THE PRESSURE IN THE INTERIOR OF A STAR

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ABSTRACT

In this paper certain integral theorems on the equilibrium of a star are proved.

I

In a recent paper[1] the author proved the following theorems:

THEOREM 1.—*In any equilibrium configuration in which the mean density inside* r *decreases outward, we have the inequality*

$$\tfrac{1}{2}G(\tfrac{4}{3}\pi)^{1/3}\bar{\rho}^{4/3}(r)M^{2/3}(r) \leqslant P_c - P \leqslant \tfrac{1}{2}G(\tfrac{4}{3}\pi)^{1/3}\rho_c^{4/3}M^{2/3}(r) \, , \tag{1}$$

where $\bar{\rho}$ (r) *denotes the mean density inside* r, ρ_c *the central density*, P_c *the central pressure, and* M(r) *the mass inclosed inside* r.

THEOREM 2.—*If* $(1 - \beta_c)$ *is the ratio of the radiation pressure to the total pressure at the center of a wholly gaseous configuration, then under the conditions of Theorem 1*

$$1 - \beta_c \leqslant 1 - \beta^* \, , \tag{2}$$

where $(1-\beta^*)$ *satisfies the quartic equation*

$$M = \left(\frac{6}{\pi}\right)^{1/2}\left[\left(\frac{k}{\mu H}\right)^4 \frac{3}{a}\,\frac{1-\beta^*}{\beta^{*4}}\right]^{1/2}\frac{1}{G^{3/2}} \, . \tag{3}$$

THEOREM 3.—*If* I_ν *is the integral defined by*

$$I_\nu = \int_0^R \frac{GM(r)dM(r)}{r^\nu} \, , \qquad (\nu < 6) \, , \tag{4}$$

where R *denotes the radius of the configuration, then*

$$\frac{3}{6-\nu}\,G(\tfrac{4}{3}\pi\bar{\rho})^{\nu/3}M^{(6-\nu)/3} \leqslant I_\nu \leqslant \frac{3}{6-\nu}\,G(\tfrac{4}{3}\pi\rho_c)^{\nu/3}M^{(6-\nu)/3} \, . \tag{5}$$

[1] *M.N.*, **96**, 644, 1936. See also E. A. Milne, *ibid.*, 179.

II

In this paper we shall prove certain additional theorems on the equilibrium of a star.

THEOREM 4.—*Under the conditions of Theorem 1 we have*

$$\frac{P_c}{\rho_c^{\nu/3}} \leqslant \frac{1}{6-\nu} (\tfrac{4}{3}\pi)^{(\nu-3)/3} G R^{\nu-4} M^{(6-\nu)/3} , \tag{6}$$

provided

$$6 > \nu \geqslant 4 , \tag{7}$$

where (6) *is a strict inequality for* $\nu > 4$.

Proof: We have equation 7 (*loc. cit.*)

$$dP = -\frac{G}{4\pi} \frac{M(r)dM(r)}{r^4} . \tag{8}$$

Introducing equation (8) in equation (4), we have

$$I_\nu = -4\pi \int_0^R \frac{dP}{r^{\nu-4}} . \tag{9}$$

Since $\nu \geqslant 4$, we clearly have

$$I_\nu \geqslant -\frac{4\pi}{R^{\nu-4}} \int_0^R dP = \frac{4\pi}{R^{\nu-4}} P_c . \tag{10}$$

In equation (10) we have the equality sign only for the case $\nu = 4$. For $\nu > 4$ we have a strict inequality.

Combining equation (10) with the inequality of Theorem 3, we have

$$\frac{4\pi}{R^{\nu-4}} P_c \leqslant I_\nu \leqslant \frac{3}{6-\nu} G(\tfrac{4}{3}\pi\rho_c)^{\nu/3} M^{(6-\nu)/3} \tag{11}$$

or

$$\frac{P_c}{\rho_c^{\nu/3}} \leqslant \frac{1}{6-\nu} (\tfrac{4}{3}\pi)^{(\nu-3)/3} G R^{\nu-4} M^{(6-\nu)/3} . \tag{12}$$

Again, equation (12) is a strict inequality for $\nu > 4$. This proves the theorem.

III

Comments on Theorem 4.—When $\nu = 4$, equation (12) reduces to an inequality from which Theorem 2 would immediately follow. If we write

$$\nu = 3\left(1 + \frac{1}{n}\right) , \tag{13}$$

then we can re-write equation (12) as

$$\frac{P_c}{\rho_c^{(n+1)/n}} \leqslant S_n G\, R^{(3-n)/n} M^{(n-1)/n}, \qquad (1 < n \leqslant 3) , \tag{14}$$

where S_n stands for the numerical coefficient

$$S_n = (\tfrac{4}{3}\pi)^{1/n} \frac{n}{3(n-1)} . \tag{15}$$

Equations (14) and (15) bring out the very general "critical" nature of $n = 1$ and $n = 3$, a circumstance only very partially disclosed in the theory of polytropes. For, if we consider a polytrope of index n, then of course

$$\frac{P}{\rho^{(n+1)/n}} = \text{Constant} = \frac{P_c}{\rho_c^{(n+1)/n}} . \tag{16}$$

For the polytropic case (16) we have

$$\frac{P_c}{\rho_c^{(n+1)/n}} = T_n G\; R^{(3-n)/n} M^{(n-1)/n} , \tag{17}$$

where

$$T_n = \frac{1}{(n+1)} \left(\frac{4\pi}{\omega_n^{n-1}}\right)^{1/n} , \tag{18}$$

$$\omega_n = -\left(\xi^{\frac{n+1}{n-1}} \frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1} . \tag{19}$$

The symbols ξ and θ stand for the Emden variables, θ_n is *any* Emden solution of index n, and the quantity in brackets in equation (19) has to be taken at the first zero $\xi = \xi_1$ of the Emden solution.[2]

[2] An Emden solution of index n is a function which satisfies Emden's equation of index n and is finite at the origin. Equation (19) can be evaluated at the boundary of any Emden solution because it is homology invariant.

Table I gives the values of S_n and T_n for different values of n.

TABLE I

n	S_n	T_n	S_n/T_n
3.0	0.806	0.364	2.214
2.5	0.985	.351	2.803
2.0	1.364	.365	3.741
1.5	2.599	.424	6.125
1.0	∞	0.637	∞

IV

THEOREM 5.—*If* $I_{\sigma,\nu}$ *stands for the integral*

$$I_{\sigma,\nu} = \int_0^R \frac{GM^\sigma(r)dM(r)}{r^\nu}, \qquad [3(\sigma+1) > \nu], \tag{20}$$

then under the conditions of Theorem 1

$$\frac{3}{3\sigma+3-\nu}\frac{GM^{\sigma+1}}{\xi^\nu} \geqslant I_{\sigma,\nu} \geqslant \frac{3}{3\sigma+3-\nu}\frac{GM^{\sigma+1}}{R^\nu}, \tag{21}$$

where ξ *is defined by the relation*

$$\tfrac{4}{3}\pi\rho_c\xi^3 = M. \tag{22}$$

Proof: Since

$$r^\nu = \left[\frac{M(r)}{\tfrac{4}{3}\pi\bar{\rho}(r)}\right]^{\nu/3}, \tag{23}$$

we have from equation (20) that

$$I_{\sigma,\nu} = G(\tfrac{4}{3}\pi)^{\nu/3}\int_0^R \bar{\rho}^{\nu/3}(r)M^{(3\sigma-\nu)/3}dM(r). \tag{24}$$

Since we have assumed that $\bar{\rho}(r)$ decreases outward, it is clear that the minimum value of equation (24) is obtained by replacing $\bar{\rho}(r)$ by its minimum $\bar{\rho}$ (the mean density for the whole configuration) and taking it out of the integral sign. In the same way the maximum value of equation (24) is obtained by replacing $\bar{\rho}(r)$ by its maximum value ρ_c and taking it out of the integral sign. One thus finds that

$$\left.\begin{aligned} \frac{3}{3\sigma+3-\nu} G(\tfrac{4}{3}\pi\bar{\rho})^{\nu/3} M^{(3\sigma+3-\nu)/3} &\leqslant I_{\sigma,\nu} \\ &\leqslant \frac{3}{3\sigma+3-\nu} G(\tfrac{4}{3}\pi\rho_c)^{\nu/3} \cdot M^{(3\sigma+3-\nu)/3} , \end{aligned}\right\} \quad (25)$$

which is easily seen to be equivalent to equation (21).

V

THEOREM 6.—*If* $\bar{P}_{p,q}$ *is the mean pressure defined by*

$$M^p R^q \bar{P}_{p,q} = \int_0^R P d(M^p(r) r^q) , \qquad (p,\ q > 0) , \qquad (26)$$

then under the conditions of Theorem 1

$$\bar{P}_{p,q} \geqslant \frac{3}{4\pi} \frac{1}{3p+q+2} \frac{GM^2}{R^4} . \qquad (27)$$

Proof: Integrating equation (26) by parts, we have

$$M^p R^q \bar{P}_{p,q} = -\int_0^R M^p(r) r^q dP . \qquad (28)$$

By equation (8) we have

$$M^p R^q \bar{P}_{p,q} = \frac{1}{4\pi}\int_0^R \frac{GM^{p+1}(r) dM(r)}{r^{4-q}} , \qquad (29)$$

$$= \frac{1}{4\pi} I_{p+1,\,4-q} . \qquad (30)$$

Hence, by Theorem 5 we easily find that

$$\bar{P}_{p,q} \geqslant \frac{3}{4\pi} \frac{1}{3p+q+2} \frac{GM^2}{R^4} , \qquad (31)$$

which proves the theorem.

VI

THEOREM 7.—*In a wholly gaseous configuration in which the mean density inside* r, *the temperature, and the ratio of the radiation pressure to the total pressure decrease outward, we have*

$$T_c > \frac{1}{5}\frac{\mu H}{k}\frac{GM}{R}\beta^* , \tag{32}$$

where T_c *is the central temperature,* β^* *has the same meaning as in Theorem 2 and* μ, H, *and* k *are, respectively, the mean molecular weight (assumed constant in the whole configuration), the mass of the proton, and the Boltzmann constant.*

Proof: Let $\bar{P}$ denote the mean pressure defined by

$$R^3\bar{P} = \int_0^R P d(r^3) . \tag{33}$$

By equation (31)

$$\bar{P} \geqslant \frac{3}{20\pi}\frac{GM^2}{R^4} = \frac{1}{5}(\tfrac{4}{3}\pi)^{1/3}G\bar{\rho}^{4/3}M^{2/3} . \tag{34}$$

In a wholly gaseous configuration

$$P = \frac{k}{\mu H}\beta^{-1}\rho T . \tag{35}$$

Hence

$$\bar{P} = \frac{k}{\mu H}\overline{(\beta^{-1}\rho T)} . \tag{36}$$

Since $(1 - \beta)$ is assumed to decrease outward, β must increase outward, and hence

$$\bar{P} < \frac{k}{\mu H}\beta_c^{-1}T_c\bar{\rho} , \tag{37}$$

where $\bar{\rho}$ is now the mean density defined in the usual way, since the means we are now taking are weighted according to the volume element; compare equation (33). Hence

$$T_c > \frac{1}{5}\frac{\mu H}{k}G(\tfrac{4}{3}\pi\bar{\rho})^{1/3}M^{2/3}\beta_c , \tag{38}$$

or

$$T_c > \tfrac{1}{5} \frac{\mu H}{k} \frac{GM}{R} \beta_c . \tag{39}$$

But by Theorem 2

$$\beta_c \geqslant \beta^* , \tag{40}$$

where β^* satisfies the quartic equation (3). Hence, combining equations (39) and (40), we have

$$T_c > \tfrac{1}{5} \frac{\mu H}{k} \frac{GM}{R} \beta^* , \tag{41}$$

which proves the theorem.

The inequality equation (41) is not a "best possible" one, but it has the advantage of not neglecting the radiation pressure and is, in fact, the first of the kind to be established.

We notice that the *mean* temperature $\bar{T}$, defined by

$$M\bar{T} = \int_0^R T dM(r) ,$$

satisfies the same inequality as T_c. For

$$M\bar{T} = \int_0^R T dM(r) = \frac{\mu H}{k} \int_0^R \beta P \rho^{-1} dM(r) .$$

$$= \frac{\mu H}{k} \int_0^R \beta P dv ,$$

where dv is the volume element. But since β is assumed to increase outward,

$$M\bar{T} \geqslant \frac{\mu H}{k} \beta_c \int_0^R P dv = \tfrac{1}{3} \frac{\mu H}{k} \beta_c \Omega ,$$

where Ω is the negative potential energy of the configuration. But by Theorem 3,

$$\Omega = I_1 \geqslant \tfrac{3}{5} \frac{GM^2}{R} .$$

Hence

$$\bar{T} \geqslant \frac{1}{5}\frac{\mu H}{k}\frac{GM}{R}\beta_c \geqslant \frac{1}{5}\frac{\mu H}{k}\frac{GM}{R}\beta^* .$$

VII

Corollary to Theorem 7.—In a wholly gaseous configuration in which the mean density inside r *and the ratio of the radiation pressure to the total pressure decreases outward, we have*

$$M < \left(2.5\frac{\overline{(\rho^{4/3})}}{(\bar{\rho})^{4/3}}\right)^{3/2}\left(\frac{6}{\pi}\right)^{1/2}\left[\left(\frac{k}{\mu H}\right)^4\frac{3}{a}\frac{1-\beta_c}{\beta_c^4}\right]^{1/2}\frac{1}{G^{3/2}} . \tag{42}$$

Proof: Equation (35) can also be written

$$P = \left[\left(\frac{k}{\mu H}\right)^4\frac{3}{a}\frac{1-\beta}{\beta^4}\right]^{1/3}\rho^{4/3} , \tag{43}$$

so that

$$\bar{P} = \overline{\left[\left(\frac{k}{\mu H}\right)^4\frac{3}{a}\frac{1-\beta}{\beta^4}\right]^{1/3}\rho^{4/3}} , \tag{44}$$

or, since $(1 - \beta)$ decreases outward,

$$\bar{P} < \left[\left(\frac{k}{\mu H}\right)^4\frac{3}{a}\frac{1-\beta_c}{\beta_c^4}\right]^{1/3}\overline{(\rho^{4/3})} . \tag{45}$$

Combining equations (45) and (34), we have

$$\left[\left(\frac{k}{\mu H}\right)^4\frac{3}{a}\frac{1-\beta_c}{\beta_c^4}\right]^{1/3}\overline{(\rho^{4/3})} > \frac{1}{5}\left(\frac{4}{3}\pi\right)^{1/3}G\bar{\rho}^{4/3}M^{2/3} , \tag{46}$$

which, after some minor transformation, goes over into equation (42).

It may be noticed that in equation (42) we cannot replace $\overline{(\rho^{4/3})}/(\bar{\rho})^{4/3}$ by unity, since

$$\overline{(\rho^{4/3})} > (\bar{\rho})^{4/3} . \tag{47}$$

Equation (42) is not a "best possible" inequality, but it is much "sharper" than the inequality established previously.[3]

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[3] Equation (19′) (*loc. cit.*).