The Schwinger SU(3) Construction-II: Relations between Heisenberg-Weyl and SU(3) Coherent States

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Abstract

The Schwinger oscillator operator representation of $SU(3)$, studied in a previous paper from the representation theory point of view, is analysed to discuss the intimate relationships between standard oscillator coherent state systems and systems of $SU(3)$ coherent states. Both $SU(3)$ standard coherent states, based on choice of highest weight vector as fiducial vector, and certain other specific systems of generalised coherent states, are found to be relevant. A complete analysis is presented, covering all the oscillator coherent states without exception, and amounting to $SU(3)$ harmonic analysis of these states.

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I. INTRODUCTION

In a previous paper [1] we have presented an analysis of the reducible unitary representation(UR) of $SU(3)$ that is obtained by a generalisation of the well-known Schwinger oscillator operator construction in the case of $SU(2)$ [2]. This construction, based on six independent pairs of oscillator operators, is a minimal one in the sense that all unitary irreducible representations (UIR) of $SU(3)$ are obtained without exception. However in contrast to the $SU(2)$ case there is an unavoidable multiplicity in that each UIR occurs a denumerably infinite number of times. A systematic way to handle this multiplicity, based on the use of the non compact group $Sp(2, R)$, has been developed; its salient features are recapitulated in the next Section.

The aim of the present paper is to extend this study and discuss various properties of coherent states in this framework. The use of oscillator operators automatically brings in the Heisenberg-Weyl (H-W) group with a dimension appropriate to the number of independent oscillators or degrees of freedom. And it is indeed in the context of this group that the standard coherent states in quantum mechanics were originally defined and applied to a very large number of problems [3]. On the other hand, the basic kinematic relations for any system of independent oscillator operators have a well-defined covariance group associated with them - a group of linear inhomogeneous transformations on the oscillator operators which leave their commutation relations invariant. The homogeneous part of this covariance group is the metaplectic group of appropriate dimension, containing a unitary group as its maximal compact subgroup. Thus for n oscillators or n canonical pairs of degrees of freedom, we encounter the groups $Mp(2n)$, $U(n)$ and $SU(n)$, and certain of their UR's, in a natural way [4].

Now the original concept of coherent states has been generalised from the H-W case to a general Lie group, and it consists of the orbit of a chosen fiducial vector under group action in any UIR of the group [5]. The usual coherent states arise by the action of the elements of the H-W group on the Fock vacuum. Given all this, it is natural and to be expected that via the Schwinger type construction we have an intricate interplay between the familiar H-W coherent states, and certain systems of coherent states associated with the groups $Mp(2n), U(n)$ and $SU(n)$.

In passing we may also mention that with this generalisation, even for the H-W group we have not only the originally defined coherent states, which may be called Standard Coherent States (SCS), but other systems of generalised coherent states (GCS) [6]. These are based on choices of states other than the Fock vacuum as the fiducial state. Similarly, for the unitary group $SU(n)$, within any given UIR the SCS are obtained when the highest weight state is used as the fiducial state, while for other choices we have systems of GCS [7]. It is therefore of interest to see how these various systems of coherent states for different groups get interconnected via the Schwinger construction. This is the main aim of the present work, in the particular case of the H-W group for six oscillators, and $SU(3)$.

A brief outline of this work is as follows. Our earlier work [1] has shown how in a natural manner we can identify and isolate a subspace $\mathcal{H}_0$ carrying a complete and multiplicity-free UR of $SU(3)$ (a 'Generating Representation' for $SU(3)$), within the full Schwinger representation characterised by infinite multiplicity. As this decomposition, in which the compact generator J_0 of $Sp(2, R)$ plays a crucial role, provides the starting point of the present work, to set the notation and to make the paper reasonably self-contained, we briefly recapitulate the relevant details of [1] in Section II. In Section III, we recall the largely familiar interconnections between H-W and $U(1)$ and $SU(2)$ coherent states, to highlight some special features of the Klauder resolution of the identity and its modifications. This helps set the stage for a unified analysis of the relations between the appropriate H-W SCS and $SU(3)$ SCS and GCS carried out in detail in Sections IV, V, and VI. Section IV contains a detailed classification of the orbits of H-W SCS under $SU(3)$ action; we identify both generic orbits of maximal dimension, and non generic lower order ones. The rest of Section IV carries out the $SU(3)$ harmonic analysis of generic orbits lying in the subspace $\mathcal{H}_0$. In Section V we examine the remaining generic orbits, lying in subspaces $\mathcal{H}_\kappa$ which are generalisations of $\mathcal{H}_0$ and are labelled by a complex parameter κ. Some calculational details pertaining to this Section are put together in an appendix. Section VI contains an analysis of the $SU(3)$ content of a family of H-W SCS belonging to a non generic orbit under $SU(3)$ action. Some concluding remarks are presented in Section VII.

II. REVIEW OF SCHWINGER CONSTRUCTION FOR $SU(3)$

This construction uses six independent sets of oscillator creation and annihilation operators $\hat{a}_j^\dagger, \hat{b}_j^\dagger, \hat{a}_j, \hat{b}_j, j = 1, 2, 3$, among which the only non vanishing commutators are

$$[\hat{a}_j, \hat{a}_k^\dagger] = [\hat{b}_j, \hat{b}_k^\dagger] = \delta_{jk},\ j, k = 1, 2, 3. \tag{2.1}$$

The Hilbert space $\mathcal{H}$ carrying an irreducible representation of these operators is the tensor product $\mathcal{H} = \mathcal{H}^{(a)} \times \mathcal{H}^{(b)}$, where $\mathcal{H}^{(a)}$ and $\mathcal{H}^{(b)}$ are the individual Hilbert spaces carrying irreducible representations of the independent sets $\hat{a}_j, \hat{a}_j^\dagger$ and $\hat{b}_j, \hat{b}_j^\dagger$ respectively. The Schwinger UR of $SU(3)$ acts on $\mathcal{H}$, and its hermitian generators are [1]

$$\begin{aligned} Q_\alpha &= Q_\alpha^{(a)} + Q_\alpha^{(b)}, \\ Q_\alpha^{(a)} &= \frac{1}{2}\hat{a}^\dagger \lambda_\alpha \hat{a},\ Q_\alpha^{(b)} = -\frac{1}{2}\hat{b}^\dagger \lambda_\alpha^*\, \hat{b},\ \alpha = 1, 2, \ldots, 8. \end{aligned} \tag{2.2}$$

Here $\frac{1}{2}\ \lambda_\alpha$ are the eight hermitian traceless 3×3 matrices generating the defining UIR $(1, 0)$ of $SU(3)$ [8].(For ease in writing, the UIR's of $SU(3)$ will be denoted by (p, q) where $p, q = 0, 1, 2, \ldots,$ independently, instead of the more elaborate notation $D^{(p,q)}$).

The independent mutually commuting generators $Q_\alpha^{(a)}, Q_\alpha^{(b)}$ lead to specific multiplicity-free UR's $\mathcal{U}^{(a)}(A), \mathcal{U}^{(b)}(A)$ of $SU(3)$ on $\mathcal{H}^{(a)}, \mathcal{H}^{(b)}$ respectively. Here A is a general matrix in the UIR $(1, 0)$. The UR $\mathcal{U}^{(a)}(A)$ is a direct sum of the 'triangular' UIR's $(p, 0)$ of $SU(3)$, for $p = 0, 1, 2, \ldots$; and similarly $\mathcal{U}^{(b)}(A)$ is a direct sum of the conjugate 'triangular' UIR's $(0, q)$. We indicate this by

$$\begin{aligned} \mathcal{U}^{(a)} &= \sum_{p=0,1,\ldots}^{\infty} \oplus\ (p, 0), \\ \mathcal{U}^{(b)} &= \sum_{q=0,1,\ldots}^{\infty} \oplus\ (0, q). \end{aligned} \tag{2.3}$$

The total generators Q_α defined in eqn(2.2) then generate the product UR $\mathcal{U}(A) = \mathcal{U}^{(a)}(A) \times \mathcal{U}^{(b)}(A)$ on $\mathcal{H}$, and this is the Schwinger UR of $SU(3)$. It does contain every UIR (p,q) of $SU(3)$, but each one occurs an infinite number of times. This can be seen from the Clebsch-Gordan decomposition of the direct product $(p,0)\times(0,q)$ of two triangular UIR's [9]:

$$(p,0)\times(0,q) = \sum_{\rho=0,1,\ldots}^{r} \oplus\ (p-\rho, q-\rho),\ r = \min\,(p,q), \tag{2.4}$$

which is multiplicity-free. Applying this to each pair in the product $\mathcal{U}^{(a)} \times \mathcal{U}^{(b)}$ we easily reach the stated conclusion.

An efficient way to handle this infinite multiplicity is based on the use of the semi-simple non compact Lie group $Sp(2,R)$, more specifically some of its UIR's belonging to the positive discrete class [10]. In the present context the hermitian $Sp(2,R)$ generators and their commutation relations are:

$$\begin{aligned} J_0 &= \frac{1}{2}\left(\hat{a}_j^\dagger \hat{a}_j + \hat{b}_j^\dagger \hat{b}_j + 3\right), \\ K_1 &= \frac{1}{2}\left(\hat{a}_j^\dagger \hat{b}_j^\dagger + \hat{a}_j \hat{b}_j\right), \\ K_2 &= \frac{-i}{2}\left(\hat{a}_j^\dagger \hat{b}_j^\dagger - \hat{a}_j \hat{b}_j\right); \end{aligned} \tag{2.5a}$$

$$[J_0, K_1] = i\ K_2,\ [J_0, K_2] = -i\ K_1,\ [K_1, K_2] = -i\ J_0. \tag{2.5b}$$

The crucial property is that the $SU(3)$ and the $Sp(2,R)$ generators mutually commute:

$$[J_0 \text{ or } K_1 \text{ or } K_2,\ Q_\alpha] = 0. \tag{2.6}$$

Thus the two UR's commute as well, and $Sp(2,R)$ is just large enough to be able to completely lift the degeneracy or multiplicity of $SU(3)$ UIR's. In other words, the UIR's of the product group $SU(3)\times Sp(2,R)$ that occur in $\mathcal{H}$ do so in a multiplicity-free manner. This is reflected at the Hilbert space level in the following manner. We first decompose the individual Hilbert spaces $\mathcal{H}^{(a)}, \mathcal{H}^{(b)}$ into mutually orthogonal subspaces reflecting the decompositions (2.3):

$$\mathcal{H}^{(a)} = \sum_{p=0,1,\ldots}^{\infty} \oplus\, \mathcal{H}^{(p,0)},$$
$$\mathcal{H}^{(b)} = \sum_{q=0,1,\ldots}^{\infty} \oplus\, \mathcal{H}^{(0,q)}. \tag{2.7}$$

The subspace $\mathcal{H}^{(p,0)} \subset \mathcal{H}^{(a)}$ is of dimension $d(p,0) = \frac{1}{2}(p+1)(p+2)$; consists of all eigenvectors in $\mathcal{H}^{(a)}$ of the total a- type number operator $\hat{a}_j^\dagger \hat{a}_j$ with eigenvalue p; and carries the UIR $(p,0)$ of $SU(3)$. Similarly the subspace $\mathcal{H}^{(0,q)} \subset \mathcal{H}^{(b)}$ is of dimension $d(0,q) = \frac{1}{2}(q+1)(q+2)$; consists of all eigenvectors in $\mathcal{H}^{(b)}$ of the total b-type number operator $\hat{b}_j^\dagger \hat{b}_j$ with eigenvalue q; and carries the UIR $(0,q)$ of $SU(3)$. After forming the direct product $\mathcal{H}^{(a)} \times \mathcal{H}^{(b)}$, using eqn.(2.7) and the Clebsch-Gordan decomposition (2.4), we arrive at an orthogonal subspace decomposition for $\mathcal{H} = \mathcal{H}^{(a)} \times \mathcal{H}^{(b)}$:

$$\begin{aligned}
\mathcal{H} &= \sum_{p,q=0,1,\ldots}^{\infty} \oplus\, \mathcal{H}^{(p,0)} \times \mathcal{H}^{(0,q)} \\
&= \sum_{p,q=0,1,\ldots}^{\infty} \sum_{\rho=0,1,\ldots}^{\infty} \oplus\, \mathcal{H}^{(p,q;\rho)}, \\
\mathcal{H}^{(p,q;\rho)} &\subset \mathcal{H}^{(p+\rho,0)} \times \mathcal{H}^{(0,q+\rho)}.
\end{aligned} \tag{2.8}$$

For each $\rho, \mathcal{H}^{(p,q;\rho)}$ is of dimension $d(p,q) = \frac{1}{2}(p+1)(q+1)(p+q+2)$ and carries the ρth occurrence of the UIR (p,q) of $SU(3)$. For $\rho' \neq \rho, \mathcal{H}^{(p,q;\rho')}$ and $\mathcal{H}^{(p,q;\rho)}$ are mutually orthogonal subspaces; and if $p' \neq p$ and/or $q' \neq q$, again $\mathcal{H}^{(p',q';\rho)}$ and $\mathcal{H}^{(p,q;\rho)}$ are mutually orthogonal. An orthonormal basis for $\mathcal{H}$ consists of vectors labelled as follows:

$$\begin{aligned}
&|p,q\ ;\ I,M,Y;m>: \\
&p,q = 0,1,2,\ldots, \\
&\quad m = k, k+1, k+2, \ldots, \\
&\quad k = \frac{1}{2}(p+q+3) = \frac{3}{2}, 2, \frac{5}{2}, \ldots.
\end{aligned} \tag{2.9}$$

Here I, M, Y are 'magnetic quantum numbers' within the UIR (p,q) of $SU(3)$, with well-known ranges [11]; and m is the eigenvalue of the $Sp(2,R)$ generator J_0. The total numbers of a-type quanta and of b-type quanta in the state displayed in eqn.(2.9) are:

$$N_a = \text{eigenvalue of } \hat{a}_j^\dagger \hat{a}_j = p + m - k,$$
$$N_b = \text{eigenvalue of } \hat{b}_j^\dagger \hat{b}_j = q + m - k. \tag{2.10}$$

For fixed p, q and m, as I, M, Y vary within the UIR (p, q) of $SU(3)$, we obtain an orthonormal basis for $\mathcal{H}^{(p,q;m-k)}$. Switching to $\rho = m - k$ we can say:

$$\mathcal{H}^{(p,q;\rho)} = Sp\{|p, q; I, M, Y; k + \rho > |p, q, \rho \text{ fixed }, I, M, Y \text{ varying}\} \tag{2.11}$$

On the other hand, if we keep p, q, I, M, Y fixed and let m vary, we get an orthonormal basis for a subspace of $\mathcal{H}$ carrying the infinite dimensional positive discrete class UIR $D_k^{(+)}$ of $Sp(2, R)$ [10]. In other words, each of these UIR's $D_k^{(+)}$ of $Sp(2, R)$ occurs $d(2k - 3, 0) + d(2k - 4, 1) + \ldots + d(1, 2k - 4) + d(2k - 3)$ times, being the sum of the dimensions of the $SU(3)$ UIR's $(2k-3, 0), (2k-4, 1), \ldots (1, 2k-4), (0, 2k-3)$. (The range of $2k$ is $3, 4, 5, \ldots$). Since our main interest is in UR's and UIR's of $SU(3)$, and we wish to use UIR's of $Sp(2, R)$ mainly to keep track of the multiplicities of the former, we do not introduce special notations for the subspaces of $\mathcal{H}$ carrying the various $Sp(2, R)$ UIR's. However we do note that, as stated earlier, each of the UIR's $(p, q) \times D^{(+)}_{\frac{1}{2}(p+q+3)}$ of $SU(3) \times Sp(2, R)$ appears just once in $\mathcal{H}$, for $p, q = 0, 1, 2, \ldots$.

At the generator level we can say that when the $SU(3)$ generators Q_α act on $|p, q; I, M, Y; m >$, they alter only the quantum numbers I, M, Y in a manner known from the representation theory of $SU(3)$ [12]; while the actions by the $Sp(2, R)$ generators J_0, K_1, K_2 lead only to changes in the quantum number m according to the UIR $D_k^{(+)}$ [10].

It is in this manner that the $Sp(2, R)$ structure helps us handle the multiplicity problem of UIR's of $SU(3)$ which is an unavoidable feature of the Schwinger construction. One can now look for a natural subspace of $\mathcal{H}, \mathcal{H}_0$ say, such that it carries every UIR (p, q) of $SU(3)$ exactly once. This can be done if we restrict ourselves to the 'ground state' within each $Sp(2, R)$ UIR $D_k^{(+)}$, namely if we set $m = k$. This amounts to picking up the 'first' occurrence of each UIR (p, q) of $SU(3)$ corresponding to $\rho = 0$, or to the 'leading piece' in the reduction of each tensor product $\mathcal{H}^{(p,0)} \times \mathcal{H}^{(0,q)}$:

$$
\begin{aligned}
\mathcal{H}_0 &= \sum_{p,q=0,1,\dots}^{\infty} \oplus\, \mathcal{H}^{(p,q;0)} \\
&= Sp\{|p,q;I,M,Y;k> |p,q,I,M,Y \text{ varying}\} \\
&= \{|\psi>\in\ \mathcal{H}|(K_1 - i\ K_2)|\psi>= 0\}.
\end{aligned}
\tag{2.12}
$$

The UR of $SU(3)$ carried by $\mathcal{H}_0, \mathcal{D}_0$ say, may be called a Generating Representation for this group, in the sense that each UIR is present, and exactly once:

$$
\mathcal{D}_0 = \sum_{p,q=0,1,\dots}^{\infty} \oplus\, (p,q). \tag{2.13}
$$

It now turns out that just this property is also present in the UR $\mathcal{D}^{(\mathrm{ind},0)}_{SU(2)}$ of $SU(3)$ induced from the trivial one-dimensional UIR of the canonical $SU(2)$ subgroup [13]. The corresponding Hilbert space is denoted by $\mathcal{H}^{(\mathrm{ind},0)}_{SU(2)}$. (Hereafter, for simplicity, the superscript zero and the subscript $SU(2)$ will be omitted.) We can set up a one-to-one mapping between $\mathcal{H}_0$ and $\mathcal{H}^{(\mathrm{ind})}$ preserving scalar products and $SU(3)$ actions, thus realising the equivalence of $\mathcal{D}_0$ and $\mathcal{D}^{(\mathrm{ind})}$. First we describe $\mathcal{H}_0$ and $\mathcal{D}_0$ more explicitly. Denote by $|\underline{0},\underline{0}>$ the Fock vacuum in $\mathcal{H}$ annihilated by $\hat{a}_j$ and $\hat{b}_j, j = 1,2,3$. Then a general vector in $\mathcal{H}_0$ is a collection of symmetric traceless tensors with respect to $SU(3)$, one for each UIR (p,q):

$$
\begin{aligned}
&|\psi>\in\ \mathcal{H}_0\ : \\
&\qquad |\psi>\ = \sum_{p,q=0,1,\dots}^{\infty} \psi^{j_1\dots j_p}_{k_1\dots k_q}\ \hat{a}^\dagger_{j_1}\dots\hat{a}^\dagger_{j_p}\ \hat{b}^\dagger_{k_1}\dots\hat{b}^\dagger_{k_q}\ |\underline{0},\underline{0}>;
\end{aligned}
\tag{2.14a}
$$

$$
\psi^{j_{P(1)}\cdots j_{P(p)}}_{k_{Q(1)}\cdots k_{Q(q)}} = \psi^{j_1\dots j_p}_{k_1\dots k_q}\ ,\ P\in S_p,\ Q\in S_q; \tag{2.14b}
$$

$$
\psi^{j\ j_2\dots j_p}_{j\ k_2\dots k_q} = 0 \tag{2.14c}
$$

$$
<\psi|\psi>\ = \parallel \psi \parallel^2 = \sum_{p,q=0,1,\dots}^{\infty} p!\ q!\ \psi^{j_1\dots j_p}_{k_1\dots k_q}{}^{*}\psi^{j_1\dots j_p}_{k_1\dots k_q}; \tag{2.14d}
$$

$$
\mathcal{D}_0(A)|\psi>=|\psi'>,
$$

$$\psi'^{\,j_1\dots j_p}_{k_1\dots k_q} = A^{j_1}{}_{\ell_1}\dots A^{j_p}{}_{\ell_p}\, A^{k_1}{}^{*}_{m_1}\dots A^{k_q}{}^{*}_{m_q}\dots\psi^{\ell_1\dots\ell_p}_{m_1\dots m_q}. \tag{2.14e}$$

Here S_p and S_q are the permutation groups on p and on q objects respectively. Turning to $\mathcal{H}^{(\mathrm{ind})}$ and $\mathcal{D}^{(\mathrm{ind})}$, the former consists of complex square integrable functions on the coset space $SU(3)/SU(2)$, namely the unit sphere in $\mathcal{C}^3$ [14]:

$$\mathcal{H}^{(\mathrm{ind})} = \left\{\psi(\underline{\xi}) \in \mathcal{C}, \underline{\xi} \in \mathcal{C}^3 |\parallel \psi \parallel^2 = \int \prod_{j=1}^{3}\left(\frac{d^2\xi_j}{\pi}\right)\delta\left(\underline{\xi}^\dagger\underline{\xi}-1\right)|\psi(\underline{\xi})|^2\right\}. \tag{2.15}$$

The group action is by change of argument:

$$\begin{aligned}\mathcal{D}^{(\mathrm{ind})}(A)\psi &= \psi',\\ \psi'(\underline{\xi}) &= \psi(A^{-1}\underline{\xi})\end{aligned} \tag{2.16}$$

.

Then the one-to-one mapping between $\mathcal{H}_0$ and $\mathcal{H}^{(\mathrm{ind})}$ consistent with the two norm definitions $(2.14d, 2.15)$ and the two group actions $(2.14e, 2.16)$ is:

$$|\psi> \; = \left\{\psi^{j_1\dots j_p}_{k_1\dots k_q}\right\} \in \; \mathcal{H}_0 \longleftrightarrow$$

$$\psi(\underline{\xi}) = \sum_{p,q=0,1,\dots}^{\infty} \sqrt{(p+q+2)!}\;\; \psi^{j_1\dots j_p}_{k_1\dots k_q}\, \xi_{j_1}\dots\xi_{j_p}\, \xi^*_{k_1}\dots\xi^*_{k_q} \in \; \mathcal{H}^{(\mathrm{ind})} \tag{2.17}$$

The fact that $\psi(\underline{\xi}) \in \; \mathcal{H}^{(\mathrm{ind})}$ is expressible in this way in terms of <u>traceless</u> symmetric tensors is a consequence of the constraint $\underline{\xi}^\dagger\underline{\xi} = 1$.

In this way we see how the Schwinger UR $\mathcal{U}(A)$ of $SU(3)$ contains within it a multiplicity-free UR $\mathcal{D}_0$ including every UIR of $SU(3)$, which is also accessible by the method of induced representations. We will see later that in fact there is a continuously infinite family of subspaces $\mathcal{H}_\kappa \subset \mathcal{H}$, labelled by a complex number κ, such that each $\mathcal{H}_\kappa$ is $SU(3)$ invariant and carries a UR $\mathcal{D}_\kappa$ of $SU(3)$ which, like $\mathcal{D}_0$, is multiplicity free and contains each UIR (p, q) without exception.

III. INTERPLAY BETWEEN HEISENBERG-WEYL AND UNITARY GROUP COHERENT STATES - ONE AND TWO DEGREES OF FREEDOM

We now turn to an examination of the interconnections between H-W coherent states and unitary group coherent states. In each case there are both standard and generalised coherent state systems. In this Section we look at the cases of $n = 1$ and $n = 2$ degrees of freedom, the relevant unitary groups being $U(1)$ and $SU(2)$ and there being no multiplicity problems. We review briefly some known material but highlighting some special aspects. This material is then used as guidance when we take up in the next Section the case $n = 6$ and the Schwinger $SU(3)$ construction.

One degree of freedom

It is convenient to be able to switch between the use of non hermitian creation and annihilation operators $\hat{a}^\dagger, \hat{a}$ and their hermitian position and momentum components $\hat{q}, \hat{p}$:

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}),\ \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p}). \tag{3.1}$$

For one degree of freedom, the canonical commutation relation

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger] &= 1, \\ [\hat{q}, \hat{p}] &= i, \end{aligned} \tag{3.2}$$

is preserved under the linear inhomogeneous transformation

$$\begin{aligned} &\begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} \to \begin{pmatrix} \hat{q}' \\ \hat{p}' \end{pmatrix} = S \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} + \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}; \\ &S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ ,\ \ ad - bc = 1;\ q_0, p_0 \in \mathcal{R}. \end{aligned} \tag{3.3}$$

Here S is an element of $Sp(2, R) = SL(2, R)$, and these transformations constitute the semi direct product of $Sp(2, R)$ with the two-dimensional Abelian group of phase-space translations. However, as is well known, these transformations are realised on the Hilbert space $\mathcal{H}$, on which $\hat{a}^\dagger, \hat{a}$ or $\hat{q}, \hat{p}$ act irreducibly, by unitary transformations forming a faithful

UIR of a group $G^{(1)}$ which is the semi-direct product of the metaplectic group $Mp(2)$ with the H-W group [15]:

$$G^{(1)} = Mp(2) \times \{\text{H-W group}\}. \tag{3.4}$$

Each factor here is a three parameter Lie group, so $G^{(1)}$ is a six-parameter Lie group. The H-W group is the invariant subgroup; it is non Abelian because of the nonzero right hand sides in the commutators (3.2). Its generators are $\hat{q}, \hat{p}$ and the unit operator on $\mathcal{H}$. The homogeneous part $Mp(2)$ is a double cover of $Sp(2, R)$; its generators are hermitian quadratic expressions in $\hat{a}^\dagger$ and $\hat{a}$, or in $\hat{q}$ and $\hat{p}$ [16]. In particular the $U(1)$ generator is $\frac{1}{2}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$, and this is the analogue of J_0 in the $Sp(2, R)$ Lie algebra (2.5).

As stated above, $\mathcal{H}$ carries a particular UIR of $G^{(1)}$. Upon restriction to the H-W subgroup, this representation remains irreducible; it is the result of exponentiating the well-known unique Stone-von Neumann representation of the commutation relations (3.2) [17]. On the other hand, upon restriction to the $Mp(2)$ subgroup, we get a direct sum of two UIR's of the positive discrete class, namely $D^{(+)}_{1/4}$ and $D^{(+)}_{3/4}$ [18]. These act on the subspaces $\mathcal{H}^{(\pm)}$ of $\mathcal{H}$ consisting of even/odd parity states or Schrodinger wave functions. The nontrivial H-W generators $\hat{q}$ and $\hat{p}$ intertwine these two UIR's of $Mp(2)$.

With this background, we collect some remarks regarding various systems of coherent states. As both $G^{(1)}$ and the H-W group are represented irreducibly on $\mathcal{H}$, for any choice of a (normalised) fiducial vector $\psi_0 \in \mathcal{H}$ we can build up a family of $G^{(1)}$ - GCS or a family of H-W GCS [5]. These are the orbits of ψ_0 under $G^{(1)}$ action and under H-W action respectively, and the latter orbit is a subset of the former. In the case of $Mp(2)$, we can construct systems of GCS separately in $\mathcal{H}^{(+)}$ and in $\mathcal{H}^{(-)}$, associated with any choices of fiducial vectors in these subspaces. Examples are the single mode squeezed coherent states and their variations [18].

Now let us limit ourselves to H-W coherent states, and to their behaviours under the maximal compact $U(1)$ subgroup of $Mp(2)$. As mentioned earlier the generator of this $U(1)$ is $\frac{1}{2}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$. However for simplicity we shall work with

$$\overline{U}(\alpha) = e^{-i\alpha\hat{a}^\dagger\hat{a}}\ ,\ 0 \leq \alpha < 2\pi. \tag{3.5}$$

Conjugation by $\overline{U}(\alpha)$ has these effects on $\hat{a}, \hat{a}^\dagger$, and the unitary phase space displacement operators $D(z)$ which represent elements of the H-W group:

$$\begin{aligned}
\overline{U}(\alpha)\ \hat{a}\ \overline{U}(\alpha)^{-1} &= e^{i\alpha}\hat{a}, \\
\overline{U}(\alpha)\ \hat{a}^\dagger\ \overline{U}(\alpha)^{-1} &= e^{-i\alpha}\hat{a}^\dagger; \\
D(z) &= \exp\left(z\hat{a}^\dagger - z^*\hat{a}\right), \\
\overline{U}(\alpha)\ D(z)\ \overline{U}(\alpha)^{-1} &= D\left(e^{-i\alpha}z\right).
\end{aligned} \tag{3.6}$$

The H-W SCS correspond to the choice of the Fock vacuum $|0>$ as the fiducial vector [3]:

$$|z>= D(z)|0>\ ,\ z \in\ \mathcal{C}. \tag{3.7}$$

Invariance of $|0>$ under $\overline{U}(\alpha)$ action then leads to the behaviour

$$\overline{U}(\alpha)\ |z>= |\ e^{-i\alpha}z>. \tag{3.8}$$

These states enjoy the well-known Klauder formula for resolution of the identity operator:

$$\int_{\mathcal{C}} \frac{d^2z}{\pi}\ |z><z| = 1 \text{ on } \mathcal{H}. \tag{3.9}$$

This can be viewed as a consequence of the Schur lemma and the square integrability of the Stone-von Neumann UIR of the H-W group [19], since the uniform integration measure on the complex plane in (3.9) is essentially the invariant measure on the H-W group.

We now examine two variations of these familiar results. By eqn.(3.8), the left hand side of eqn.(3.9) is explicitly $U(1)$-invariant. We can consider including some nontrivial function $f(z^*z)$ inside the integral, which would maintain $U(1)$ invariance, and define the operator

$$A(f) = \int_{\mathcal{C}} \frac{d^2z}{\pi}\ f(z^*z)\ |z><z|. \tag{3.10}$$

As long as $f(z^*z)$ is not a constant, the integration measure here is no longer the invariant measure on the H-W group, so the Schur lemma is not available. Formally,

$$f(z^* z) \neq \text{ constant } \Longleftrightarrow D(z)\; A(f) \neq A(f)\; D(z), \tag{3.11}$$

so there is no reason to expect $A(f)$ to be a multiple of the identity. However, $U(1)$ invariance,

$$\overline{U}(\alpha)\; A(f) = A(f)\; \overline{U}(\alpha), \tag{3.12}$$

implies that $A(f)$ is a linear combination of projections on to the various Fock states, and indeed we find:

$$A(f) = \sum_{n=0}^{\infty} \int_0^{\infty} dx\; f(x)\; x^n\; e^{-x} \cdot \frac{|n><n|}{n!}. \tag{3.13}$$

Clearly the only choice of f leading to the Klauder formula (3.9) is $f = 1$. On the other hand, if we choose $f(z^* z) = \delta\left(z^* z - r_0^2\right)$ for some real positive r_0, we are limiting ourselves to a subset of H-W SCS lying on a circle in the complex plane. This is essentially the $U(1)$ group manifold; and if $r_0 = 1$ we have exactly the manifold S^1, that is, we have a $U(1)$-worth of H-W SCS. In this case we find:

$$\begin{aligned}
f(x) &= \delta\left(x - r_0^2\right) : \\
A(f) &= \int \frac{d^2 z}{\pi}\; \delta\left(z^* z - r_0^2\right)\; |z><z| \\
&= \int_0^{2\pi} \frac{d\theta}{2\pi}\; |r_0\; e^{i\theta} >< r_0\; e^{i\theta}| \\
&= \sum_{n=0}^{\infty}\; e^{-r_0^2}\; \frac{r_0^{2n}}{n!}\; |n><n| \\
&= e^{-r_0^2} \cdot r_0^{2\hat{N}}/\hat{N}!, \\
\hat{N} &= \hat{a}^\dagger \hat{a}.
\end{aligned} \tag{3.14}$$

This means that even though the subset of H-W SCS $\left\{|r_0 e^{i\theta}>, 0 \leq \theta < 2\pi\right\}$ lying on a circle in the complex plane is 'total' [20], and each Fock state $|n>$ can be projected out of this subset as

$$|n> = e^{r_0^2/2} \cdot \sqrt{n!}\; r_0^{-n} \cdot \int_0^{2\pi} \frac{d\theta}{2\pi} \cdot e^{-in\theta} \cdot |r_0 e^{i\theta}>, \tag{3.15}$$

we cannot obtain a Klauder-type resolution of the identity using them. Thus this $U(1)$-worth of SCS does not form a system of GCS in the Klauder sense.

The next variation we consider is replacing the Fock vacuum $|0>$ by a generic unit vector $|\psi_0>\in\ \mathcal{H}$ as fiducial vector. We then get a family of H-W GCS [21]:

$$|z;\psi_0>= D(z)\ |\psi_0>,\ z\in\ \mathcal{C}. \tag{3.16}$$

Once again, Schur lemma leads to the Klauder resolution of the identity,

$$\int\ \frac{d^2z}{\pi}\ |z;\psi_0><z;\ \psi_0| = c.1, \tag{3.17}$$

for some constant c; and square integrability ensures that c is finite. If in the manner of eqn.(3.10) we next define

$$A(f;\psi_0) = \int\ \frac{d^2z}{\pi}\ f(z^*z)\ |z;\psi_0><z;\psi_0|, \tag{3.18}$$

then on the one hand we do not expect $A(f;\psi_0)$ to be a multiple of the unit operator since we lose Schur lemma; and on the other hand we do not even expect $A(f;\psi_0)$ to commute with $\overline{U}(\alpha)$. That is, in general $A(f;\psi_0)$ is not a linear combination of the projections $|n><n|$ on to the Fock states. The exceptions are when $|\psi_0>$ is an eigenstate of $\hat{a}^\dagger\hat{a}$, ie., a Fock state $|n_0>$ for some integer n_0. This possibility arises because $U(1)$ is Abelian, and its UIR's are all one-dimensional. In that case we find [22]:

$$\begin{aligned}
|\psi_0> \ &= |n_0>: \\
\overline{U}(\alpha)|z;n_0> \ &= e^{-i\alpha n_0}|e^{-i\alpha}z;n_0>, \\
\overline{U}(\alpha)A(f;n_0) &= A(f;n_0)\overline{U}(\alpha); \\
A(f;n_0) &= \sum_{n=0}^{\infty}\ C_{n,n_0}(f)\ |n><n|, \\
C_{n,n_0}(f) &= \frac{n_<!}{n_>!}\int_0^\infty dx\ f(x)\ x^{|n-n_0|}e^{-x}\left(L_{n_<}^{|n-n_0|}(x)\right)^2, \\
n_> &= \max\ (n,n_0), n_< = \min\ (n,n_0).
\end{aligned} \tag{3.19}$$

When $n_0 = 0$ we recover eqn.(3.13). If we next choose $f(z^*z) = \delta\left(z^*z - r_0^2\right)$, thus limiting ourselves to a $U(1)$-worth of H-W GCS, we find in place of eqn.(3.14)):

$$
\begin{aligned}
f(x) &= \delta\left(x - r_0^2\right) : \\
A(f; n_0) &= \int \frac{d^2 z}{\pi}\ \delta\left(z^* z - r_0^2\right) |z; n_0 >< z; n_0| \\
&= \int_0^{2\pi} \frac{d\theta}{2\pi}\ |r_0 e^{i\theta}; n_0 >< r_0 e^{i\theta}; n_0| \\
&= \sum_{n=0}^{\infty} e^{-r_0^2} r_0^{2n} \left(L_{n_<}^{|n-n_0|}\left(r_0^2\right)\right)^2 \frac{n_<!}{n_>!}\ |n >< n|.
\end{aligned}
\tag{3.20}
$$

The main result of these considerations is that with SCS or GCS for the H-W group for one degree of freedom, we can get a Klauder type resolution of the identity only if we use the invariant measure on the group, but understandably not if we limit ourselves to a subset amounting to a $U(1)$-worth of these states.

Two degrees of freedom

Here we are interested in the interplay between coherent state systems for the relevant five-parameter H-W group, and the unitary groups $U(2)$ and $SU(2)$ which were the subject of the original Schwinger construction.

The non vanishing commutators in non hermitian and hermitian forms are

$$
\begin{aligned}
&[\hat{a}_r, \hat{a}_s^\dagger] = \delta_{rs}, \\
&[\hat{q}_r, \hat{p}_s] = i\ \delta_{rs}, \quad r, s = 1, 2.
\end{aligned}
\tag{3.21}
$$

There is no cause for confusion if again we write $\mathcal{H}$ for the Hilbert space carrying the irreducible Stone-von Neumann representation of these relations. The largest natural invariance group now acts on the four $\hat{q}$'s and $\hat{p}$'s as follows:

$$
\begin{pmatrix} \hat{q}_r \\ \hat{p}_r \end{pmatrix} \longrightarrow \begin{pmatrix} \hat{q}_r' \\ \hat{p}_r' \end{pmatrix} = S \begin{pmatrix} \hat{q}_r \\ \hat{p}_r \end{pmatrix} + \begin{pmatrix} q_{r,0} \\ p_{r,0} \end{pmatrix}.
\tag{3.22}
$$

Here $S \in\ Sp(4, R)$ is a four-dimensional real symplectic matrix, and $q_{r,0}, p_{r,0}$ denote an Abelian phase space translation [23]. These fourteen parameter transformations preserve (3.21). They make up the semi direct product of $Sp(4, R)$, which is ten dimensional, with the four dimensional Abelian translations. On the space $\mathcal{H}$, however, these transformations are realised as a faithful UIR of the fifteen-parameter semi direct product

$$G^{(2)} = Mp(4) \times \{\text{H-W group}\}. \tag{3.23}$$

Here the invariant subgroup is the five-parameter non Abelian H-W group appropriate for two degrees of freedom, while the homogeneous part is the metaplectic group $Mp(4)$, a double cover of $Sp(4, R)$. The generators of the former are $\hat{q}_r, \hat{p}_r$ and the unit operator, while those of the latter are hermitian symmetrised quadratics in $\hat{q}_r, \hat{p}_r$.

The Hilbert space $\mathcal{H}$ carries a UIR of $G^{(2)}$, which remains irreducible when restricted to the H-W group. On the other hand, $Mp(4)$ is represented by the direct sum of two UIR's, one each on the subspaces of even and odd parity states in $\mathcal{H}$. The general statements that can be made about GCS with respect to $G^{(2)}, Mp(4)$ and the H-W group are similar to those in the one degree of freedom case. Once again, our main interest is in the connections between H-W and $SU(2)$ coherent state systems.

The maximal compact subgroup of $Mp(4)$ is $U(2)$. The $SU(2)$ part of $U(2)$ has the generators and commutation relations (Schwinger construction)

$$\begin{aligned} J_j &= \frac{1}{2}\hat{a}^\dagger \sigma_j \hat{a}, \\ [J_j, J_k] &= i\ \epsilon_{jk\ell} J_\ell, \quad j, k = 1, 2, 3. \end{aligned} \tag{3.24}$$

The $U(1)$ part of $U(2)$ has as generator the total number operator

$$\begin{aligned} \hat{N} &= \hat{N}_1 + \hat{N}_2, \\ \hat{N}_r &= \hat{a}_r^\dagger \hat{a}_r, \\ [J_j, \hat{N}] &= 0. \end{aligned} \tag{3.25}$$

For general $u \in U(2)$, we write $\overline{U}(u)$ for the corresponding unitary operator on $\mathcal{H}$, generated by $J_j, \hat{N}$. Then in place of eqn.(3.6) we now have:

$$\begin{aligned} \overline{U}(u)\hat{a}\overline{U}(u)^{-1} &= u^{-1}\hat{a}, \\ \overline{U}(u)\hat{a}^\dagger\overline{U}(u)^{-1} &= \hat{a}^\dagger u; \\ D(\underline{z}) &= \exp\left(\hat{a}^\dagger \underline{z} - \underline{z}^\dagger \hat{a}\right), \\ \overline{U}(u)D(\underline{z})\overline{U}(u)^{-1} &= D(u\underline{z}). \end{aligned} \tag{3.26}$$

Here $\underline{z} = (z_1, z_2)^T$ is a complex two-component column vector, while $\hat{a}$ and $\hat{a}^\dagger$ are written as column and row vectors respectively.

The reduction of $\overline{U}(u)$ into UIR's is accomplished by the break-up of $\mathcal{H}$ into the mutually orthogonal eigenspaces $\mathcal{H}^{(j)}$ of $\hat{N}$ with eigenvalues $2j$, where $j = 0, 1/2, 1, \ldots$. The orthonormal Fock basis for $\mathcal{H}$ is made up of the simultaneous eigenvectors of $\hat{N}_1$ and $\hat{N}_2$:

$$|n_1, n_2> \; = \frac{\left(\hat{a}_1^\dagger\right)^{n_1} \left(\hat{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1! n_2!}} \; |0, 0>,$$

$$\hat{N}_r \; |n_1, n_2> \; = n_r \; |n_1, n_2>, \quad r = 1, 2. \tag{3.27}$$

For the purposes of reduction of $\overline{U}$, with no danger of confusion we use vectors labelled $|j, m>$ and defined in terms of these Fock states by

$$\begin{aligned} |j, m> \; &= |n_1, n_2>, \\ n_1 = \frac{1}{2}(j+m), n_2 &= \frac{1}{2}(j-m), \\ j = 0, 1/2, 1, \ldots, \quad m &= j, j-1, \ldots, -j. \end{aligned} \tag{3.28}$$

Then the subspaces $\mathcal{H}^{(j)}$ are given by

$$\begin{aligned} \mathcal{H}^{(j)} &= Sp\{|j, m> |j \text{ fixed}, \; m = j, j-1, \ldots, -j\} \\ j &= 0, 1/2, 1, \ldots. \end{aligned} \tag{3.29}$$

The operators $\overline{U}(u)$ leave each $\mathcal{H}^{(j)}$, of dimension $(2j+1)$, invariant, and reduce thereon to the spin j UIR of $SU(2)$, along with the value $2j$ for the $U(1)$ generator $\hat{N}$. This is the known multiplicity- free reduction of the $SU(2)$ Schwinger construction [2]. The projection operator P_j onto the subspace $\mathcal{H}^{(j)}$, which will be needed later, is

$$P_j = \sum_{m=-j}^{+j} \; |j, m >< j, m| = \delta_{\hat{N}, 2j}. \tag{3.30}$$

The H-W SCS use the Fock vacuum $|0, 0>$ as the fiducial vector:

$$|\underline{z} >= D(\underline{z}) \; |0, 0>, \tag{3.31}$$

and on account of eqn.(3.26) they have the $U(2)$ behaviour

$$\overline{U}(u)|\underline{z}>=|\ u\ \underline{z}>. \tag{3.32}$$

This is because $|0,0>$ is invariant under $U(2)$ action; in fact it is the only such vector in $\mathcal{H}$. Therefore the general H-W SCS $|\underline{z}>$ is obtainable by suitable $U(2)$ action from a SCS for the first degree of freedom alone:

$$\begin{aligned}|\underline{z}> &= \overline{U}(u)\ |\underline{z}^{(0)}>,\ \text{suitable}\ u \in\ U(2),\\ \underline{z}^{(0)} &= r\begin{pmatrix}1\\0\end{pmatrix},\\ r^2 &= \underline{z}^{\dagger}\underline{z},\ \ 0 \leq r < \infty.\end{aligned} \tag{3.33}$$

To bring out the connection between these H-W SCS and $SU(2)$ SCS (identified below) in the clearest possible manner, we parametrise $\underline{z}$ and define elements $A(\theta,\phi) \in SU(2)$ in a coordinated manner:

$$\begin{aligned}\underline{z} &= e^{i\alpha}A(\theta,\phi)\underline{z}^{(0)},\\ A(\theta,\phi) &= e^{\frac{-i}{2}\phi\sigma_3}e^{\frac{-i}{2}\theta\sigma_2} \in\ SU(2),\\ &0 \leq \theta \leq \pi,\ \ 0 \leq \alpha,\ \phi \leq 2\pi;\\ z_1 &= r\ e^{i\alpha}e^{-i\phi/2}\cos\theta/2,\ z_2 = r\ e^{i\alpha}e^{i\phi/2}\sin\theta/2.\end{aligned} \tag{3.34}$$

We view θ,ϕ as spherical polar angles on S^2. Then eqn.(3.33) assumes the more detailed form

$$\begin{aligned}|\underline{z}> &= e^{i\alpha\hat{N}}\overline{U}(A(\theta,\phi))|\underline{z}^{(0)}>,\\ |\underline{z}^{(0)}> &= e^{r\left(\hat{a}_1^{\dagger}-\hat{a}_1\right)}|0,0>\\ &= e^{-\frac{1}{2}r^2}\sum_{j=0,1/2,1,\ldots}^{\infty}\frac{r^{2j}}{\sqrt{2j!}}\ |j,j>.\end{aligned} \tag{3.35}$$

The component of $|\underline{z}^{(0)}>$ within $\mathcal{H}^{(j)}$ is a multiple of $|j,j>$, the highest weight vector in the spin j UIR of $SU(2)$. By definition, the $SU(2)$ SCS in any UIR are based on the choice of highest weight vector (or any $SU(2)$ transform of it) as fiducial vector [24]. This vector is the eigenvector of J_3 with maximum eigenvalue j, so any $SU(2)$ transform of it is an eigenvector of a suitable combination of J_k with the same (maximum) eigenvalue. These remarks lead to the following notations for $SU(2)$ SCS:

$$\overline{U}(A(\theta,\phi))\ |j,j>\equiv |j,\hat{n}(\theta,\phi)>$$

$$=\sum_{m=-j}^{j}\sqrt{\frac{2j!}{(j+m)!(j-m)!}}\ e^{-im\phi}(\cos\theta/2)^{j+m}(\sin\theta/2)^{j-m}|j,m>,$$

$$\hat{n}(\theta,\phi)\cdot\vec{J}\ |j,\hat{n}(\theta,\phi)>\ =j|j,\hat{n}(\theta,\phi)>,$$
$$\hat{n}(\theta,\phi)=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)=\frac{1}{r^2}\ \underline{z}^{\dagger}\underline{\sigma}\ \underline{z}\ \in\ S^2. \tag{3.36}$$

Thus the family of $SU(2)$ SCS in the spin j UIR is $\{|j,\hat{n}(\theta,\phi)>\}$, one for each point on S^2 which is the coset space $SU(2)/U(1)$. For these states we have the well-known properties

$$<j,\hat{n}(\theta',\phi')|j,\hat{n}(\theta,\phi)>\ =\Big(\cos\theta'/2\ \cos\theta/2\ e^{i(\phi'-\phi)/2}+\sin\theta'/2\ \sin\theta/2\ e^{i(\phi-\phi')/2}\Big)^{2j}, \tag{3.37a}$$
$$A\in\ SU(2):\overline{U}(A)|j,\hat{n}>\ =e^{i\omega(A;\hat{n})}|j,\ R(A)\hat{n}>, \tag{3.37b}$$

where $R(A)\in\ SO(3)$ is the image of $A\in\ SU(2)$ under the $SU(2)\rightarrow SO(3)$ homomorphism, and $\omega(A;\hat{n})$ is a (Wigner) phase angle [25]. Combining eqns.(3.35, 3.36) we get the connection between H-W and $SU(2)$ SCS:

$$|\underline{z}>=e^{-\frac{1}{2}r^2}\sum_{j=0,1/2,1,\ldots}^{\infty}\frac{(r\ e^{i\alpha})^{2j}}{\sqrt{2j!}}\ |j,\hat{n}(\theta,\phi)>. \tag{3.38}$$

We trace this direct connection to the simple $U(2)$ action (3.32), and the expansion (3.35) of $|\underline{z}^{(0)}>$ in terms of $SU(2)$ highest weight states.

We now look at the Klauder resolution of unity for the H-W SCS, highlighting the $SU(2)$ SCS structure. Using the parametrisation (3.34) for $\underline{z}$ we find:

$$\int\frac{d^2z_1}{\pi}\frac{d^2z_2}{\pi}\ |\underline{z}><\underline{z}|=\frac{1}{4\pi^2}\int_0^{\infty}r^3dr\int_0^{2\pi}d\alpha\int_{S^2}d\Omega(\theta,\phi)|\ \underline{z}><\underline{z}|$$

$$= \frac{1}{4\pi^2} \int r^3 dr\ d\alpha\ d\Omega(\theta,\phi) \sum_{j,j'=0,1/2,1,\dots}^{\infty} e^{-r^2} r^{2(j+j')} e^{2i\alpha(j-j')} \times$$
$$|j,\hat{n}(\theta,\phi)><j',\hat{n}(\theta,\phi)|/\sqrt{2j!\ 2j'!}$$

$$= \frac{1}{2\pi} \sum_{j=0,1/2,1,\dots}^{\infty} \frac{1}{2j!} \int_0^\infty r^3 dr\ e^{-r^2} r^{4j} \int_{S^2} d\Omega(\theta,\phi)\ |j,\hat{n}(\theta,\phi)><j,\hat{n}(\theta,\phi)|. \tag{3.39}$$

Here $d\Omega(\theta,\phi)$ is the element of solid angle on S^2. Using eqn.(3.37b) we see that the integral over S^2 results in an operator invariant under the spin j UIR of $SU(2)$ appearing on $\mathcal{H}^{(j)}$, therefore by Schur lemma for this UIR we have:

$$\int_{S^2} d\Omega(\theta,\phi)\ |j,\hat{n}(\theta,\phi)><j,\hat{n}(\theta,\phi)| = \frac{4\pi}{2j+1}\ P_j. \tag{3.40}$$

Substituting this in eqn.(3.39) we get

$$\begin{aligned}\int \frac{d^2z_1}{\pi}\frac{d^2z_2}{\pi}\ |\underline{z}><\underline{z}| &= 2 \sum_{j=0,1/2,1,\dots}^{\infty} \frac{1}{(2j+1)!} \int_0^\infty r^3 dr\ e^{-r^2}\cdot r^{4j}\cdot P_j \\ &= \sum_{j=0,1/2,1,\dots}^{\infty} P_j \\ &= 1 \text{ on } \mathcal{H}.\end{aligned} \tag{3.41}$$

This is known and expected on account of the Schur lemma for the H-W UIR, since the integration measure is the invariant one on the H-W group. At the same time we can immediately trace the consequences of modifying the measure in a $U(2)$-invariant way, when we lose the possibility of using the lemma for the H-W UIR:

$$A(f) = \int \frac{d^2z_1}{\pi}\frac{d^2z_2}{\pi}\ f(\underline{z}^\dagger\underline{z})\ |\underline{z}><\underline{z}|$$

$$= \sum_{j=0,1/2,1,\dots}^{\infty} \int_0^\infty dx\ f(x) x^{2j+1}\ e^{-x}\ \frac{P_j}{(2j+1)!},$$

$$f \neq \text{constant} \Longleftrightarrow D(\underline{z})\ A(f) \neq A(f)\ D(\underline{z}). \tag{3.42}$$

With the particular choice $f(x) = \delta\left(x - r_0^2\right)$ for real positive r_0, we limit ourselves to an "$SU(2)$-worth" of H-W SCS, and in that case we have:

$$f(x) = \delta\left(x - r_0^2\right):$$

$$A(f) = \int \frac{d^2 z_1}{\pi}\,\frac{d^2 z_2}{\pi}\,\delta\left(\underline{z}^\dagger \underline{z} - r_0^2\right)\,|\underline{z}><\underline{z}|$$

$$= r_0^2 \int_0^{2\pi} \frac{d\alpha}{2\pi} \cdot \int_{S^2} \frac{d\Omega(\theta,\phi)}{4\pi} \cdot \left| e^{i\alpha} A(\theta,\phi) \begin{pmatrix} r_0 \\ 0 \end{pmatrix} \right\rangle \left\langle e^{i\alpha} A(\theta,\phi) \begin{pmatrix} r_0 \\ 0 \end{pmatrix} \right|$$

$$= \sum_{j=0,1/2,1,\dots}^{\infty} e^{-r_0^2} \cdot \frac{\left(r_0^2\right)^{2j+1}}{(2j+1)!}\, P_j. \tag{3.43}$$

The structure of these results (3.42, 3.43) is as expected since $A(f)$ does commute with $\overline{U}(u)$.

Lastly we consider briefly some aspects of H-W GCS in the case of two degrees of freedom. These arise by replacing the Fock vacuum $|0,0>$ by some other (normalised) vector $|\psi_0> \in \mathcal{H}$ as fiducial vector:

$$|\underline{z};\psi_0>= D(\underline{z})\,|\psi_0>. \tag{3.44}$$

Schur lemma and square integrability of the H-W UIR ensure the Klauder formula

$$\int \frac{d^2 z_1}{\pi}\,\frac{d^2 z_2}{\pi}\,|\underline{z};\psi_0><\underline{z};\psi_0| = c.1, \tag{3.45}$$

for some finite constant c. However, if $|\psi_0> \neq |0,0>$, we never have any simple behaviour for these GCS under $U(2)$ action. This is in contrast to eqn.(3.19) in the case of one degree of freedom. The reason is that the only one-dimensional UIR of $SU(2)$ is the trivial UIR, all others are of dimension two or greater. This can be traced to the non Abelian nature of $SU(2)$, in contrast to $U(1)$. For this reason we are unable to obtain $|\underline{z};\psi_0>$ for general $\underline{z}$ from some specially chosen and simpler state $|\underline{z}^{(0)};\psi_0>$ via $U(2)$ action; so the possibility of relating H-W GCS to some sequence of $SU(2)$ GCS's within each subspace $\mathcal{H}^{(j)}$ is also lost. Going one step further, if we consider a modified $U(2)$-invariant measure in place of the translation invariant one in eqn.(3.45), but for a GCS system, and if we define

$$A(f;\psi_0) = \int \frac{d^2 z_1}{\pi}\,\frac{d^2 z_2}{\pi}\,f(\underline{z}^\dagger \underline{z})\,|\underline{z};\psi_0><\underline{z};\psi_0|, \tag{3.46}$$

for $|\psi_0> \neq |0,0>$, this will not commute with $\overline{U}(u)$ and will not reduce to a linear combination of the projections P_j.

IV. RELATION BETWEEN H-W AND $SU(3)$ SCS, RESTRICTION TO $\mathcal{H}_0$

Now that we have explored the relationships between H-W SCS and unitary group SCS for one and two degrees of freedom, we proceed to the $SU(3)$ Schwinger construction recalled in Section 2, and the corresponding H-W SCS for six oscillators. Here we invert the order of development as compared to the previous Section. We recall first the definition of $SU(3)$ SCS within each UIR, then proceed to the H-W system. The specific new feature is the multiplicity problem, to be handled using $Sp(2,R)$.

$\underline{SU(3)\text{ Standard Coherent States}}$

The familiar orthonormal basis states within the UIR (p,q) of $SU(3)$, corresponding to the canonical subgroup chain $U(1) \subset U(2) \subset SU(3)$, consist of a set of isospin-hypercharge multiplets (cf.eqns.$(2.9, 2.11)$) [26]:

$$\begin{aligned}
&|p,q\ ;\ IMY\rangle,\\
&\quad I=\frac{1}{2}(r+s),\ Y=\frac{2}{3}(q-p)+r-s,\\
&\quad M=I,I-1,\ldots,\ -I+1,-I,\\
&\quad 0\leq r\leq p,\ 0\leq s\leq q.
\end{aligned}\tag{4.1}$$

The highest weight state is the one with maximum possible value of M:

$$|p,q;\frac{1}{2}(p+q),\ \frac{1}{2}(p+q),\ \frac{1}{3}(p-q)\rangle.\tag{4.2}$$

In terms of the realisation of the UIR (p,q) via irreducible tensors $T=\left\{T^{j_1\ldots j_p}_{k_1\ldots k_q}\right\}$, this state corresponds to the component

$$T^{11\ldots1}_{22\ldots2}.\tag{4.3}$$

From this one can see that the stability group (upto phase factors) of the state (4.2) is a subgroup $H\subset SU(3)$ dependent on p and q. Disregarding the trivial UIR $(0,0)$, we have:

$$p \geq 1, q = 0 : H = U(2) \text{ on dimensions } 2, 3; \tag{4.4a}$$

$$p = 0, q \geq 1 : H = U(2) \text{ on dimensions } 1, 3; \tag{4.4b}$$

$$p, q \geq 1 : H = \text{diagonal subgroup of } SU(3) \tag{4.4c}$$

(Here the dimensions 1,2,3 refer to the space of the defining UIR $(1, 0)$). In eqn.(4.4a) (eqn.(4.4b)), a $U(2)$ transformation on dimensions 2 and 3 (1 and 3) is to be accompanied by a phase change in dimension 1(2) to preserve unimodularity of the $SU(3)$ transformation. The dimensionalities of these three stability groups are four, four and two respectively.

The $SU(3)$ SCS within the UIR (p, q) are the states obtained by acting with all $SU(3)$ elements on the highest weight state (4.2). They may be written as $|p, q; A >$, $A \in SU(3)$:

$$|p, q; A >= \overline{U}(A)|p, q; \frac{1}{2}(p+q),\ \frac{1}{2}(p+q), \frac{1}{3}(p-q) > . \tag{4.5}$$

Therefore in the UIR's $(p, 0)$ and $(0, q)$, they form four- parameter continuous families of normalised states; while in (p, q) with $p, q \geq 1$ we have six-parameter continuous families. Referring to eqn.(4.4) we have:

$$h \in\ H : |p, q; Ah >= e^{i\varphi(h)}|p, q; A >, \tag{4.6}$$

for some phase $\varphi(h)$.

These $SU(3)$ SCS have been studied in detail in ref. [27] , individually within each UIR. As we see below, the Schwinger construction helps us generate them collectively and explore some of their properties in an efficient manner, just as in eqn.(3.38) we have a construction of the $SU(2)$ SCS in all its UIR's at one stroke.

If within the UIR (p, q) we choose as fiducial vector some vector other than the highest weight vector (4.2) or any $SU(3)$ transform of it, then we obtain a family of $SU(3)$ GCS. For the present we consider only SCS's, turning to particular GCS's in subsequent Sections.

In the Hilbert space $\mathcal{H}$ of the $SU(3)$ Schwinger construction the 'first' occurrence of the UIR (p, q) is in the subspace $\mathcal{H}^{(p,q;0)} \subset \mathcal{H}_0$ which is annihilated by K_-. The corresponding highest weight state (4.2), using the complete notation of eqn.(2.9) and recalling eqn.(4.3), is:

$$|p,q;\frac{1}{2}(p+q),\ \frac{1}{2}(p+q),\frac{1}{3}(p-q);\ \frac{1}{2}(p+q+3)>=$$

$$\frac{\left(\hat{a}_1^\dagger\right)^p\left(\hat{b}_2^\dagger\right)^q}{\sqrt{p!q!}}\ |\underline{0},\underline{0}>\in\ \mathcal{H}^{(p,q;0)}\subset\mathcal{H}_0. \tag{4.7}$$

It follows that all these highest weight states, one for each UIR (p,q), are generated by the special H-W SCS

$$|z_1,0,0;0,w_2,0>=D(z_1,0,0,0,w_2,0)|\underline{0},\underline{0}>\in\ \mathcal{H}_0,$$

$$\begin{aligned}D(\underline{z},\underline{w})&=\exp\left(\underline{z}\cdot\underline{\hat{a}}^\dagger-\underline{z}^*\cdot\underline{\hat{a}}+\underline{w}\cdot\underline{\hat{b}}^\dagger-\underline{w}^*\cdot\underline{\hat{b}}\right)\\&=\exp\left(-\frac{1}{2}\underline{z}^\dagger\underline{z}\ -\frac{1}{2}\underline{w}^\dagger\underline{w}\ +\underline{z}\cdot\underline{\hat{a}}^\dagger\ +\underline{w}\cdot\underline{\hat{b}}^\dagger\right).\end{aligned} \tag{4.8}$$

Here $\underline{z}$ and $\underline{w}$ are independent complex 3-vectors, and $D(\underline{z},\underline{w})$ are the displacement operators for the six-oscillator system of the Schwinger construction. Indeed we have:

$$|z_1,0,0;0,w_2,0>\ =e^{-\frac{1}{2}|z_1|^2-\frac{1}{2}|w_2|^2}\sum_{p,q=0}^{\infty}\frac{z_1^p w_2^q}{\sqrt{p!q!}}\times$$

$$|p,q;\frac{1}{2}(p+q),\frac{1}{2}(p+q),\frac{1}{3}(p-q);\frac{1}{2}(p+q+3)>, \tag{4.9}$$

which is analogous to the second of eqns.(3.35). We will use this below.

$SU(3)$ analysis of the H-W SCS

For the six oscillator system used in the Schwinger $SU(3)$ construction the H-W SCS are labelled by two complex three-dimensional vectors $\underline{z}$ and $\underline{w}$, thus the pair $(\underline{z},\underline{w})$ is a point in $\mathcal{C}^6$. They are obtained by applying the displacement operators $D(\underline{z},\underline{w})$ to the Fock vacuum $|\underline{0},\underline{0}>$ as fiducial vector:

$$|\underline{z},\underline{w}>=D(\underline{z},\underline{w})\ |\underline{0},\underline{0}>. \tag{4.10}$$

We see from eqn.(2.5) that they are eigenstates of the $Sp(2,R)$ lowering operator $K_- = K_1 - iK_2$:

$$\begin{aligned}\hat{a}_j|\underline{z},\underline{w}> &= z_j|\underline{z},\underline{w}>,\\ \hat{b}_j|\underline{z},\underline{w}> &= w_j|\underline{z},\underline{w}>,\\ K_-|\underline{z},\underline{w}> &= \underline{z}^T\underline{w}|\underline{z},\underline{w}> .\end{aligned} \tag{4.11}$$

Therefore only those SCS $|\underline{z},\underline{w}>$ for which $\underline{z}^T\underline{w} = 0$ belong to $\mathcal{H}_0$. The complete set of SCS obeys the Klauder resolution of the identity,

$$\int\limits_{C^6}\prod_{j=1}^{3}\left(\frac{d^2z_j}{\pi}\ \frac{d^2w_j}{\pi}\right)\ |\underline{z},\underline{w}\rangle\langle\underline{z},\underline{w}| = 1 \text{ on } \mathcal{H}, \tag{4.12}$$

the integration measure being the invariant one on the H-W group.

We now explore the behaviour of these SCS under $SU(3)$ action. From the manner in which the generators Q_α are constructed in eqn.(2.2) we have:

$$A\ \in\ SU(3) : \mathcal{U}(A)\ D(\underline{z},\underline{w})\,\mathcal{U}(A)^{-1} = D(A\underline{z}, A^*\underline{w}), \tag{4.13}$$

from which it follows that

$$\mathcal{U}(A)|\underline{z},\underline{w}>= |A\underline{z}, A^*\underline{w}> . \tag{4.14}$$

The independent invariants under this action are $\underline{z}^\dagger\underline{z}, \underline{w}^\dagger\underline{w}$ and $\underline{z}^T\underline{w}$, the last being the eigenvalue of K_-. We describe them using four real independent parameters u, v, x, y as

$$\begin{aligned}&\underline{z}^\dagger\underline{z} = u^2, \underline{w}^\dagger\underline{w} = v^2\ ,\ \underline{z}^T\underline{w} = uv(x+iy),\\ &u, v \geq 0, 0 \leq x^2+y^2 \leq 1.\end{aligned} \tag{4.15}$$

The upper bound on x^2+y^2 is an expression of the Cauchy-Schwarz inequality. For each set of values of (u, v, x, y), the SCS $|\underline{z},\underline{w}>$ form an orbit under $SU(3)$ action. On each orbit we can choose a convenient representative point $\left(\underline{z}^{(0)},\underline{w}^{(0)}\right)$, with any other point $(\underline{z},\underline{w})$ on the orbit arising from $\left(\underline{z}^{(0)},\underline{w}^{(0)}\right)$ via suitable $SU(3)$ action as $\left(A\underline{z}^{(0)}, A^*\underline{w}^{(0)}\right)$. The complete list of orbits, representative points, stability subgroups $H\left(\underline{z}^{(0)},\underline{w}^{(0)}\right)\subset SU(3)$ and orbit dimensions are as follows (with x,y omitted when irrelevant):

$$
\begin{aligned}
&a)\ \ \vartheta_1=\{u,v|u=v=0\},(\underline{z}^{(0)},\underline{w}^{(0)})=(\underline{0},\underline{0}),H=SU(3),\text{dimension }0;\\
&b)\ \ \vartheta_2=\{u,v|u>0,v=0\},\underline{z}^{(0)}=u(1,0,0)^T,\underline{w}^{(0)}=\underline{0},H=SU(2),\ \text{dimension }5;\\
&c)\ \ \vartheta_3=\{u,v|u=0,v>0\},\underline{z}^{(0)}=\underline{0},\underline{w}^{(0)}=v(0,1,0)^T,H=SU(2),\ \text{dimension }5;\\
&d)\ \ \vartheta_4=\{u,v,x,y|u,v>0,0\leq x^2+y^2<1\},\\
&\qquad\qquad \underline{z}^{(0)}=u(1,0,0)^T,\underline{w}^{(0)}=v\left(x+iy,\sqrt{1-x^2-y^2},0\right)^T,H=\{e\},\ \text{dimension }8;\\
&e)\ \ \vartheta_5=\{u,v,x,y|u,v>0,\ x^2+y^2=1\},\\
&\qquad\qquad \underline{z}^{(0)}=u(0,0,1)^T,\ \underline{w}^{(0)}=v(x+iy)(0,0,1)^T,H=SU(2),\ \text{dimension }5.
\end{aligned}
\tag{4.16}
$$

We add some comments: Class (a) comprises just the Fock vacuum $|\underline{0},\underline{0}>$, invariant under $SU(3)$ and forming a trivial orbit by itself. Classes (b) and (c) form collections of orbits with one of $\underline{z}$ and $\underline{w}$ vanishing identically, so these are simply SCS for systems of three oscillators. Class (d) is a four parameter family consisting of generic orbits. Each orbit in this Class is eight dimensional and is essentially the $SU(3)$ group manifold. Class (e) is a limiting form, as $x^2+y^2\rightarrow 1$, of Class (d); in these orbits, $\underline{w}$ is a complex multiple of $\underline{z}^*$. However the limit is a singular one, as is evident from the rise in the dimension of H from zero to three, and the drop in orbit dimension from eight to five. This is why we have listed Class (e) separately. Moreover, the representative point $(z^{(0)},w^{(0)})$ in this class has been chosen so that the stability group $SU(2)$ acts on dimensions 1 and 2, thus coinciding with the subgroup relevant for the canonical basis (4.21). Disregarding Class (a), and recalling that $\mathcal{C}^6$ is of real dimension 12, we see that Classes (b), (c), (d), (e) are non overlapping regions in $\mathcal{C}^6$ of real dimensions 6, 6, 12 and 8 respectively. Thus almost all of $\mathcal{C}^6$ is covered by orbits of Class (d).

Based on this orbit structure, we now express the Klauder resolution of the identity, eqn.(4.12), in a manner similar to eqn.(3.39), namely as an integration over the $SU(3)$ manifold followed by an integration over the invariants (4.15). (The difference compared to the case of two degrees of freedom is that here we integrate over the whole of $SU(3)$, not just over a coset space such as $SU(2)/U(1) = S^2$ in eqn.(3.39)). In this process we can limit ourselves to Class (d) orbits which are generic, as long as we do not at any later stage alter the integrand of eqn.(4.12) by inserting a Dirac delta function with support in one of the exceptional orbits in eqn.(4.16). To obtain a general pair $(\underline{z}, \underline{w})$ from $\left(\underline{z}^{(0)}, \underline{w}^{(0)}\right)$ in eqn.(4.16) Class (d), we need to parametrise (almost all) elements of $SU(3)$ in a convenient manner. Here we use the fact that, except on a set of vanishing measure, each $A \in SU(3)$ is uniquely determined by a pair $\left(\underline{\hat{\eta}}, \underline{\hat{\zeta}}\right)$, where $\underline{\hat{\eta}}$ is a complex three-component unit vector and $\underline{\hat{\zeta}}$ is a complex two- component unit vector [28]:

$$\begin{aligned} &\underline{\hat{\eta}} = (\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3)^T \ , \ \underline{\hat{\zeta}} = \left(\hat{\zeta}_2, \hat{\zeta}_3\right)^T, \\ &\underline{\hat{\eta}}^\dagger \underline{\hat{\eta}} = \underline{\hat{\zeta}}^\dagger \underline{\hat{\zeta}} = 1. \end{aligned} \tag{4.17}$$

Then we have:

$$A \in \ SU(3) \Longleftrightarrow A = A\left(\underline{\hat{\eta}}, \underline{\hat{\zeta}}\right) = A_3\left(\underline{\hat{\eta}}\right) \ A_2\left(\underline{\hat{\zeta}}\right),$$

$$\begin{aligned} A_3(\hat{\eta}) &= \begin{pmatrix} \hat{\eta}_1 & \rho_1 & 0 \\ \hat{\eta}_2 & -\hat{\eta}_2\hat{\eta}_1^*/\rho_1 & \hat{\eta}_3^*/\rho_1 \\ \hat{\eta}_3 & -\hat{\eta}_3\hat{\eta}_1^*/\rho_1 & -\hat{\eta}_2^*/\rho_1 \end{pmatrix} \in \ SU(3), \\ \rho_1 &= \left(1 - |\hat{\eta}_1|^2\right)^{1/2}; \\ A_2\left(\underline{\hat{\zeta}}\right) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \hat{\zeta}_2 & -\hat{\zeta}_3^* \\ 0 & \hat{\zeta}_3 & \hat{\zeta}_2^* \end{pmatrix} \in \ SU(2) \subset SU(3). \end{aligned} \tag{4.18}$$

For each $\underline{\hat{\eta}}$ (provided $|\hat{\eta}_1| < 1$), $A_3\left(\underline{\hat{\eta}}\right)$ is a particular $SU(3)$ element completely determined by its first column which is $\underline{\hat{\eta}}$; and for each $\underline{\hat{\zeta}}$, $A_2\left(\underline{\hat{\zeta}}\right)$ is an element in the $SU(2)$ subgroup leaving $\underline{z}^{(0)}$ invariant. We can picture $\underline{\hat{\eta}}$ and $\underline{\hat{\zeta}}$ as representing points on $S^5 \subset \mathcal{R}^6$ and $S^3 \subset \mathcal{R}^4$ respectively. Then the normalised invariant volume element on $SU(3)$ is a numerical factor times the product of the solid angle elements on S^5 and S^3:

$$dA\left(\underline{\hat{\eta}},\underline{\hat{\zeta}}\right) = (2\pi^5)^{-1}\, d\Omega_5\left(\underline{\hat{\eta}}\right)\, d\Omega_3\left(\underline{\hat{\zeta}}\right),$$
$$\int\limits_{SU(3)} dA = 1. \tag{4.19}$$

The expressions for $(\underline{z},\underline{w})$ in terms of $A\left(\underline{\hat{\eta}},\underline{\hat{\zeta}}\right)$ and $\left(\underline{z}^{(0)},\underline{w}^{(0)}\right)$ are:

$$\underline{z} = A\left(\underline{\hat{\eta}},\underline{\hat{\zeta}}\right)\,\underline{z}^{(0)}(u) = u\,\underline{\hat{\eta}},$$
$$\underline{w} = A\left(\underline{\hat{\eta}},\underline{\hat{\zeta}}\right)^*\,\underline{w}^{(0)}(v,x,y) = v\,A_3\left(\underline{\hat{\eta}}\right)^*\begin{pmatrix} x+iy \\ \sqrt{1-x^2-y^2}\quad \hat{\zeta}_2^* \\ \sqrt{1-x^2-y^2}\quad \hat{\zeta}_3^* \end{pmatrix}. \tag{4.20}$$

These are the generalisations of eqn.(3.34). Straight forward computations of the Jacobians yield:

$$\prod_{j=1}^{3}\left(d^2z_j\,d^2w_j\right) = u^5v^5(1-x^2-y^2)du\,dv\,dx\,dy\,d\Omega_5\left(\underline{\hat{\eta}}\right)\,d\Omega_3\left(\underline{\hat{\zeta}}\right). \tag{4.21}$$

We can now rewrite the Klauder result (4.12) as:

$$\frac{2}{\pi}\int\limits_0^\infty u^5du\int\limits_0^\infty v^5\,dv\int\limits_{x^2+y^2\le 1}(1-x^2-y^2)dx\,dy\int\limits_{SU(3)} dA\,\mathcal{U}(A)|\underline{z}^{(0)}(u),\underline{w}^{(0)}(v,x,y)\rangle\times$$
$$\langle\underline{z}^{(0)}(u),\underline{w}^{(0)}(v,x,y)|\mathcal{U}(A)^{-1} = 1 \text{ on } \mathcal{H}. \tag{4.22}$$

This is the analogue of (the initial form of) eqn.(3.39).

In the spirit of eqns.(3.10, 3.42) we can now consider modifications of eqn.(4.22) by including in the integrand a function of the $SU(3)$ invariants. Thus we define

$$A(f) = \int\prod_{j=1}^{3}\left(\frac{d^2z_j}{\pi}\,\frac{d^2w_j}{\pi}\right)\,f(u,v,x,y)\,|\underline{z},\underline{w}><\underline{z},\underline{w}|$$
$$= \frac{2}{\pi}\int\limits_0^\infty u^5du\int\limits_0^\infty v^5dv\int\limits_{x^2+y^2\le 1}(1-x^2-y^2)dx\,dy\,f(u,v,x,y)\int\limits_{SU(3)} dA\;\times$$
$$\mathcal{U}(A)|\underline{z}^{(0)}(u),\underline{w}^{(0)}(v,x,y)><\underline{z}^{(0)}(u),\underline{w}^{(0)}(v,x,y)|\mathcal{U}(A)^{-1}. \tag{4.23}$$

Such an operator definitely obeys

$$\mathcal{U}(A)\, A(f) = A(f)\, \mathcal{U}(A),\ \text{all}\ A \in\ SU(3). \tag{4.24}$$

However, as long as $f(u, v, x, y)$ is nontrivial, the measure in eqn.(4.23) is not the invariant one on the H-W group, we do not have recourse to Schur lemma for the UIR of this group, and $A(f)$ is not proportional to the identity operator on $\mathcal{H}$. The presence of (infinite!) multiplicity in the reduction of $\mathcal{U}(A)$ on $\mathcal{H}$ into UIR's of $SU(3)$ means furthermore that we do not immediately get for $A(f)$ a simple combination of $SU(3)$-invariant projections as we did in eqns.(3.42, 3.43) with $SU(2)$.

The restriction to $\mathcal{H}_0$

Now we limit ourselves to the SCS $|\underline{z}, \underline{w} >$ belonging to $\mathcal{H}_0 \subset \mathcal{H}$, as in this subspace the multiplicity problem is avoided. As noted following eqn. (4.11), the condition $\underline{z}^T \underline{w} = 0$ ensures $|\underline{z}, \underline{w} > \in\ \mathcal{H}_0$. This happens in Classes (a), (b), (c) of eqn.(4.16) in a trivial manner, and in Class (d) when $x = y = 0$. The former can be disregarded as being sets of vanishing measure.

We deal first with vector level relations in $\mathcal{H}_0$, then look at modifications of $A(f)$ in eqn.(4.23). We begin with eqn.(4.9). For the highest weight states of $SU(3)$ UIR's occurring there, we introduce a simpler notation:

$$|p, q; \frac{1}{2}(p+q), \frac{1}{2}(p+q); \frac{1}{3}(p-q); \frac{1}{2}(p+q+3)\rangle \equiv |p, q; \frac{1}{2}(p+q), \frac{1}{2}(p+q); \frac{1}{3}(p-q)\rangle_0$$

$$\in\ \mathcal{H}^{(p,q;0)} \subset \mathcal{H}_0. \tag{4.25}$$

We have omitted the $Sp(2, R)$ quantum number m as it is superfluous within $\mathcal{H}_0$. Then eqn.(4.9) takes the form

$$\underline{z}^{(0)}(u) = u(1, 0, 0)^T,\ \underline{w}^{(0)}(v, 0, 0) = v(0, 1, 0)^T\ :$$

$$|\underline{z}^{(0)}(u), \underline{w}^{(0)}(v, 0, 0)\rangle = e^{-\frac{1}{2}(u^2+v^2)} \sum_{p,q=0}^{\infty} \frac{u^p\ v^q}{\sqrt{p!q!}}$$

$$|p, q; \frac{1}{2}(p+q), \frac{1}{2}(p+q), \frac{1}{3}(p-q)\rangle_0\ \in\ \mathcal{H}_0. \tag{4.26}$$

In place of eqn.(4.5), the $SU(3)$ SCS within each UIR (p,q) contained in $\mathcal{H}_0$ can be written as

$$A \in \ SU(3) : |p, q; A\rangle_0 = \mathcal{U}(A)|p, q; \frac{1}{2}(p+q), \frac{1}{2}(p+q), \frac{1}{3}(p-q)\rangle_0 \ \in \ \mathcal{H}^{(p,q;0)} \tag{4.27}$$

Applying $\mathcal{U}(A)$ for general $A \in \ SU(3)$ to both sides of eqn.(4.26) we get a result linking those H-W SCS that lie in $\mathcal{H}_0$, and the $SU(3)$ SCS (4.27) within each UIR (p,q) in $\mathcal{H}_0$:

$$A \in \ SU(3), \underline{z} = A\underline{z}^{(0)}(u), \underline{w} = A^*\underline{w}^{(0)}(v,0,0) :$$

$$|\underline{z}, \underline{w}\rangle = e^{-\frac{1}{2}(u^2+v^2)} \sum_{p,q=0}^{\infty} \frac{u^p \ v^q}{\sqrt{p!q!}} \ |p, q; A\rangle_0 \ \in \ \mathcal{H}_0. \tag{4.28}$$

This is the $SU(3)$ analogue to the $SU(2)$ relation (3.38).

Now we turn to the operator $A(f)$ in eqn.(4.23) and make the choice

$$f(u, v, x, y) = f_0(u, v) \ \delta(x) \ \delta(y). \tag{4.29}$$

This leads to

$$A(f_0) = \frac{2}{\pi} \int_0^\infty u^5 du \int_0^\infty v^5 dv \ f_0(u, v) \int_{SU(3)} dA \, \mathcal{U}(A)$$

$$|\underline{z}^{(0)}(u), \underline{w}^{(0)}(v,0,0)\rangle\langle\underline{z}^{(0)}(u), \underline{w}^{(0)}(v,0,0) \ |\mathcal{U}(A)^{-1}. \tag{4.30}$$

Such an operator obeys the following:

$$\begin{aligned} \psi \in \mathcal{H}_0^\perp &: A(f_0)\psi = 0; \\ \psi \in \mathcal{H}_0 &: A(f_0)\psi \in \mathcal{H}_0; \\ A \in \ SU(3) &: \mathcal{U}(A) \ A(f_0) = A(f_0) \, \mathcal{U}(A). \end{aligned} \tag{4.31}$$

Therefore $A(f_0)$ must be a linear combination of the projection operators $P^{(p,q;0)}$ onto the subspaces $\mathcal{H}^{(p,q;0)} \subset \mathcal{H}_0$; it is here that we exploit the multiplicity-free reduction of the $SU(3)$ UR $\mathcal{D}_0$ on $\mathcal{H}_0$. To get $A(f_0)$ explicitly, we use the following immediate consequences of Schur lemma applied to $SU(3)$, the multiplicity-free nature of $\mathcal{D}_0$, and the orthogonality of inequivalent UIR's:

$$\int_{SU(3)} dA \; |p,q;A\rangle_0 \; {}_0\langle p',q';A| = \delta_{p'p}\delta_{q'q} \; P^{(p,q;0)}/d(p,q). \tag{4.32}$$

Then a combination of eqns.(4.30, 4.26, 4.27, 4.32) immediately gives:

$$\begin{aligned} A(f_0) &= \sum_{p,q=0}^{\infty} C(f_0;p,q) \; P^{(p,q;0)}, \\ C(f_0;p,q) &= \{p!q! \; d(p,q)\}^{-1} \cdot \frac{2}{\pi} \cdot \int_0^{\infty} u^5 du \int_0^{\infty} v^5 dv \; f_0(u,v) u^{2p} v^{2q} e^{-(u^2+v^2)}. \end{aligned} \tag{4.33}$$

This is an $SU(3)$ analogue of the $SU(2)$ result (3.42), but it is valid only after the restriction to $\mathcal{H}_0$. On account of the freedom still remaining in eqns.(4.30, 4.33) in the choice of the function $f_0(u,v)$, we see that the H-W SCS occurring there are overcomplete in $\mathcal{H}_0$. If we wish to limit ourselves to an exact "$SU(3)$- worth" of H-W SCS within $\mathcal{H}_0$, then we have the analogue to eqn.(3.43):

$$\begin{aligned} f_0(u,v) &= \delta(u-u_0)\;\delta(v-v_0): \\ A(f_0) &= \int \prod_{j=1}^{3} \left(\frac{d^2 z_j}{\pi} \; \frac{d^2 w_j}{\pi} \right) \delta(u-u_0)\delta(v-v_0)\delta(x)\delta(y) |\underline{z},\underline{w}\rangle\langle\underline{z},\underline{w}| \\ &= e^{-\left(u_0^2+v_0^2\right)} \cdot \frac{2}{\pi} \sum_{p,q=0}^{\infty} u_0^{2p+5} v_0^{2q+5} P^{(p,q;0)}/p!q!d(p,q). \end{aligned} \tag{4.34}$$

The point to be emphasised is how far this result departs from being the identity operator in $\mathcal{H}_0$, leave alone in $\mathcal{H}$, but understandably so.

Description in $\mathcal{H}^{(\mathrm{ind})}$

As recalled in Section II, and established in detail in I, the multiplicity-free UR $\mathcal{D}_0$ of $SU(3)$ on $\mathcal{H}_0$ is equivalent to an induced UR $\mathcal{D}^{(\mathrm{ind})}$ of $SU(3)$, namely the one arising from the trivial representation of an $SU(2)$ subgroup of $SU(3)$. The isomorphism between $\mathcal{H}_0$ and $\mathcal{H}^{(\mathrm{ind})}$ carrying $\mathcal{D}^{(\mathrm{ind})}$, consistent with the two group actions, is given in eqn.(2.17). It is of interest to see what wave functions $\psi(\underline{\xi}) \in \mathcal{H}^{(\mathrm{ind})}$ one obtains for the various vectors in $\mathcal{H}_0$ that have played a role earlier in this Section. We now give these wave functions and comment briefly on them.

For the highest weight state in the $SU(3)$ UIR (p,q) on $\mathcal{H}^{(p,q;0)}$, and the associated $SU(3)$ SCS, we find the following wavefunctions in $\mathcal{H}^{(\mathrm{ind})}$:

$$|p,q;\frac{1}{2}(p+q),\frac{1}{2}(p+q),\frac{1}{3}(p-q)\rangle_0 \longrightarrow \sqrt{\frac{(p+q+2)!}{p!q!}}\,(\xi_1)^p\,(\xi_2^*)^q\,;$$

$$|p,q;A\rangle_0 = \mathcal{U}(A)|p,q;\frac{1}{2}(p+q),\frac{1}{2}(p+q),\frac{1}{3}(p-q)\rangle_0 \longrightarrow$$

$$\sqrt{\frac{(p+q+2)!}{p!q!}}\,\left(A_{j1}^*\xi_j\right)^p (A_{k2}\xi_k^*)^q\,. \tag{4.35}$$

For the H-W SCS in $\mathcal{H}_0$ generating these states within each UIR we have:

$$|\underline{z}^{(0)}(u),\underline{w}^{(0)}(v,0,0)\rangle \longrightarrow e^{-\frac{1}{2}(u^2+v^2)}\sum_{p,q=0}^{\infty}\sqrt{(p+q+2)!}\,\frac{(u\xi_1)^p}{p!}\,\frac{(v\xi_2^*)^q}{q!};$$

$$|\underline{z},\underline{w}\rangle = \mathcal{U}(A)|\underline{z}^{(0}(u),\underline{w}^{(0)}(v,0,0)\rangle \longrightarrow$$

$$e^{-\frac{1}{2}(u^2+v^2)}\sum_{p,q=0}^{\infty}\sqrt{(p+q+2)!}\,\frac{\left(uA_{j1}^*\xi_j\right)^p}{p!}\,\frac{(vA_{k2}\xi_k^*)^q}{q!}. \tag{4.36}$$

The principal comment we may make is that these particular H-W SCS do not have wave functions in $\mathcal{H}^{(\mathrm{ind})}$ in the form of any simple expressions involving exponential functions. The reason for this can be traced to the factorial in eqn.(2.17) as compared to eqn.(2.14a). Another way of understanding this situation is to realise that $\mathcal{H}_0$ (and so $\mathcal{H}^{(\mathrm{ind})}$ as well) is too small to carry a representation of the H-W system used in the Schwinger $SU(3)$ construction; in addition the argument $\underline{\xi}$ in $\psi(\underline{\xi})$ is a complex unit vector in three dimensions rather than a variable in all of $\mathcal{C}^3$.

V. GENERAL EIGENSPACES $\mathcal{H}_\kappa$ OF K_-

The subspace $\mathcal{H}_0 \subset \mathcal{H}$ carrying the multiplicity-free UR $\mathcal{D}_0$ of $SU(3)$, the focus of analysis in the preceding Section, is spanned by those H-W SCS $|\underline{z},\underline{w}>$ for which $\underline{z}^T\underline{w}=0$, and belonging to a particular collection of orbits under Class (d) of eqn.(4.16):

$$\mathcal{H}_0 = Sp\{|\underline{z}, \underline{w}> |\underline{z}, \underline{w} \in \mathcal{C}^3,\ \underline{z}^T \underline{w} = 0\}. \tag{5.1}$$

As noted earlier, these SCS are actually over complete within $\mathcal{H}_0$. Since, by eqn.(4.11), $\underline{z}^T \underline{w}$ is the eigenvalue of the $SU(3)$ invariant $Sp(2, R)$ lowering operator K_-, this means that $\mathcal{H}_0$ is spanned by those H-W SCS that are eigenvectors of K_- with eigenvalue zero. Moreover, eqns.(4.26, 4.28) show that these H-W SCS are directly connected to the $SU(3)$ SCS within each $SU(3)$ UIR (p, q), carried by $\mathcal{H}^{(p,q;0)} \subset \mathcal{H}_0$.

It now turns out that a somewhat similar situation exists involving eigenvectors of K_- corresponding to nonzero eigenvalues as well, but with one major difference: we encounter certain specific $SU(3)$ GCS systems. This also connects up with a certain class of coherent states within the UIR's $D_k^{(+)}$ of $Sp(2, R)$. We analyse these matters in this Section.It turns out that H-W SCS of both Classes (d) and (e) are involved.

We begin by generalising eqn.(5.1) and defining a subspace $\mathcal{H}_\kappa \subset \mathcal{H}$, for any complex number κ, as consisting of eigenvectors of K_- with eigenvalue κ; equally well it is the span of all those H-W SCS which obey this condition:

$$\begin{aligned}\mathcal{H}_\kappa &= \{|\psi> \in \mathcal{H} | K_- |\psi> = \kappa |\psi>\} \\ &= Sp\{|\underline{z}, \underline{w}> |\underline{z}, \underline{w} \in \mathcal{C}^3,\ \underline{z}^T \underline{w} = \kappa\} \subset \mathcal{H}.\end{aligned} \tag{5.2}$$

These H-W SCS comprise a particular subset of Class (d) orbits in eqn.(4.16); for $\kappa = 0$ we get back $\mathcal{H}_0$. It is important to remark that even though κ varies over a continuum, each $\mathcal{H}_\kappa$ consists of bona fide (ie., normalisable) vectors in $\mathcal{H}$; and for $\kappa' \neq \kappa$, $\mathcal{H}_{\kappa'}$ and $\mathcal{H}_\kappa$ are not mutually orthogonal. As in the case of the oscillator annihilation operator, these are consequences of K_- being non hermitian. Since K_- is $SU(3)$ invariant, each $\mathcal{H}_\kappa$ is $SU(3)$ invariant as well:

$$A \in\ SU(3),\ |\psi> \in\ \mathcal{H}_\kappa \Longrightarrow \mathcal{U}(A)|\psi> \in\ \mathcal{H}_\kappa. \tag{5.3}$$

Therefore the UR $\mathcal{U}(A)$ of $SU(3)$ on $\mathcal{H}$, when restricted to $\mathcal{H}_\kappa$ leads to a UR $\mathcal{D}_\kappa$ acting on $\mathcal{H}_\kappa$. We will see that this UR contains each UIR (p, q) exactly once, just like $\mathcal{D}_0$ on $\mathcal{H}_0$. Thus it is also multiplicity-free and complete.

To exhibit these properties, we first recall the construction of eigenvectors of K_- in any discrete class UIR $D_k^{(+)}$ of $Sp(2,R)$ [29]. (Though the following results are valid for all real $k>0$, we require only the cases $k=3/2,2,5/2\ldots$). As in eqn.(I.3.24,25), denote the eigenvectors of J_0 in $D_k^{(+)}$ by $|k,m>$. Then we have the well-known results:

$$
\begin{aligned}
|k,\kappa> &= \left({}_0F_1(2k;|\kappa|^2)\right)^{-1/2}\sum_{m=k}^{\infty}(\Gamma(2k)/(m-k)!\Gamma(m+k))^{1/2}\,\kappa^{m-k}|k,m>,\\
K_-|k,\kappa> &= \kappa|k,\kappa>,\quad \kappa\in\mathcal{C}; \qquad (5.4a)\\
<k,\kappa'|k,\kappa> &= {}_0F_1(2k;\kappa'^*\kappa)\Big/\left\{{}_0F_1(2k;|\kappa'|^2)\,{}_0F_1(2k;|\kappa|^2)\right\}^{1/2}; \qquad (5.4b)
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathcal{C}}\frac{d^2\kappa}{\pi}\,\sigma(|\kappa|^2)|k,\kappa><k,\kappa|=1,\\
&\sigma(|\kappa|^2)=\frac{2}{\Gamma(2k)}\,{}_0F_1(2k;|\kappa|^2)|\kappa|^{2k-1}K_{\frac{1}{2}-k}(2|\kappa|), \qquad (5.4c)
\end{aligned}
$$

where $K_\nu(z)$ denotes modified Bessel function of the third kind. (For simplicity the k-dependence of the weight function σ is omitted). We note that even though these states $|k,\kappa>$ within $D_k^{(+)}$ do not form an $Sp(2,R)$ orbit, they do furnish a Klauder-type resolution of the identity.

We now exploit this construction in the present context. We begin with two facts: (a) the vectors $|p,q;IMY;m>$, as all labels vary, form an orthonormal basis for the total Hilbert space $\mathcal{H}$; (b) if we keep p,q,IMY fixed and allow only m to vary, we get an orthonormal basis for a subspace carrying just the UIR $D_k^{(+)}$ of $Sp(2,R)$. Therefore, in view of the construction (5.4), within each such subspace we can define and have:

$$
\begin{aligned}
|p,q;IMY\rangle_\kappa &= \left\{{}_0F_1(2k;|\kappa|^2)\right\}^{-1/2}\sum_{m=k}^{\infty}((2k-1)!/(m-k)!(m+k-1)!)^{1/2}\times\\
&\qquad \kappa^{m-k}|p,q;IMY;m\rangle,\\
K_-|p,q;IMY\rangle_\kappa &= \kappa|p,q;IMY\rangle_\kappa;\\
{}_{\kappa'}\langle p',q';I'M'Y'|p,q;IMY\rangle_\kappa &= \delta_{p'p}\delta_{q'q}\delta_{I'I}\delta_{M'M}\delta_{Y'Y}\times\\
&\qquad {}_0F_1\left(2k;\kappa'^*\kappa\right)\Big/\left\{{}_0F_1(2k;|\kappa'|^2)\,{}_0F_1(2k;|\kappa|^2)\right\}^{1/2}. \qquad (5.5)
\end{aligned}
$$

(For fixed p,q,IMY we also have a resolution of the appropriate identity in the form of eqn.(5.4c), but we omit it). For $\kappa=0$ we recover the orthonormal basis for $\mathcal{H}_0$. However for

$\kappa \neq 0$, these vectors are not eigenvectors of the total a-type and b-type number operators $\hat{N}^{(a)}, \hat{N}^{(b)}$. It is now evident that if we keep κ fixed, allow $pqIMY$ to vary, and recall that the range $\mathcal{C}$ of κ is k-independent, we get an orthonormal basis for $\mathcal{H}_\kappa$:

$$\mathcal{H}_\kappa = Sp\{|p,q;IMY\rangle_\kappa | \kappa \text{ fixed, } pqIMY \text{ varying}\},$$

$$_\kappa\langle p',q';I'M'Y'|p,q;IMY\rangle_\kappa = \delta_{p'p}\delta_{q'q}\delta_{I'I}\delta_{M'M}\delta_{Y'Y}. \tag{5.6}$$

It is also clear that each UIR (p,q) of $SU(3)$, carried by the $d(p,q)$ vectors $|p,q;IMY\rangle_\kappa \in \mathcal{H}_\kappa$ as IMY alone vary, appears exactly once in $\mathcal{H}_\kappa$. In other words, $\mathcal{D}_\kappa$ is multiplicity-free. In eqns.(5.2, 5.6) we have three equally good ways of identifying the subspace $\mathcal{H}_\kappa \subset \mathcal{H}$.

We next relate the orthonormal basis vectors (5.6) for $\mathcal{H}_\kappa$ to the corresponding ones for $\mathcal{H}_0$ in eqn.(2.12), in a compact manner. For this we use eqn. (I.3.25b) valid within each UIR $D_k^{(+)}$ of $Sp(2,R)$, along with $K_+ = \underline{\hat{a}}^\dagger \cdot \underline{\hat{b}}^\dagger$:

$$|p,q;IMY;m\rangle = ((2k-1)!/(m-k)!(m+k-1)!)^{1/2}\left(\underline{\hat{a}}^\dagger \cdot \underline{\hat{b}}^\dagger\right)^{m-k}|p,q;IMY;k\rangle;$$

$$\begin{aligned}|p,q;IMY\rangle_\kappa &= \left\{{}_0F_1(2k;|\kappa|^2)\right\}^{-1/2}\sum_{m=k}^{\infty}\frac{(2k-1)!}{(m-k)!(m+k-1)!}\times \\ &\qquad \left(\kappa\underline{\hat{a}}^\dagger \cdot \underline{\hat{b}}^\dagger\right)^{m-k}|p,q;IMY;k\rangle \\ &= A^\dagger_{k,\kappa}|p,q;IMY\rangle_0,\end{aligned}$$

$$\begin{aligned}A^\dagger_{k,\kappa} &= \left\{{}_0F_1(2k;|\kappa|^2)\right\}^{-1/2}\sum_{m=k}^{\infty}\frac{(2k-1)!}{(m-k)!(m+k-1)!}\left(\kappa\underline{\hat{a}}^\dagger \cdot \underline{\hat{b}}^\dagger\right)^{m-k} \\ &= {}_0F_1\left(2k;\kappa\underline{\hat{a}}^\dagger \cdot \underline{\hat{b}}^\dagger\right)\Big/\left\{{}_0F_1(2k;|\kappa|^2)\right\}^{1/2}.\end{aligned} \tag{5.7}$$

It is important to notice that there is a dependence on $k = \frac{1}{2}(p+q+3)$ in the operator $A^\dagger_{k,\kappa}$; so the basis vectors $|p,q;IMY\rangle_\kappa$ for $\mathcal{H}_\kappa$ do not arise from the basis vectors $|p,q;IMY\rangle_0$ for

$\mathcal{H}_0$ by application of a single operator dependent on κ alone. In spite of this, we will see below the usefulness of the connection (5.7).

We now obtain an expansion of the H-W SCS $|\underline{z}, \underline{w} >$ with $\underline{z}^T \underline{w} = \kappa$, in the orthonormal basis (5.6) for $\mathcal{H}_\kappa$. Thus we seek analogues to eqns.(4.26, 4.28), as well as to eqns.(4.27, 4.32), in the case of $\mathcal{H}_0$. Given $|\underline{z}, \underline{w}\rangle \in \mathcal{H}_\kappa$, by a suitable $SU(3)$ transformation we can relate it to a standard state $|\underline{z}^{(0)}(u), \underline{w}^{(0)}(v,x,y)\rangle$ on its orbit. We parametrise the latter as in eqn.(4.16) Class (d)(We are assuming here for definiteness that $x^2+y^2<1$, the possibility $x^2+y^2=1$ which is of vanishing measure being handled in the next Section):

$$\begin{aligned}
\underline{z}^{(0)}(u) &= u(1,0,0)^T, \\
\underline{w}^{(0)}(v,x,y) &= v\left(x+iy, \sqrt{1-x^2-y^2}, 0\right)^T, \\
uv(x+iy) &= \kappa.
\end{aligned} \tag{5.8}$$

We develop first the replacement for eqn.(4.26). The point of interest is to see which vector within each UIR (p,q) in $\mathcal{D}_\kappa$ appears, in place of the higher weight vector present in eqn.(4.26). Thanks to eqn.(5.7), the relevant overlap simplifies to a calculation in $\mathcal{H}_0$:

$$\begin{aligned}
{}_\kappa\langle p,q;IMY|\underline{z}^{(0)}(u), \underline{w}^{(0)}(v,x,y)\rangle &= {}_0\langle p,q;IMY|A_{k,\kappa}|\underline{z}^{(0)}(u), \underline{w}^{(0)}(v,x,y)\rangle \\
&= \left\{{}_0F_1(2k;|\kappa|^2)\right\}^{1/2} \langle p,q;IMY;k|\underline{z}^{(0)}(u), \underline{w}^{(0)}(v,x,y)\rangle.
\end{aligned} \tag{5.9}$$

Here the bra vector, in $\mathcal{H}_0$, is an eigenvector of $\hat{N}^{(a)}, \hat{N}^{(b)}$ with eigenvalues p,q respectively. This leads to further simplification:

$$\begin{aligned}
\langle p,q;IMY;k|\underline{z}^{(0)}(u), \underline{w}^{(0)}(v,x,y)\rangle &= e^{-\frac{1}{2}(u^2+v^2)} \frac{u^p}{p!} \frac{v^q}{q!} \times \\
\langle p,q;IMY;k| \left(\hat{a}_1^\dagger\right)^p & \left((x+iy)\,\hat{b}_1^\dagger + \sqrt{1-x^2-y^2}\,\hat{b}_2^\dagger\right)^q |\underline{0}, \underline{0}\rangle.
\end{aligned} \tag{5.10}$$

The ket vector here has hypercharge $\frac{1}{3}(p-q)$, as does the highest weight state in (p,q), so this overlap is nonzero only if $Y = \frac{1}{3}(p-q)$. This then determines the possible values of I:

$$I = I_0, I_0 - 1, \ldots, \frac{1}{2}|p-q|,$$
$$I_0 = \frac{1}{2}(p+q). \tag{5.11}$$

Notice that I_0 is the highest possible value of I in the UIR (p,q). For the bra vector in eqn.(5.10) we have the explicit expression (eqn.(I.A.9)):

$$\langle p,q;IMY;k| = \mathcal{N}_{pqIY}((I+M)!(I-M)!/2I!)^{1/2} \sum_{n=0}^{(p-r,q-s)_<} \sum_{L=0}^{I-M} \times$$

$$\frac{(-1)^{n+I-M-L}}{(r+s+n+1)!} \langle \underline{0},\underline{0}| \frac{(\hat{a}_\alpha \hat{b}_\alpha)^n}{n!} \frac{\hat{a}_1^{r-L}}{(r-L)!} \frac{\hat{a}_2^L}{L!} \times$$

$$\frac{\hat{b}_1^{I-M-L}}{(I-M-L)!} \frac{\hat{b}_2^{s-I+M+L}}{(s-I+M+L)!} \frac{\hat{a}_3^{p-r-n}}{(p-r-n)!} \frac{\hat{b}_3^{q-s-n}}{(q-s-n)!},$$

$$\hat{a}_\alpha\, \hat{b}_\alpha = \hat{a}_1\, \hat{b}_1 + \hat{a}_2\, \hat{b}_2,$$

$$\mathcal{N}_{pqIY} = \{r!s!(r+s+1)!(p-r)!(q-s)!(p+s+1)!(q+r+1)!/(p+q+1)!\}^{1/2},$$

$$r = I + \frac{Y}{2} + \frac{1}{3}(p-q),\ S = I - \frac{Y}{2} + \frac{1}{3}(q-p). \tag{5.12}$$

Use of this in eqn.(5.10) leads to further simplifications. The condition $Y = \frac{1}{3}(p-q)$ gives:

$$r = I + M_0,\ s = I - M_0,$$
$$p - r = q - s = I_0 - I,$$
$$M_0 = \frac{1}{2}(p-q). \tag{5.13}$$

Then, in the sums over n and L in eqn.(5.12), only the terms $n = p-r = q-s$ and $L = 0$ survive. Using all this, the scalar product in eqn.(5.9) can be explicitly computed:

$$_\kappa\langle p,q;IMY|\underline{z}^{(0)}(u), \underline{w}^{(0)}(v,x,y)\rangle = \left\{{}_0F_1(2k;|\kappa|^2)\right\}^{1/2} \cdot e^{-\frac{1}{2}(u^2+v^2)} \cdot u^p\, v^q \times$$

$$\delta_{Y,\frac{1}{3}(p-q)} \frac{(-1)^{I_0-M}}{(M-M_0)!} \{(2I+1)(I+M)!(I-M_0)!/(2I_0+1)!(I-M)!(I+M_0)!\}^{1/2} \times$$

$$(x+iy)^{I_0-M}(1-x^2-y^2)^{\frac{1}{2}(M-M_0)},$$

$$u\,v(x+i\,y)=\kappa. \tag{5.14}$$

We see that, provided $Y=\frac{1}{3}(p-q)$ and $M\geq M_0$, this overlap is nonzero for all values of I in the range (5.11). This shows how far the projection of $|\underline{z}^{(0)}(u),\underline{w}^{(0)}(v,x,y)\rangle$ onto the subspace of $\mathcal{H}_\kappa$ carrying the UIR (p,q) differs from the highest weight state.

We can now obtain the replacement for the previous eqn.(4.26). It is unavoidably somewhat more complicated. Using eqn.(5.14) and with $\kappa=uv(x+iy)$, we have:

$$|\underline{z}^{(0)}(u),\underline{w}^{(0)}(v,x,y)\rangle=e^{-\frac{1}{2}(u^2+v^2)}\sum_{p,q=0}^{\infty}u^p\,v^q\left\{{}_0F_1(2k;|\kappa|^2)/(p+q+1)!\right\}^{1/2}\times$$

$$\mathcal{N}'(p,q;|\kappa|/uv)\;|p,q;\kappa/uv\rangle_\kappa,$$

$$\mathcal{N}'(p,q;|\kappa|/uv)\;|p,q;\kappa/uv\rangle_\kappa=\sum_{I=|M_0|}^{I_0}\;\sum_{M=M_0}^{I}\frac{(-1)^{I_0-M}}{(M-M_0)!}\,(\kappa/uv)^{I_0-M}\times$$

$$\left(1-\frac{|\kappa|^2}{u^2v^2}\right)^{\frac{1}{2}(M-M_0)}\{(2I+1)(I-M_0)!(I+M)!/(I+M_0)!(I-M)!\}^{1/2}$$
$$|p,q;I,M,\frac{1}{3}(p-q)\rangle_\kappa,$$

$$\mathcal{N}'(p,q;|\kappa|/uv)=\left\{\sum_{I=|M_0|}^{I_0}\;\sum_{M=M_0}^{I}\frac{(2I+1)(I-M_0)!(I+M)!}{(I+M_0)!(I-M)!(M-M_0)!^2}\right.$$
$$\left.(|\kappa|/uv)^{2(I_0-M)}\left(1-\frac{|\kappa|^2}{u^2v^2}\right)^{M-M_0}\right\}^{1/2}, \tag{5.15}$$

which, as shown in the Appendix, can be compactly written as

$$\mathcal{N}'(p,q;|\kappa|/uv)=\{(I_0-|M_0|+1)(I_0+|M_0|+1)$$
$${}_2F_1(-(I_0-|M_0|),-(I_0+|M_0|),2,1-|\kappa|^2/(u^2v^2))\Big\}^{1/2} \tag{5.16}$$

The normalisation factor $\mathcal{N}'(p,q;|\kappa|/uv)$ has been defined so as to make the vector $|p,q;\kappa/uv\rangle_\kappa$ have unit norm; this vector lies in the subspace of $\mathcal{H}_\kappa$ carrying the (single occurrence of the) UIR (p,q) in $\mathcal{D}_\kappa$. Now we apply $\mathcal{U}(A)$ to both sides of eqn.(5.15) and get the replacements for eqns.(4.27, 4.28):

$$A \in \ SU(3), \ \underline{z} = A \ \underline{z}^{(0)}(u), \ \underline{w} = A^* \underline{w}^{(0)}(v,x,y), \underline{z}^T \underline{w} = \kappa :$$

$$\begin{aligned} |\underline{z}, \underline{w}\rangle = e^{-\frac{1}{2}(u^2+v^2)} \sum_{p,q=0}^{\infty} u^p \ v^q \\ \left\{ {}_0F_1(2k; |\kappa|^2)/(p+q+1)! \right\}^{1/2} \mathcal{N}'(p,q; |\kappa|/uv) \times \\ |p,q; \kappa/uv; A\rangle_\kappa, \\ |p,q;\kappa/uv;A\rangle_\kappa = \mathcal{U}(A) \ |p,q;\kappa/uv\rangle_\kappa. \end{aligned} \tag{5.17}$$

We see that for $(\underline{z}, \underline{w}) \in \mathcal{C}^6$ with given u, v, κ, corresponding to Class (d) in eqn.(4.16), the H-W SCS $|\underline{z}, \underline{w} >$ is expressible in terms of a sequence of $SU(3)$ GCS, all contained in $\mathcal{H}_\kappa$. The $SU(3)$ GCS within the UIR (p,q) use $|p,q;\kappa/uv\rangle_\kappa$ as the fiducial vector, and this is very different from the highest weight vector. For this family of $SU(3)$ GCS we have in place of eqn.(4.32):

$$\int_{SU(3)} dA \ |p,q;\kappa/uv;A\rangle_{\kappa \ \kappa}\langle p',q';\kappa/uv;A| = \delta_{p'p}\delta_{q'q} \ \frac{P^{(p,q;\kappa)}}{d(p,q)} \ , \tag{5.18}$$

where $P^{(p,q;\kappa)}$ is the projection operator onto the subspace of $\mathcal{H}_\kappa$ carrying the UIR (p,q). This follows from Schur lemma for $SU(3)$ UIR's, and the fact that $\mathcal{D}_\kappa$ is multiplicity-free.

With these replacements for eqns.(4.26, 4.27, 4.28, 4.32) in hand, we can study the analogue of the operator $A(f_0)$ in eqn.(4.30). We begin with the general definition (4.23) of $A(f)$ and choose

$$\begin{aligned} f(u,v,x,y) &= f_0(u,v)\delta^{(2)}(x+iy-\kappa/uv) \\ &= f_0(u,v)\delta(x - \text{Re } \kappa/uv)\delta/y - \text{Im } \kappa/uv). \end{aligned} \tag{5.19}$$

This achieves the restriction to $\mathcal{H}_\kappa$. We then define

$$A(f_0) = \int \prod_{j=1}^{3} \left(\frac{d^2 z_j}{\pi} \ \frac{d^2 w_j}{\pi} \right) f_0(u,v)\delta^{(2)}(x+iy-\kappa/uv)|\underline{z},\underline{w}\rangle\langle\underline{z},\underline{w}|$$

$$= \frac{2}{\pi} \int_0^\infty u^5 du \int_0^\infty v^5 dv f_0(u,v) \theta(uv - |\kappa|) \left(1 - \frac{|\kappa|^2}{u^2v^2}\right) \times$$

$$\int_{SU(3)} dA\, \mathcal{U}(A) |\underline{z}^{(0)}(u), \underline{w}^{(0)}(v,x,y)\rangle \langle \underline{z}^{(0)}(u), \underline{w}^{(0)}(v,x,y)| \mathcal{U}(A)^{-1}, \tag{5.20}$$

it being understood in the last expression that $x + iy = \kappa/uv$. We can now use eqns.(5.17, 5.18) here and get the final result replacing eqn.(4.33):

$$A(f_0) = \sum_{p,q=0}^{\infty} C(f_0; p, q; \kappa)\ P^{(p,q;\kappa)},$$

$$C(f_0; p, q; \kappa) = \frac{2}{\pi} \left\{ {}_0F_1(2k; |\kappa|^2)/(p+q+1)! d(p,q) \right\} \int_0^\infty u^5 du \int_0^\infty v^5 dv\ f_0(u,v) \times$$

$$\theta(uv - |\kappa|) \left(| - \frac{|\kappa|^2}{u^2v^2}\right) e^{-(u^2+v^2)} u^{2p} v^{2q} \{\mathcal{N}'(p,q;|\kappa|/uv)\}^2. \tag{5.21}$$

The freedom remaining in the choice of $f_0(u,v)$ displays the overcompleteness, within $\mathcal{H}_\kappa$, of the H-W SCS belonging to $\mathcal{H}_\kappa$. To limit ourselves to an exact "$SU(3)$- worth" of these states, we choose $f_0(u,v)$ to be the product of two delta functions. Then we get a generalisation of eqn.(4.34):

$$f_0(u,v) = \delta(u - u_0)\delta(v - v_0)\ ,\ u_0 v_0 > |\kappa| :$$

$$A(f_0) = \int \prod_{j=1}^{3} \left(\frac{d^2 z_j}{\pi} \frac{d^2 w_j}{\pi} \right) \delta(u - u_0)\delta(v - v_0)\delta^{(2)}\left(x + iy - \frac{\kappa}{uv}\right) |\underline{z}, \underline{w}\rangle \langle \underline{z}, \underline{w}|$$

$$= \frac{2}{\pi} \cdot e^{-\left(u_0^2 + v_0^2\right)} \sum_{p,q=0}^{\infty} \left\{ {}_0F_1(2k; |\kappa|^2)/(p+q+1)! d(p,q) \right\} \times$$

$$u_0^{2p+5} v_0^{2q+5} \left(1 - \frac{|\kappa|^2}{u_0^2 v_0^2}\right) \{\mathcal{N}'(p,q;|\kappa|/u_0v_0)\}^2\ \ P^{(p,q;\kappa)}. \tag{5.22}$$

In this manner all the results found in the preceeding Section for the subspace $\mathcal{H}_0 \subset \mathcal{H}$, the null space of K_-, generalise to a general eigenspace $\mathcal{H}_\kappa \subset \mathcal{H}$ of K_-. Here again, limiting oneself to an exact "$SU(3)$-worth" of H-W SCS does give us a total set of vectors, but they do not obey the Klauder resolution of the identity within $\mathcal{H}_\kappa$

VI. H-W SCS OF CLASS(E) AND THEIR $SU(3)$ CONTENT

In the listing of $SU(3)$ orbits of H-W SCS given in eqn.(4.16), it was pointed out that only Classes (d) and (e) involve all six oscillators of the Schwinger $SU(3)$ construction in a nontrivial manner. Furthermore, of these, only the former are generic. As we have seen, Class (d) orbits form a four-parameter continuous family, each orbit being of dimension eight. In contrast, Class (e) orbits are a three parameter family, with each orbit of dimension five. Another characteristic is that each H-W SCS $|\underline{z}, \underline{w}>$ in Class (d) is such that the complex three-vectors $\underline{z}^*$ and $\underline{w}$ are linearly independent; on the other hand, if $|\underline{z}, \underline{w}>$ is in Class (e), then $\underline{w}$ is a (complex) multiple of $\underline{z}^*$.

In Sections IV and V we have analysed in detail the $SU(3)$ structure and representation content of H-W SCS on all Class (d) orbits, for $\underline{z}^T\underline{w} = 0$ and $\underline{z}^T\underline{w} = \kappa \neq 0$ respectively. Now we turn to a similar analysis of the Class (e) orbits [30]. There is however a difficulty in handling this case by starting with the Klauder resolution of the identity, eqns.(4.12), and then modifying the integrand by inserting some function of the $SU(3)$ invariants with the aim of restricting the integration to a chosen subset of orbits. We are unable to use the methods of Sections IV and V here. The reason is that in terms of the $SU(3)$ invariant parameters u, v, x, y in eqn.(4.15), Class (e) corresponds to $x^2 + y^2 = 1$; while in the volume element (4.21) on the H-W group there is an explicit factor $(1 - x^2 - y^2)$. For this reason, we handle Class (e) orbits more directly, guided however by the results in Class (d).

A convenient representative point on a general Class (e) orbit is given by the pair of complex three-vectors

$$
\begin{aligned}
\underline{z}^{(0)}(u) &= u(0,0,1)^T \ , \ u > 0, \\
\underline{w}^{(0)}(ve^{i\alpha}) &= ve^{i\alpha}(0,0,1)^T \ , \ v > 0, \ , \ 0 \leq \alpha < 2\pi, \\
\underline{z}^{(0)}(u)^T\underline{w}^{(0)}(ve^{i\alpha}) &= uve^{i\alpha}.
\end{aligned} \tag{6.1}
$$

(As mentioned earlier in Section IV, the reason for choosing this configuration is that the corresponding stability group is the $SU(2)$ subgroup acting on dimensions 1 and 2 in the defining representation $(1,0)$, and it is just this subgroup that is involved in the canonical basis vectors $|p,q;IMY\rangle$ in a general $SU(3)$ UIR (p,q)). Acting with a general $A \in SU(3)$, we reach a general point $(\underline{z},\underline{w})$ on the orbit given by

$$\begin{aligned} \underline{z} &= A\,\underline{z}^{(0)}(u), \\ \underline{w} &= A^* \underline{w}^{(0)}(ve^{i\alpha}) = \left(\frac{ve^{i\alpha}}{u}\right)\,\underline{z}^*, \\ \underline{z}^T \underline{w} &= uve^{i\alpha} = e^{i\alpha}\left(\underline{z}^\dagger \underline{z}\;\underline{w}^\dagger \underline{w}\right)^{1/2}. \end{aligned} \tag{6.2}$$

The H-W SCS $|\underline{z}^{(0)}(u),\underline{w}^{(0)}(ve^{i\alpha})\rangle$ is of course given by

$$|\underline{z}^{(0)}(u),\underline{w}^{(0)}(ve^{i\alpha})\rangle = e^{-\frac{1}{2}(u^2+v^2)+u\hat{a}_3^\dagger+ve^{i\alpha}\hat{b}_3^\dagger}\;|\underline{0},\underline{0}>. \tag{6.3}$$

We can expand this in the orthonormal basis $|p,q;IMY;m>$ for $\mathcal{H}$, recognising that the only states that appear have $I=M=0, Y=\frac{2}{3}(q-p)$ for various (p,q). We need the results (I.A.6, I.A.7):

$$\begin{aligned} |p,q;0,0,\frac{2}{3}(q-p);k\rangle &= p!q!\{(p+1)(q+1)/(p+q+1)!\}^{1/2}\times \\ &\sum_{n=0}^{(p,q)_<} \frac{(-1)^n}{(n+1)!}\,\frac{\left(\hat{a}_\alpha^\dagger\hat{b}_\alpha^\dagger\right)^n}{n!}\,\frac{\left(\hat{a}_3^\dagger\right)^{p-n}}{(p-n)!}\,\frac{\left(\hat{b}_3^\dagger\right)^{q-n}}{(q-n)!}\;|\underline{0},\underline{0}\rangle, \\ \hat{a}_\alpha^\dagger\hat{b}_\alpha^\dagger &= \hat{a}_1^\dagger\hat{b}_1^\dagger+\hat{a}_2^\dagger\hat{b}_2^\dagger; \end{aligned} \tag{6.4a}$$

$$\begin{aligned} |p,q;0,0,\frac{2}{3}(q-p);m\rangle &= \{(2k-1)!/(m-k)!(m+k-1)!\}^{1/2}\cdot\left(\underline{\hat{a}}^\dagger\cdot\underline{\hat{b}}^\dagger\right)^{m-k}\times \\ &\quad |p,q;0,0,\frac{2}{3}(q-p);k\rangle. \end{aligned} \tag{6.4b}$$

We can now easily compute the desired overlap:

$$\langle p,q;0,0,\frac{2}{3}(q-p);m|\underline{z}^{(0)}(u),\underline{w}^{(0)}(ve^{i\alpha})\rangle = \{(2k-1)!/(m-k)!(m+k-1)!\}^{1/2}\times$$

$$(uve^{i\alpha})^{m-k}\langle p,q;0,0,\frac{2}{3}(q-p);k|\underline{z}^{(0)}(u),\underline{w}^{(0)}(ve^{i\alpha})\rangle$$

$$= \{(2k-1)!/(m-k)!(m+k-1)!\}^{1/2}(uve^{i\alpha})^{m-k}\cdot p!q!\{(p+1)(q+1)/(p+q+1)!\}^{1/2}\times$$

$$\langle\underline{0},\underline{0}|\frac{\hat{a}_3{}^p}{p!}\frac{\hat{b}_3{}^q}{q!}\,|\underline{z}^{(0)}(u),\underline{w}^{(0)}(ve^{i\alpha})\rangle$$

$$\{(p+1)(q+1)(2k-1)!/(p+q+1)!(m-k)!(m+k-1)!\}^{1/2}\,(uve^{i\alpha})^{m-k}\times$$

$$e^{-\frac{1}{2}(u^2+v^2)}\,u^p(ve^{i\alpha})^q. \tag{6.5}$$

In the second step here, when using the expansion (6.4a), only the term $n=0$ contributes. We therefore have the expansion of the representative Class (e) H-W SCS in the $SU(3)\times Sp(2,R)$ basis:

$$|\underline{z}^{(0)}(u),\underline{w}^{(0)}(ve^{i\alpha})\rangle = e^{-\frac{1}{2}(u^2+v^2)}\sum_{p,q=0}^{\infty}\{(p+1)(q+1)/(p+q+1)!\}^{1/2}\,u^p(ve^{i\alpha})^q\times$$

$$\sum_{m=k}^{\infty}\{(2k-1)!/(m-k)!(m+k-1)!\}^{1/2}(uve^{i\alpha})^{m-k}|p,q;0,0,\frac{2}{3}(q-p);m\rangle$$

$$= e^{-\frac{1}{2}(u^2+v^2)}\sum_{p,q=0}^{\infty}\{(p+1)(q+1/(p+q+1)!\}^{1/2}\,u^p(ve^{i\alpha})^q\left\{{}_0F_1(2k;u^2v^2)\right\}^{1/2}\times$$

$$|p,q;0,0,\frac{2}{3}(q-p)\rangle_{uve^{i\alpha}}. \tag{6.6}$$

As we would expect, this expansion involves just the K_- eigenstate defined in eqn.(5.5), namely the $I=M=0, Y=\frac{2}{3}(q-p)$ member of the orthonormal basis $\{|p,q;IMY\rangle_{uve^{i\alpha}}\}$ for $\mathcal{H}_{uve^{i\alpha}}$. As in the case of the $SU(3)$ SCS, where the fiducial vector within the UIR (p,q) is the single highest weight vector $|p,q;\frac{1}{2}(p+q),\frac{1}{2}(p+q),\frac{1}{3}(p-q)\rangle$, here too a single vector of the canonical basis appears as fiducial vector, but it is of course not the highest weight state.

Now within each UIR (p,q) contained in the UR $\mathcal{D}_{uve^{i\alpha}}$ on $\mathcal{H}_{uve^{i\alpha}}$, we define the family of $SU(3)$ GCS:

$$A\in\ SU(3): |p,q;0,0,\frac{2}{3}(q-p);A\rangle_{uve^{i\alpha}} = \mathcal{U}(A)|p,q;0,0,\frac{2}{3}(q-p)\rangle_{uve^{i\alpha}} \tag{6.7}$$

Then applying $\mathcal{U}(A)$ to both sides of eqn.(6.6) we have the general connection between Class (e) H-W SCS and the $SU(3)$ GCS (6.7):

$$|A\underline{z}^{(0)}(u),\ A^*\underline{w}^{(0)}(ve^{i\alpha})\rangle = e^{-\frac{1}{2}(u^2+v^2)} \sum_{p,q=0}^{\infty} \{(p+1)(q+1)/(p+q+1)!\}^{1/2} \times$$

$$u^p(ve^{i\alpha})^q \left\{{}_0F_1(2k;u^2v^2)\right\}^{1/2} |p,q;0,0,\frac{2}{3}(q-p);A\rangle_{uve^{i\alpha}} \tag{6.8}$$

We recognise that eqns.$(6.6, 6.7, 6.8)$ are replacements for eqns. $(4.26, 4.27, 4.28)$ and eqns.$(5.15, 5.17)$ of Class (d).

Keeping $uve^{i\alpha}$ fixed, the $SU(3)$ GCS (6.7) all belong to $\mathcal{H}_{uve^{i\alpha}}$, and from Schur lemma they obey the analogues to eqns.$(4.32, 5.18)$:

$$\int\limits_{SU(3)} dA|p,q;0,0,\frac{2}{3}(q-p);A\rangle_{uve^{i\alpha}}\ {}_{uve^{i\alpha}}\langle p',q';0,0,\frac{2}{3}(q'-p');A| = \delta_{p'p}\delta_{q'q}\,\frac{P^{(p,q;uve^{i\alpha})}}{d(p,q)}, \tag{6.9}$$

Here of course we exploit the multiplicity-free reduction of $\mathcal{D}_{uve^{i\alpha}}$. It follows that for the H-W SCS (6.8) we have:

$$\int\limits_{SU(3)} dA|A\underline{z}^{(0)}(u), A^*\underline{w}^{(0)}(ve^{i\alpha})\rangle\langle A\underline{z}^{(0)}(u), A^*\underline{w}^{(0)}(ve^{i\alpha})| =$$

$$2e^{-(u^2+v^2)} \sum_{p,q=0}^{\infty} u^{2p}v^{2q}\ {}_0F_1(2k;u^2v^2)\ P^{(p,q;uve^{i\alpha})}/(p+q+2)! \tag{6.10}$$

The integration over $SU(3)$ here is in effect only over the five-dimensional coset space $SU(3)/SU(2)$, in contrast to eqns.$(4.32, 5.18)$ in Class (d).

If we write $\kappa = uve^{i\alpha}$ and allow u and v to vary reciprocally, and also keep α fixed so that κ stays fixed, we never leave the subspace $\mathcal{H}_{uve^{i\alpha}}$ and the projection operators $P^{(p,q;uve^{i\alpha})}$. Therefore we can multiply both sides of eqn.(6.10) by any function

$$f(u,v) = f_0(u)\ \delta(uv-|\kappa|), \tag{6.11}$$

and integrate over both u and v to get results similar to eqns.$(4.33, 5.21)$. Here $f_0(u)$ is free. This then shows that for each fixed κ, the Class (e) H-W SCS $|\underline{z},\underline{w}>$ with $\underline{z}^T\underline{w} = \kappa$ are overcomplete in $\mathcal{H}_\kappa$

VII. CONCLUDING REMARKS

To conclude, we have given a unified analysis of the interconnections between the Heisenberg-Weyl standard coherent states and the standard coherent states as well as certain generalised coherent states of $SU(3)$. The specific family of $SU(3)$ coherent states to be used is dependent on the type of orbit of the H-W SCS belong to. This situation is describable in detail as follows. In terms of the $SU(3)$ invariant parameters x and y, at $x = y = 0$ we have those generic Class(d) orbits which lie entirely within the subspace $\mathcal{H}_0$. For these H-W SCS, the $SU(3)$ harmonic analysis involves precisely the $SU(3)$ SCS within each UIR. For $0 < x^2 + y^2 < 1$ we deal with the subspaces $\mathcal{H}_\kappa \subset \mathcal{H}$ which generalise $\mathcal{H}_0$; the corresponding orbits consist of H-W SCS whose $SU(3)$ content brings in the $SU(3)$ GCS studied in Section V. The fiducial vectors here are rather complicated, at any rate in the canonical basis for $SU(3)$ UIR's. In the limit $x^2 + y^2 = 1$, we have the Class (e) orbits. These H-W SCS involve yet another family of $SU(3)$ GCS, though now the fiducial vectors are the unique $SU(2)$ scalar states within each $SU(3)$ UIR, and their properties are studied in Secion VI. In this entire development the group $Sp(2, R)$ plays a particularly helpful role and so does the Schur lemma wherever it is available. Indeed we have used this lemma for UIR's of the H-W group wherever possible, and after modifications of the completeness identity used it for UIR's of $SU(3)$. This systematic use of Schur lemma makes several computations much easier than otherwise. It must be emphasised that all the Heisenberg standard coherent states have been included in our study in the spirit of $SU(3)$ harmonic analysis, so that there is a satisfactory completeness in our analysis. The significant property of the discrete series UIR's of $Sp(2, R)$, which we have exploited, is worth mention. It is that while the spectrum of the compact generator J_0 depends on k, hence on the UIR, the 'spectrum' of the non hermitian lowering operator K_- is the entire complex plane, thus being UIR independent. The calculations in Section V clearly show the importance of these facts.

Appendix

We outline here the steps involved in going from (5.15) to (5.16). Eqn. (5.15).

$$\mathcal{N}'(p,q;|\kappa|/uv) = \left\{ \sum_{I=|M_0|}^{I_0} \sum_{M=M_0}^{I} \frac{(2I+1)(I-M_0)!(I+M)!}{(I+M_0)!(I-M)!(M-M_0)!^2} \right.$$

$$\left. (|\kappa|/uv)^{2(I_0-M)} \left(1 - \frac{|\kappa|^2}{u^2v^2}\right)^{M-M_0} \right\}^{1/2}, \tag{A.1}$$

can be written in terms of the Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) \equiv \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(\alpha+\beta+n+m+1)}{2^m\Gamma(\alpha+m+1)}(x-1)^m \tag{A.2}$$

as

$$\mathcal{N}'(p,q;|\kappa|/uv) = \left\{ \sum_{I=|M_0|}^{I_0} (2I+1) \left(\frac{|\kappa|}{uv}\right)^{2(I_0-M_0)} P_{I-M_0}^{(0,2M_0)} \left(\frac{2u^2v^2}{|\kappa|^2} - 1\right) \right\}^{1/2} \tag{A.3}$$

Using the fact that $P_n^{(\alpha,\beta)}$ can also be written as

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n} \sum_{m=0}^{n} \binom{n+\alpha}{m} \binom{n+\beta}{n-m} (x-1)^{n-m}(x+1)^m \tag{A.4}$$

one can show that

$$x^{M_0} P_{I-M_0}^{(0,2M_0)}(2x-1) = x^{-M_0} P_{I+M_0}^{(0,-2M_0)}(2x-1) \tag{A.5}$$

which implies that $\mathcal{N}'$ depends on M_0 only through its magnitude $|M_0|$. Replacing M_0 in the rhs of $(A.1)$ by $|M_0|$ and rewriting it as a polynomial in $(1-|\kappa|^2/u^2v^2)$, we obtain

$$\mathcal{N}'(p,q;|\kappa|/uv) = \left\{ \sum_{k=0}^{I_0-|M_0|} a_k \left(1 - \frac{|\kappa|^2}{u^2v^2}\right)^k \right\}^{1/2}, \tag{A.6}$$

where

$$a_k = \sum_{M=0}^{k} \sum_{I=M}^{I_0-|M_0|} (2I+2|M_0|+1) \binom{I+2|M_0|+M}{M} \binom{I}{M} \binom{I-|M_0|-M}{k-M} (-1)^{k-M}. \tag{A.7}$$

which, after some rearrangement, can be written as

$$a_k = \sum_{M=0}^{k} (-1)^M \binom{I_0-|M_0|+M-k}{M} \cdot$$

$$\sum_{I=0}^{I_0-|M_0|-k+M} (2I+2k+2|M_0|-2M+1) \binom{I+k-M}{I} \binom{I+2k+2|M_0|-2M}{k-M}. \tag{A.8}$$

Using the identities

$$\sum_{I=0}^{I_0-|M_0|-k+M} (2I+2k+2|M_0|-2M+1)\binom{I+k-M}{I}\binom{I+2k+2|M_0|-2M}{k-M}$$
$$= (I_0-|M_0|+1)\binom{I_0+|M_0|+k-M+1}{I_0+|M_0|}\binom{I_0-|M_0|}{k-M} \tag{A.9}$$

and

$$\sum_{M=0}^{k} (-1)^M \binom{I_0-|M_0|+M-k}{M}\binom{I_0+|M_0|+k-M+1}{I_0+|M_0|}\binom{I_0-|M_0|}{k-M}$$
$$= \binom{I_0+|M_0|+1}{k+1}\binom{I_0-|M_0|}{k} \tag{A.10}$$

we obtain

$$a_k = (k+1)\binom{I_0+|M_0|+1}{k+1}\binom{I_0-|M_0|+1}{k+1} \tag{A.11}$$

Substituting this in $(A.6)$ we finally obtain the result (5.16).

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