NOTE CONCERNING MILBURN'S NEBULA.

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ON THE HYPOTHESIS OF THE RADIAL EJECTION OF HIGH-SPEED ATOMS FOR THE WOLF-RAYET STARS AND THE NOVÆ.

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§ 1. *Introduction.*—In all recent discussions concerning the nature of the Wolf-Rayet emission a description of the observed phenomena has been attempted on the basis of a hypothesis which postulates a continual ejection of high-speed atoms by the parent star. These atoms, which are supposed to be ejected radially, are assumed to have maximum velocities ranging from 500–3000 km./sec. in the different stars. Further, it is assumed that these ejected atoms form an extensive envelope and that the broad emission bands which are characteristic of the Wolf-Rayet spectra (and also of the Novæ Spectra in their later stages of development) originate here. Though this hypothesis has been discussed in a general way by a number of investigators, and particularly Beals, yet so far no very serious attempt has been made to

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* *M.N.*, **94**, 229–230, 1934.   † *Russ. A.J.*, **11**, No. 1, p. 40, 1934.

‡ *Ibid.*, **8**, Nos. 3–4, p. 207, 1931.   § *Publ. Lick Obs.*, **18**, 1932.

‖ *Publ. A.S.P.*, **37**, 225, 1925.   ¶ *Ibid.*, **33**, 175, 1921.

deduce all the consequences of the underlying hypothesis. But to find out whether this hypothesis is at all an appropriate one for the Wolf-Rayet stars we should really develop a dynamical theory of ejection and from the deduced velocity and density variations with the distance r derive the contours of the emission bands for comparison with observation. A dynamical theory of the ejection process itself is necessary only in so far as we require the emission per unit volume and the radial velocity as functions of r, and consequently even without any underlying dynamical theory we can formally examine the type of band contours that could be expected from assumed hypothetical laws of velocity variation, and from a comparison with the observed contours infer something about the actual conditions obtaining in the stellar atmospheres where such emission bands originate. Thus, as was first pointed out by Beals, an atmosphere consisting of atoms moving out radially with uniform velocities from a parent star of relatively small dimensions should produce a flat-topped contour with discontinuously dropping sides, and Beals argues from this that the existence even of a single flat-topped contour would support the hypothesis of the radial ejection of atoms. This question of the contours of emission bands has recently been discussed in some detail by Professor Gerasimovič * in a paper of great interest. Gerasimovič finds, for instance, that in Nova Aquilæ four days after the maximum the hydrogen shell should have been moving with a velocity varying as $r^{-1/4}$.† (We give a different interpretation of Gerasimovič's result in § 6.)

Now the hypothesis of the radial ejection of atoms is a very stringent one —very stringent in the sense that we cannot make this hypothesis and make others as well—and as this does not appear to be generally recognized, we shall attempt to develop some of the consequences of the underlying hypothesis, starting with a discussion of the dynamics of the ejection process itself.

§ 2. *The Possible Ways in which the Radial Ejection of Atoms could take Place.*—If one postulates that the parent star is continually ejecting atoms then from a dynamical point of view there are not many possibilities of the ways in which this could happen. The ejection process could, in fact, take place in one of two ways :—

(A) At the boundary of the star the atoms (presumably only those with a relatively small but finite outward velocities) are "repelled" by some kind of force which is, say, f times the gravitational attraction. Unless f is very nearly unity we could reasonably assume that f is a constant, *i.e.* the repulsive force—whatever its nature—falls off like gravity inversely as the square of the distance.

As we shall show in the sequel this hypothesis includes, as a special case, the emission of particles arising from unbalanced radiation pressure, which possibility was first pointed out by Milne ‡ in a well-known paper.

(B) The atom at the boundary of a star might receive a large initial

* *Zeits. für Astrophysik*, **7**, 335, 1933.

† Actually Gerasimovič gives $r^{-1/2}$, but this is due to a slight numerical error. His value of a is $(1/3 \cdot 83)$ and not $(1/2 \cdot 17)$ as he gives.

‡ *M.N.*, **86**, 459, 1926.

outward velocity (either in a single process or in stages), and in escaping from the star be continually de-accelerated in the gravitational field of the star, the atom either escaping from the star with a finite outward velocity, or after ascending a certain distance begin to fall back towards the parent star. We could have an atmosphere of high-speed particles set up in this way.

This underlying mechanism for the ejection process includes the "explosion hypothesis," which (even in its extreme form) appears to be favoured by some investigators.* It has a definite resemblance, for instance, with a point of view recently advocated by Zanstra † in a rather different connection. But a much more satisfactory explanation of an ejection process of this character is, as was kindly pointed out to the writer by Professor H. H. Plaskett, that as the Wolf-Rayet stars (and the Novæ stars after the initial outburst) are rich in ultra-violet light, the atoms near the boundary would receive large outward momenta in the process of getting ionized—the emission lines themselves originating from a recombination process, as in Zanstra's theory of nebular luminosity. Indeed, as Professor Plaskett points out, this kind of mechanism is the simplest one that can underlie a theory which postulates a radial ejection of atoms and also applies Zanstra's method to determine the temperature of the parent star. However, from a consideration of the high-lying metastable states, Plaskett concludes that both the mechanisms A and B might be of importance in discussing the nature of the Wolf-Rayet emission if the radial ejection hypothesis should finally prove satisfactory.

The main point is that both the mechanisms A and B for the ejection process lead to perfectly definite predictions for the types of emission band contours that should be obtained, and the comparison of these predicted contours with the observed ones should settle the tenability or otherwise of the radial ejection hypothesis.

Before proceeding to the calculations one point of general interest might be mentioned. In all the recent discussions (including Gerasimovič's) the occultation effect has been tacitly neglected, but we shall see that, as a consequence of this effect, very essential asymmetries in the band contours are produced. The reason is that any dynamical theory for the ejection of the atoms would not provide stellar atmospheres so extensive as to justify our regarding the parent star as of "negligible" dimensions.

I. *The Contours of Emission Bands on the Mechanism A for the Ejection Process*

§ 3. By hypothesis, an atom at a distance r from the centre experiences an outward acceleration specified by

$$\frac{d^2r}{dt^2} = (f - 1)g\frac{a^2}{r^2}, \tag{1}$$

where a is the radius of the star and g the value of gravity at the boundary. From (1) it follows that

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* *Cf.* J. Genard, *M.N.*, **92**, 400, 1932.   † H. Zanstra, *M.N.*, **93**, 131, 1932.

$$\left(\frac{dr}{dt}\right)^2 = \text{constant} - 2(f-1)g\frac{a^2}{r}.$$ (2)

We now assume that the atom starts with a zero velocity at the boundary of the star.   This is admittedly an approximation, but since the atom later acquires large macroscopic velocities of the order of 1000 km./sec. we can reasonably neglect the small initial velocities the atoms might possess. From (2) we then have

$$\frac{dr}{dt} = \sqrt{2(f-1)ag\left(1 - \frac{a}{r}\right)}.$$ (3)

The limiting velocity with which the atom finally leaves the star is

$$v_\infty = \sqrt{2(f-1)ag}.$$ (3′)

We can therefore rewrite (3) in the form

$$v = \sqrt{1 - r^{-1}},$$ (4)

where we now measure the distances and velocities in scales where the radius of the star and the final limiting velocity v_∞ are respectively units. We shall adopt this convention consistently in Part I of this paper.

§ 4. Now, in Milne's original theory of the emission of high-speed particles the atom is at first supposed to be in equilibrium with the flux of the emergent radiation corresponding to the residual intensity in the absorption line, and an accidental outward velocity (large in this connection, but negligible in comparison with the velocity it subsequently acquires) will make it absorb in the wing of the line, where since it is now exposed to a greater flux of radiation it begins to be accelerated till it finally "climbs" out of the absorption line, being accelerated all the time and ending with a large limiting velocity.   It is, however, clear that unless the width of the absorption line is unduly large the atom would have the same limiting velocity as specified by (3′), provided we substitute for f the value

$$f = I_1/I_0,$$ (5)

where I_1 is the intensity of the emergent radiation adjacent to the line and I_0 the residual intensity in the line.   We verify that with this value of f in (3′) we have recovered Milne's original result.   But since Milne assumed that I_1 was constant outside the line it would appear that this restriction is necessary for our result to be valid.   It can, however, be proved that for the expulsion of atoms by selective radiation pressure the result (4) is valid provided (i) f at (or near) the boundary of the star differs from unity, (ii) the maximum velocity acquired is such that $vf/(f-1)c$ (c = velocity of light) is small compared with unity, and (iii) the initial velocity is very small relative to the order of the velocity it subsequently acquires.

§ 5. We therefore proceed to calculate the emission band contours for the velocity law specified by (4).   Write for the radial velocity u :

$$u = \sqrt{1 - r^{-1}} \cos\theta = v\cos\theta.$$ (6)

We have to calculate then the intensity in the emission band $i(u)du$ for a frequency interval $d\nu$ corresponding to a velocity interval du where

$$d\nu = \frac{\nu_0 du}{c}. \tag{6'}$$

In (6') ν_0 is the frequency of the line emitted by an atom stationary with respect to the observer.

Let $i(r)$ be the emission per unit volume which clearly is a function only of r. Then

$$i(u)du = 2\pi \iint i(r)r^2 \sin\theta \, dr \, d\theta, \tag{7}$$

$$(u = \text{constant}),$$

the integration being carried out over such volume elements that u lies between the interval u, $u + du$ (*i.e.*) along the curves of constant u (see fig. 1).
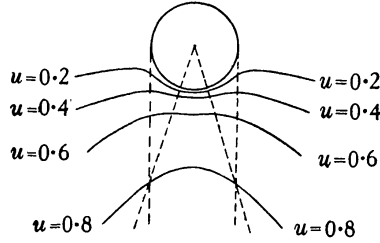

FIG. 1.—*Curves of Constant u.*

Then

$$i(u) = 2\pi \int_{\theta_1}^{\theta_2} \left[i(r)r^2 \frac{\partial r}{\partial u} \right] \sin\theta \, d\theta, \tag{8}$$

where the quantity in the brackets [ ] has to be expressed in terms of θ only by eliminating r by (6), and θ_1 and θ_2 are the appropriate limits. The question of the limits is a rather delicate matter, and we shall consider this in due course. By (6) we can rewrite (8) as

$$i(u) = \frac{4\pi}{u} \int_{\theta_1}^{\theta_2} \left[i(r)r^4 \left(1 - \frac{1}{r} \right) \right] \sin\theta \, d\theta. \tag{9}$$

To evaluate the integral we need $i(r)$ explicitly. Now, in a steady state the law (4) corresponds to a density law

$$\rho \propto \frac{1}{r^2 \sqrt{1 - r^{-1}}}. \tag{10}$$

The singularity at $r = 1$ arises on account of the approximation that the velocity is zero at $r = 1$. Actually, however, v, though small compared with unity, must be finite. We now assume that

$$i(r) \propto \rho^2 v^\beta, \tag{11}$$

v being specified by (4). Gerasimovič, in his paper already referred to, assumes the law $i(r) \propto \rho^2$, but this law does not appreciably differ from (11)

for large values of r.   The law (11) is perhaps not so artificial as it looks,* but there can be no harm at any rate in formally investigating the consequences which happen to be of some interest.   By (11) we then have (on a suitable scale)

$$i(r) = r^{-4}v^{\beta-2}. \tag{12}\dagger$$

Introducing (12) in (9) and eliminating r we have

$$i(u) = 4\pi u^{\beta-1} \int_{\theta_1}^{\theta_2} \sin\theta \sec^\beta \theta \, d\theta, \tag{13}$$

or

$$i(u) = \begin{cases} \dfrac{4\pi}{\beta-1} u^{\beta-1} \left[\sec^{\beta-1}\theta \right]_{\theta_1}^{\theta_2}, & \beta \neq 1, \\[2mm] 4\pi \left[\log\sec\theta \right]_{\theta_1}^{\theta_2}, & \beta = 1. \end{cases} \tag{14}$$

§ 6. *The Band Contour on the Violet Side of the Normal Frequency v_0.*—We now need to consider only the curves of constant u in the hemisphere presented to the observer.   Let $r = r_1 (\geqslant 1)$ be the inner boundary of the envelope responsible for the emission bands.   At $r = r_1$,

$$v = \sqrt{1 - r_1^{-1}} = v_1 \text{ (say).} \tag{15}$$

Now if $u \leqslant v_1$, obviously

$$\cos^{-1} u/v_1 \leqslant \theta \leqslant \cos^{-1} u, \tag{16}$$

which specifies the limits.   By (14) then

$$i(u) = \begin{cases} \dfrac{4\pi}{\beta-1}(1 - v_1^{\beta-1}), & \beta \neq 1, \\[2mm] -4\pi \log v_1, & \beta = 1. \end{cases} \tag{17}$$

On the other hand, if $u > v_1$ then the limits are

$$\theta_2 = \cos^{-1} u, \qquad \theta_1 = 0. \tag{18}$$

By (14) now

$$i(u) = \begin{cases} \dfrac{4\pi}{\beta-1}(1 - u^{\beta-1}), & \beta \neq 1, \\[2mm] -4\pi \log u, & \beta = 1. \end{cases} \tag{19}$$

At $u = v_1$, (19) gives $i(u)$ identical with (17).   Hence on the violet part of the band (by which we shall mean the part of the band to the violet of the

* Professor H. H. Plaskett has suggested to the writer that the law (11) could perhaps be justified physically by supposing that the mechanism of support was by resonance lines in the violet of the ionized atom, the visual luminosity arising from the recombination of the ionized atom and the free electrons giving neutral lines with an intensity distribution roughly as the density squared.

† If $\beta \geqslant 2$ there is no singularity for $i(r)$ at $r = 1$.

normal frequency ν_0) $i(u)$ is constant for $u \leqslant v_1$, and for $1 \geqslant u > v_1$ we have a sloping side. On a certain arbitrary scale of intensities we can write for the sloping sides

$$i(u) = \begin{cases} (1 - u^{\beta-1}), & \beta > 1, \\ -\log u, & \beta = 1, \\ (u^{\beta-1} - 1), & \beta < 1. \end{cases} \qquad (20)$$

If $v_1 = 1$ then (20) defines the contour on the violet side for all values of u. We have then "peaked" or "rounded" types of contours.

It is remarkable that the families of contours predicted on this model are identical with the families obtained by Gerasimovič (*loc. cit.*) on the basis of quite different laws for velocity variation.

In particular, Gerasimovič found that for Nova Aquilæ four days after the maximum the contour could be approximately represented by

$$i(u) = 1 - u. \qquad (21)^*$$

By (20) this means on our model that $\beta = 2$. Thus to explain a contour of this character we have the possibilities :

	v	$i(r) \propto$
Gerasimovič Mechanism (A)	$r^{-1/4}$ $\sqrt{1 - r^{-1}}$	$\rho^2 \propto r^{-7/2}$ $\rho^2 v^2 \propto r^{-4}$

From a physical point of view there is clearly nothing to choose between the two laws for $i(r)$, but if the expulsion of atoms did take place by an unbalanced radiation pressure (as was first suggested by Milne) then the second possibility is to be preferred.

§ 7. *The Band Contour on the Red Side of the Normal Frequency ν_0. The Occultation Effect.*—Now, the emission in a region to the red of the undisplaced frequency ν_0 is contributed by the hemisphere on the antipodal side of the star. First we will consider, for the sake of simplicity, the case where the shell begins from $r = 1$. The problem is to find the appropriate limits of θ for a given value of u (*i.e.* at a frequency displaced by an amount $u\nu_0/c$ to the red of ν_0) for substitution in (14). Because of the occultation effect we are concerned only with those parts of the curves $u = $ constant which are to the right of the tangent at $\theta = \pi/2$ (see fig. 1). The upper limit of θ is clearly the same as before :

$$\theta_2 = \cos^{-1} u,$$

and the lower limit is determined by the intersection of the tangent at $\theta = \pi/2$ with the curve $u = $ constant. At the point of intersection

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* *Note added April* 29.—Professor Gerasimovič has kindly pointed out to the writer that the equation (21) for the contour should be regarded as only tentative, and that on the law $u = r^\alpha \cos \theta$ the observations can be taken as merely indicating a negative value for the exponent α.

$$\cos\theta = \frac{\sqrt{r^2 - 1}}{r} = \frac{u}{\sqrt{1 - r^{-1}}}, \qquad (22)$$

or

$$u = \left(1 - \frac{1}{r}\right)\sqrt{1 + \frac{1}{r}}. \qquad (23)$$

For a given value of u we have to solve (23) for r. If we denote the solution by r_u then

$$\cos\theta_u = \sqrt{1 - \frac{1}{r_u^2}}. \qquad (24)$$

By (14) then

$$i(u) = \begin{cases} \dfrac{4\pi}{\beta - 1}(1 - u^{\beta-1}\sec^{\beta-1}\theta_u), & \beta \neq 1, \\[2ex] -4\pi \log u \sec\theta_u, & \beta = 1, \end{cases} \qquad (25)$$

or by (22)

$$i(u) = \begin{cases} \dfrac{4\pi}{\beta - 1}\left[1 - (1 - r_u^{-1})^{\frac{\beta-1}{2}}\right], & \beta \neq 1, \\[2ex] -2\pi \log(1 - r_u^{-1}), & \beta = 1. \end{cases} \qquad (25')$$

A more convenient method of constructing the line contours is to calculate u from (23) for different values of r and tabulate $u\sec\theta_u(=\sqrt{1 - r^{-1}})$ against u. Knowing the corresponding values of u and $u\sec\theta_u$ we can now calculate $i(u)$ for different values of u by (25).

In Table I some pairs of values of u, $u\sec\theta_u$ are given (the last column refers to the following section).

TABLE I

r	u, (v_2)	$u\sec\theta_u$, (v_1)	v_2/v_1
1·0	0·0	0·0	0·0
1·1	0·126	0·302	0·417
1·2	·226	·408	·553
1·25	·268	·447	·600
1·5	·430	·578	·745
2	·612	·707	·866
3	·770	·816	·942
4	·839	·866	·968
5	·876	·894	·977
6	·900	·913	·986
∞	1·	1·	1·

Using the results of Table I, the band contours for $\beta = 2, 3, 4$ were constructed and are illustrated graphically in fig. 2. There is a distinct asymmetry between the violet and the red parts of the contour. In fact,

if $\beta = 2$ the tangent to the line contour on the red side of the band is *vertical* at $u = 0$. The following analytical proof of this may be of interest:

If we write $r = 1 + \epsilon$ and treat ϵ as small then by (23)

$$u = \sqrt{2} \cdot \epsilon ; \qquad u \sec \theta_u = \sqrt{\epsilon},$$

or

$$u \sec \theta_u = \sqrt{\frac{u}{\sqrt{2}}}, \tag{26}$$

so that while on the violet side of ν_0

$$i(u) \sim (1 - u^{\beta - 1}), \qquad (\beta > 1), \tag{27}$$

on the red side we have

$$i(u) \sim \left[1 - (\sqrt{\tfrac{1}{2}} \cdot u)^{\frac{\beta - 1}{2}} \right] \quad \text{as} \quad u \to 0. \tag{28}$$

From (28) we deduce that if $\beta = 2$, the tangent to the contour is vertical at $u = 0$. Also, if $\beta = 3$ the tangent to the contour on the red side makes an
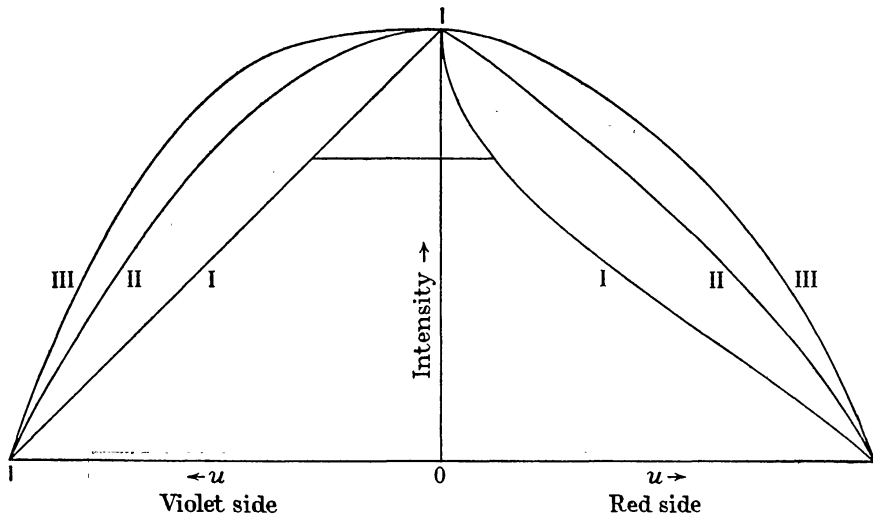

Fig. 2.—$I \to \beta = 2$; $II \to \beta = 3$; $III \to \beta = 4$.

angle $\theta = \tan^{-1} \sqrt{2} (= 54° \ 45')$ with the vertical. For $\beta > 3$ the tangents to the contour on both sides of ν_0 are horizontal.* In all cases, however, the asymmetry between the violet and the red parts of the contour consists mainly in the violet part being brighter than the corresponding red part. Gerasimovič mentions a case where the observations indicate just this kind of asymmetry, but it will be of interest to know if this asymmetry is of a general kind.

§ 8. *The Band Contour on the Red Side of* ν_0 (*continued*).—We will now consider the case where the shell begins at $r = r_1$. Then, as we have already shown, on the violet side of ν_0, $i(u)$ is constant for $u \leqslant v_1$ where v_1 is defined

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* If $\beta > 2$, $i(r)$ defined by (12) takes the value of zero both when $r = 1$ and $r = \infty$. Hence in its *emission properties* the envelope is of the "*planetary-nebula*" type if the contour is of the "rounded type"—this, of course, on the hypothesis that ejection takes place according to mechanism A.

by (15). On the red part of the band there is also a flat part but *not* to an equal extent. For $u < v_1$ the lower limit θ_1 is defined by $\theta_1 = \cos^{-1}(u/v_1)$ or θ_u (as given by equations (22) and (23)), whichever is the *greater*. Clearly, then, if v_2 be such that

$$v_1 = v_2 \sec \theta_{v_2}, \tag{29}$$

then for $u \leqslant v_2$ the contour is flat and has the same intensity as on the violet part of the band. In the last column of Table I the ratio of the extents to which the contour is flat on the red side and on the violet side of v_0 (when on the violet side it is flat to an extent $v_1/2$ of the *whole* width of the emission band) for different values of v_1 is tabulated. We see that this fraction is always less than unity. It should be finally mentioned that the ratios given in the last column of Table I are a direct consequence of the geometry and the dynamics of the problem, and is *independent* of our assumption regarding $i(r)$. Hence exact measures on line contours by determining this ratio should settle whether the radial ejection process takes place according to mechanism A or not.

II. *The Contours of Emission Bands on the Mechanism B for the Ejection Process*

§ 9. On this mechanism, according to H. H. Plaskett, the atom receives a large momentum mainly in the forward direction and is subsequently de-accelerated. Consistent with this hypothesis we should really allow some (perhaps considerable) spread in velocities, but to start with we shall suppose that all the atoms receive the same outward velocity v_0, at the boundary of the star. Let us now take as our unit of velocity the parabolic velocity of escape specified by

$$V_\infty = \sqrt{2ag}, \tag{30}$$

where, as before, a is the radius of the star and g the value of gravity at the boundary. Measuring, as before, distances in a scale where the radius of the star is unity we easily find that

$$v = \sqrt{v_0^2 + \left(\frac{1}{r} - 1\right)}. \tag{31}$$

From (31) it is clear that we should consider two distinct cases: $v_0 \leqslant 1$ and $v_0 > 1$. If $0 < v_0 < 1$ we can find $r = r_1 \,(\,> 1)$ such that

$$v_0 = \sqrt{1 - \frac{1}{r_1}}. \tag{32}$$

Then by (31), at $r = r_1$, $v = 0$ and the envelope extends from $r = 1$ to $r = r_1$. On the other hand, if $v_0 \geqslant 1$ the atom goes off to infinity with a limiting velocity

$$v_\infty = \sqrt{v_0^2 - 1}, \tag{33}$$

and the atmosphere extends to infinity.   The law (31) defines by the equation of continuity the density law

$$\rho \propto \frac{1}{r^2 \sqrt{v_0^2 + \left(\dfrac{1}{r} - 1\right)}}. \tag{34}$$

When $v_0 < 1$, then at $r = r_1$ we have a singularity, and to get over this singularity we assume, as before, that (on a certain scale)

$$i(r) = \rho^2 v^\beta. \tag{35}$$

If $v_0 > 1$ the law (35) does not appreciably differ from $i(r) \propto \rho^2$, which law is what we should expect if the emission is due to a recombination process, as in Zanstra's theory.   Further, if $v_0 < 1$, then the envelope extends to $r = r_1$ (defined in (32)), and hence $i(r)$ must tend to zero as $r \to r_1$ and a term of the form $v^\beta (\beta > 2)$ is necessary.   Proceeding as in Part I we find that

$$i(u) = \begin{cases} \dfrac{4\pi}{\beta - 1} u^{\beta-1} \left[\sec^{\beta-1} \theta \right]_{\theta_1}^{\theta_2}, & \beta \neq 1, \\[2ex] 4\pi \left[\log \sec \theta \right]_{\theta_1}^{\theta_2}, & \beta = 1. \end{cases} \tag{36}$$

Where θ_1 and θ_2 are the appropriate limits of θ on the curves $u = \text{constant}$. (See fig. 3.)

§ 10. *The Band Contour on the Violet Side of v_0.*—We have to consider two cases.

Case I $(v_0 \leqslant 1)$.—Naturally we are only concerned with values of u less than v_0.   The appropriate limits of θ are

$$\theta_2 = \cos^{-1} (u/v_0) ; \qquad \theta_1 = 0, \tag{37}$$

and by (36)

$$i(u) = \begin{cases} \dfrac{4\pi}{\beta - 1} (v_0^{\beta-1} - u^{\beta-1}), & \beta \neq 1, \\[2ex] 4\pi \log (v_0/u), & \beta = 1. \end{cases} \tag{38}$$

We have then a peaked or rounded contour on the violet side with an intensity increasing from 0 at $u = v_0$ to a maximum at $u = 0$.

Case II $(v_0 > 1)$.—If $u < v_\infty (cf.$ equation (33)) then

$$\theta_2 = \cos^{-1} (u/v_0) ; \qquad \theta_1 = \cos^{-1} (u/v_\infty), \tag{39}$$

and by (36)

$$i(u) = \begin{cases} \dfrac{4\pi}{\beta - 1} (v_0^{\beta-1} - v_\infty^{\beta-1}), & \beta \neq 1, \\[2ex] 4\pi \log (v_0/v_\infty), & \beta = 1. \end{cases} \tag{40}$$

Hence this part of the contour is flat.   On the other hand, if $v_\infty \leqslant u \leqslant v_0$ the limits are the same as (37) and the equation of the contour is specified by (38).   Hence to the violet of the normal frequency v_0 the contours have flat tops and sloping sides which are identical with those obtained on mechanism

A.   Here the existence of a flat part means that the atoms receive initial velocities greater than the velocity of escape.   If the initial velocity is less than v_0 then the contour is necessarily of a peaked or rounded character.

§ 11. *The Occultation Effect.*—As before, when we are considering the contour on the red side of v_0 we must take into account the occultation effect, and we shall show presently that this effect produces an asymmetry in the line contour of an entirely different kind than is predicted on mechanism A.   Now the main problem is, as before, to find the appropriate parts of the curves of constant u which contribute to the emission.   From fig. 3 we see that only curves with u less than a certain critical value have intersections with the tangent at $\theta = \pi/2$, which means that to the red

FIG. 3.—*Curves of Constant u (=0·1, 0·2, 0·3, 0·4) v_0 =0·5.*

of v_0 the intensity falls to zero at a frequency corresponding to a value of u less than v_0, at which place the intensity falls to zero on the violet side. This critical value of $u = u_{\mathrm{red}}$ (say) at which the intensity is zero on the red side of v_0 can be calculated quite easily as follows :—
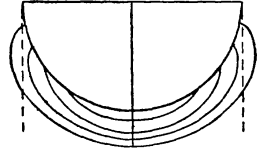
Since at the point of intersection (assuming one exists)

$$\cos \theta = \sqrt{1 - \frac{1}{r^2}} = \frac{u}{\sqrt{(v_0{}^2 - 1) + \dfrac{1}{r}}}, \tag{41}$$

we have

$$u = \sqrt{\left(1 - \frac{1}{r^2}\right)\left(v_0{}^2 - 1 + \frac{1}{r}\right)}, \tag{42}$$

u defined by (42) has a maximum, and this maximum is clearly equal to u_{red}. Now (42) is found to have a maximum when

$$r = (v_0{}^2 - 1) + \sqrt{(v_0{}^2 - 1)^2 + 3}. \tag{43}$$

This value of r substituted in (42) gives us u_{red}.   Having calculated u_{red} we can compute the ratio of the band widths to the red and to the violet of the normal frequency as a function of v_0.   In Table II the results of such calculations are summarised.

Hence, if the ejection process in the Wolf-Rayet stars follows mechanism B then this asymmetry should exist, and we can from a measure of the band contour determine v_0 *in terms of the velocity of escape* (since the ratio of the band widths on the red and on the violet sides of the normal frequency v_0 is by Table II a function of v_0 expressed in units of the velocity of escape), and knowing also its *absolute value* (determined from the band width on the violet side of v_0) we can determine an observational value of $\sqrt{ag}$, and if, in addition, we know also the mass of the star, we can deduce the value of g.   It has to be emphasised in this connection that the result of Table II is a consequence purely of the geometry and the dynamics of the problem, and is independent of our assumption regarding $i(r)$.

TABLE II

v_0	u_{red}	v_0/u_{red}
0·2	0·0281	0·141
0·4	0·1109	0·277
0·5	0·1712	0·343
0·6	0·2431	0·405
0·8	0·4166	0·521
1·0	0·6206	0·621
1·5	1·188	0·792
2·0	1·755	0·877
2·5	2·302	0·921
3·0	2·833	0·945
4·0	3·875	0·968
5·0	4·901	0·980
∞	∞	1·

Finally, the following approximate solution of (43) when $v_0 < 1$ may be noted. One finds that

$$u_{\text{red}} = \frac{1}{\sqrt{2}}v_0^2\left(1 - \tfrac{1}{8}v_0^2\right) + O(v_0^6), \qquad (44)$$

or, since on the violet side of ν_0 the width of the band is v_0, we have for our ratio tabulated in the last column of Table II approximately

$$\frac{u_{\text{red}}}{v_0} = \frac{1}{\sqrt{2}}v_0\left(1 - \tfrac{1}{8}v_0^2\right). \qquad (45)$$

This formula reproduces the results of exact calculation remarkably well when $v_0 \leqslant 1$. Thus, when $v_0 = 1$, while (45) gives a value 0·6197, the result of the exact calculation is 0·6206. Of course, when $v_0 > 1$, (45) should not be used.

§ 12. *The Band Contour on the Red Side of* ν_0.—We consider the two cases separately.

Case I $(v_0 \leqslant 1)$.—Now, because of the occultation effect only when $u < u_{\text{red}}$ have we a non-zero intensity. It is clear from fig. 3 that the curves of constant $u(< u_{\text{red}})$ intersect the tangent at $\theta = \pi/2$ at two distinct points which correspond to the two roots of the equation (42) for a given value of $u < u_{\text{red}}$. If we denote the corresponding angles by θ_1 and θ_2 we have the required limits for use in (36).

A more convenient method to construct the line contour is to calculate u for different values of r (for a given v_0) and plot u against $u \sec \theta$. From this graph we can directly read off the values of $u \sec \theta_1$ and $u \sec \theta_2$ for any given value of $u(< u_{\text{red}})$. The contour can now be constructed according to (36). Fig. 4 illustrates the calculations for $v_0 = 0·8$ and fig. 5 for the case $v_0 = 1·0$.

Now observations seem to indicate that almost all the contours of emission bands in Wolf-Rayet stars are of the "peaked" or "rounded"

$I \rightarrow u \sec \theta.$
$II \rightarrow u^2 . \sec^2 \theta.$
$(v_0 = 0.8).$

$I \rightarrow \beta = 2.$
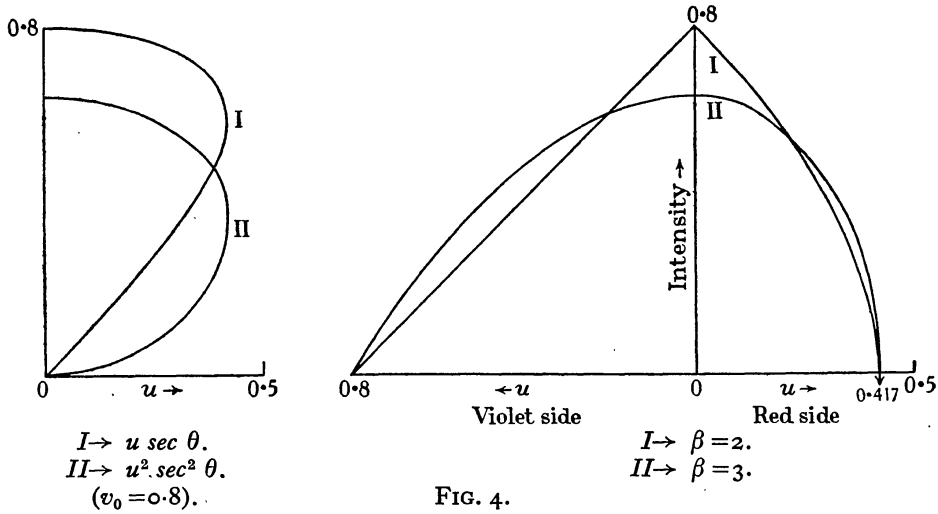$II \rightarrow \beta = 3.$

FIG. 4.

types,* and if the ejection process takes place according to mechanism B then careful measures on band contours must reveal asymmetries of the character illustrated in figs. 4 and 5.
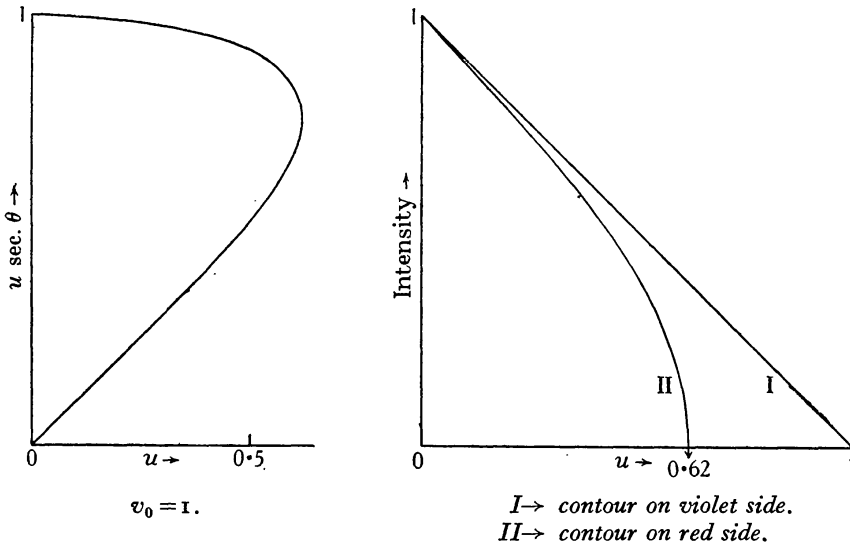

$v_0 = 1.$

$I \rightarrow$ *contour on violet side.*
$II \rightarrow$ *contour on red side.*

FIG. 5.

Case II $(v_0 > 1)$.—On the violet side of v_0 the contour is flat for $u < v_\infty (= \sqrt{v_0^2 - 1})$ and has a sloping side for $v_0 < u \leqslant v_\infty$. But on the red side the situation is quite different. If $u < v_\infty$ the curves of constant u have just *one* intersection with the tangent at $\theta = \pi/2$, and if we denote the solution by θ_u then the required limits are

$$\theta_2 = \theta_u ; \qquad \theta_1 = \cos^{-1} (u/v_\infty). \qquad (46)$$

* C. S. Beals, *Dominion Astro. Obs. Pub.*, **6**, No. 9 (in press). I am indebted to Professor H. H. Plaskett for the opportunity to see an advance manuscript copy of this paper.

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Then

$$i(u) = \begin{cases} \dfrac{4\pi}{\beta - 1}\{(u \sec \theta_u)^{\beta-1} - v_\infty^{\beta-1}\}, & \beta \neq 1, \\[2ex] 4\pi \log (u \sec \theta_u / v_\infty), & \beta = 1. \end{cases} \qquad (47)$$

Thus the contour is not flat to the red of ν_0. When $v_\infty \leqslant u \leqslant u_{\mathrm{red}}$ the curves of constant u have two intersections with the tangent at $\theta = \pi/2$, and these specify the limits θ_1 and θ_2 to be used in (36). In practice the contour can be constructed by the method sketched under Case I. In fig. 6 we illustrate the case where $v_0 = 1\cdot5$ and $\beta = 2$. We notice that to the red of ν_0 the part corresponding to the "flat top" on the violet side is of a slightly
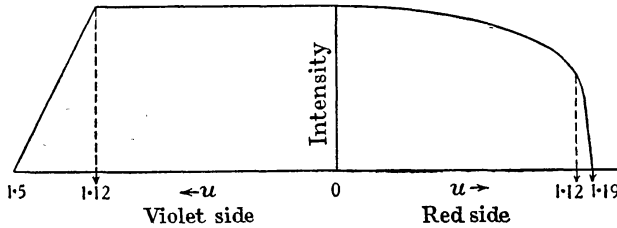

FIG. 6.—$v_0 = 1\cdot5$; $\beta = 2$.

but definitely sloping nature. (When $v_0 = 1\cdot5$ this sloping is quite appreciable and should be capable of observational verification if the ejection takes place according to mechanism B.) The part corresponding to the sloping side to the violet of ν_0 is very much steeper on the red side. The band is, in fact, wholly unsymmetrical about ν_0.

Finally, the following geometrical properties of the curves $(u, u \sec \theta)$ and their consequences might be noted :—

(i) If $v_0 \leqslant 1$ the curve passes through the origin, making an angle $\tan^{-1} v_0 \sqrt{2 - v_0^2}$ with the $u \sec \theta$ – axis. Further, the curve also crosses the $u \sec \theta$ – axis at right angles on its "upper branch." Hence for $v_0 < 1$ the contour is unsymmetrical even about the immediate neighbourhood of $u = 0$, and this in addition to the fact that the intensity on the red side falls to zero for a value of $u = u_{\mathrm{red}}$, less that at which it tends to zero on the violet side. If $v_0 = 1$ the curve passes through the origin, making an angle of $45°$ with both the axes. Hence in this case the contour is symmetrical about the neighbourhood of $u = 0$.

(ii) If $v_0 > 1$ the curve crosses the u – axis at $u = v_\infty$, making an angle of $45°$ with it. Also the curve crosses the $u \sec \theta$ – axis at right angles when $u = 0$. Hence when $v_0 \geqslant 1$ the band contours are symmetrical about the *immediate* neighbourhood of $u = 0$.

§ 13. There is one point of importance which has to be mentioned here. In calculating the line contours we have assumed that when $v_0 < 1$ only those atoms which have an outward radial velocity contribute to the emission. But when $v_0 < 1$, the atoms after "ascending" to a distance r_1 (see equation (32)) begin to fall back towards the parent star. If these "descending" atoms are equally efficient in contributing to the emission then it is clear that the asymmetry between the red and the violet parts of the band is

removed and that the contour on both sides of v_0 will be represented by summing up the intensities calculated in §§ 10, 12 for the "violet" and the "red" parts of the band separately.   But if we assume that the descending atoms are equally efficient in the emission process then it is clear that we no longer have a strict radial ejection hypothesis ; in particular the explanation generally given for the "violet absorption" will not be valid.   This, however, is not so important as the fact that when $v_0 < 1$ the density will rise rather steeply near the boundary $r = r_1$, and if the emission arises by some kind of recombination process then the probability is that an atom which has been ejected from the boundary in an ionized state will have re-captured an electron before it begins to fall back towards the parent star. Of course, we must now consider the possibility of an atom getting ionized once again when it would receive an additional outward momentum.   This introduces further complications, and it is best to avoid these, in the first instance, by neglecting the emission by the "descending" atoms.   It is, however, important to bear these reservations in mind in accepting the conclusions stated in the following section.

§ 14. *Concluding Remarks.*—The observational material is not extensive, but they appear to indicate, firstly, that "the centres of the Wolf-Rayet bands are approximately in their normal positions," * and secondly, that "flat-topped contours are exceptions rather than the rule." †

Now, on the basis of mechanism A, though the maximum intensity occurs at the centre of the band, the band contours are not symmetrical, the asymmetry consisting essentially in the violet part being brighter than the corresponding red part.   Further, the contours are of the "peaked" type if $\beta \leqslant 3$, but develop "rounded" tops for $\beta > 3$.   Even for $\beta = 4$ (see curve III, fig. 2) the contour has a more or less flat top.   It is worthy of note here that if $\beta > 2$ the envelope of the high-speed particles in its emission properties is of the "planetary nebula" type.‡

On the other hand, on the basis of mechanism B a "peaked" or "rounded" contour means that the atoms are ejected with velocities less than or equal to the velocity of escape.   Even when $v_0 = 1 \cdot 5$ (*cf.* fig. 6) the contour has a pronounced flat top on the violet side.   But when $v_0 \leqslant 1$ the contours are very unsymmetrical.   Thus for $v_0 = 1$ the band width on the red side is only 0·6 of the band width on the violet side, and one would expect that observations could easily detect a difference of 40 per cent.

Broadly speaking, then, the band contours on mechanism A show much less asymmetry than the band contours on mechanism B.   One could conclude then that the existing observational material is formally more consistent with the ejection process taking place according to mechanism A

* C. S. Beals, *Pub. Dom. Astro. Ob.*, **4**, No. 17.

† *Ibid.*, **6**, No. 9 (in press).

‡ If we write $\beta = n + 2$ then on a certain scale

$$i(r) = \frac{1}{r^4}\left(1 - \frac{1}{r}\right)^{n/2},$$

from which we deduce that $i(r)$ attains its maximum when $r = (n + 8)/8$.

than according to mechanism B. But if the ejection process takes place according to mechanism A then one meets with a number of difficulties. For instance, the limiting velocity must depend on the atom. Actually, however, it appears that there is not much difference between the band widths of the different atoms. There are, in fact, a number of other difficulties, but this is not the place to go into such matters. The purpose of this paper was merely to work out the formal consequences of the radial ejection hypothesis.

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