., after the isothermal core has grown to include the maximum possible mass. When this stage has been reached, the liberation of energy from the carbon cycle must cease, and we should expect the star to adjust itself to a contractive model ($\epsilon\alpha T$) and evolve according to the Helmholtz-Kelvin time scale. But this gravitational contraction cannot proceed indefinitely, for with continued contraction the temperature at the interface between the central regions exhausted of hydrogen and the outer envelope containing hydrogen will steadily increase. And, when the temperature exceeds a certain value, the nuclear reactions will start, and the carbon cycle will again become operative. At first, the energy liberated by the nuclear processes will be small compared to the gravitational liberation. But very soon, because of the high-temperature sensitiveness of the nuclear reactions, the energy liberation from the carbon cycle will exceed the liberation of energy by the gravitational contraction. In other words, the central regions must again tend to become isothermal. However, no equilibrium configuration is possible under these circumstances: for the isothermal core would have to contain a greater fraction of the total mass than is possible under equilibrium conditions. It therefore appears difficult to escape the conclusion that beyond this point the star must evolve through nonequilibrium configurations. It is difficult to visualize what form these nonequilibrium transformations will take; but, whatever their precise nature, they must depend critically on whether the mass of the star is greater or less than the upper limit M_3 ($= 5.7 \mu^{-2} \odot$) to the mass of degenerate configurations.⁹ For masses less than M_3 the nonequilibrium transformations need not take particularly violent forms, as finite degenerate white-dwarf states exist for these stars. However, when $M > M_3$, the star must eject the excess mass first, before it can evolve through a sequence of composite models consisting of degenerate cores and gaseous envelopes toward the completely degenerate state. Our present conclusions tend to confirm a suggestion made by one of us (S. C.) on different occasions that the supernova phenomenon may result from the inability of a star of mass greater than M_3 to settle down to the final state of complete degeneracy without getting rid of the excess mass.¹⁰

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⁹ *An Introduction to the Study of Stellar Structure*, p. 423, Chicago, 1939.

¹⁰ At a symposium held at the Yerkes Observatory in the fall of 1941, Dr. R. Minkowski stated that the analysis of his spectroscopic observations on the Crab nebula supports this suggestion.