Geometric invariants for resolutions of Hilbert modules [1]

Ronald G. Douglas and Gadadhar Misra

0 Introduction

The development of complex function theory beyond the one-variable case required new techniques and approaches, not just an extension of what had worked already. The same is proving true of multi–variate operator theory. Still it is reasonable to start by seeking to understand in the larger context results that have proved important and useful in the study of single operator theory. For that reason the first author showed [5] how to frame the canonical model theory of Sz.-Nagy and Foias for contraction operators in the language of Hilbert module resolutions over the disk algebra. This point of view made clear why a straightforward extension of model theory failed in the multi–variate case. Since the appropriate algebra in this case would be higher dimensional, one would expect module resolutions, if they existed, to be of longer length and hence not expressible as a canonical model. While work on this topic has shed light on what one might expect to be true (cf. [6]), useful results are still scarce.

Now one can raise two kinds of questions about module resolutions. The first concerns their existence, that is, under what hypotheses on a given Hilbert module does a resolution by "nice" Hilbert modules exist. Here little progress has been made except to understand better what being a nice module should mean. The current working definition [6] requires the spectral sheaf to be locally free or a vector bundle with acyclic Koszul cohomology. (We are assuming all modules are finitely generated.) We do not intend to explore this issue here.

The second question asks how, given such a resolution, does one extract information from it about the module one is trying to study. In the case considered by Sz.-Nagy and Foias, this problem was solved [5] by localizing the connecting homomorphism in the resolution to recover the characteristic operator function. However, here one can impose conditions effectively making the resolution unique. In the multi-variate case, this is not possible.

One of the important impetuses for the development of homological algebra was the problem of extracting information from module resolutions in the context of pure algebra. The techniques developed there, however, will not carry over without some adaptation since the algebraic hypotheses required are seldom satisfied in the context of Hilbert modules. While developing an adaptation of the algebraic theory to the multi–variate operator theory context is important, here we are exploring another approach. We seek to build on the complex geometric structure inherent in the situation and attempt to relate curvatures and other geometric quantities. Such an approach can be viewed as generalizing that which leads to the characteristic operator function in the one-dimensional case.

At present, we do not understand the situation well enough to formulate theorems, let alone prove them. One of the difficulties is the scarcity of concrete examples with calculations of the associated geometric objects such as curvature. The statement of Theorem 1.4 which relates the curvature in

[1] Research supported in part by grants from the National Science Foundation. The second author was also supported by a travel grant from the CDE.

a quotient module under very special circumstances to that of the module, became apparent only after the calculation of particular cases outlined in sections 2.4 and 2.5. Moreover, these calculations themselves suggest the possibility that the difference of the curvatures, corresponding to a moudle and a submodule defined by a zero set, may depend only on the geometry of the zero set and its embedding. Some of the other examples support such a conjecture but the family discussed in section 2.1 shows it can't be true without further restrictions. What the nature of such hypotheses might be or, indeed, if there are any remains a mystery.

There is another way to arrive at the existence of such a conjecture. In the single-variable case, the alternating difference of the curvatures and a curvature–like contribution from the connecting map, yields an atomic measure with support equal to the spectrum of the quotient module. One seeks an analogue of such a formula in the multi–variable case assuming that the other connecting map contributes nothing since it is co–isometric. In various examples, such a formula is shown to hold. One can also consider these expressions restricted to the zero set. A chief purpose of the calculations in section 2 is to determine if such formulae hold and in what generality.

Although Theorem 1.4 provides a good method for calculating the curvature of a quotient module, the approach requires the submodule to be prime-like; it does not work in the presence of nilpotents. Coming to grips with such cases requires first that one generalize the notion of functional Hilbert space since the quotient module will not be one in the sense of Aronszajn if the submodule is not prime-like. Further, one will need to consider geometric invariants that arise from higher-order localization, that is, localizations by modules with one point spectrum but which are not one-dimensional. We take some preliminary steps in this direction in the last section.

Here we consider some natural examples of short exact sequences of Hilbert modules that are built from a family of modules, which contains the Hardy and Bergman modules, and a submodule arising from the closure of a principal ideal. The quotient module is supported on the zero set of the ideal and we seek to relate its curvature to that of the curvatures of the other two modules and an analogous quantity defined from the connecting homomorphism. All are considered as currents. Although we attempt to suggest more general conclusions and to formulate conjectures, this paper records work in progress.

1 Restrictions

Let Ω be a bounded domain $\mathbb{C}^m$ and $K : \Omega \times \Omega \to \mathbb{C}$ be a function which is holomorphic in the first variable and anti - holomorphic in the second variable. If K is also a non negative (nnd) function in the sense that the matrix $(\!(K(z_i, z_j))\!)$ is nnd for all finite subsets $\{z_1, \ldots, z_n\} \subseteq \Omega$, then K determines a unique Hilbert space $\mathcal{M}$ of holomorphic functions on Ω such that

$$\langle f, K(\cdot, w)\rangle = f(w) \text{ for all } f \in \mathcal{M},\ w \in \Omega. \qquad (1.1)$$

In the usual terminology (cf. [1]), we say that $\mathcal{M}$ is a functional Hilbert space and K is said to be the reproducing kernel for $\mathcal{M}$.

Let $\mathcal{A}(\Omega)$ denote the closure of the polynomials in the supremum norm on Ω. We assume throughout that the algebra $\mathcal{A}(\Omega)$ acts boundedly on the Hilbert space $\mathcal{M}$. This means that the pointwise product $f \cdot h$ is in $\mathcal{M}$ for each $f \in \mathcal{A}(\Omega)$ and each $h \in \mathcal{M}$. Note that the closed graph theorem ensures the boundedness of the operator $(f, h) \to f \cdot h$ so that $\mathcal{M}$ is a Hilbert module in the sense of [7]. Let $\mathcal{M}_0$ denote the closure in $\mathcal{M}$ of an ideal $\mathcal{I} \subseteq \mathcal{A}(\Omega)$. If $\mathcal{Z}$ denotes the set of common zeros of the functions in $\mathcal{I}$, then the case when $\mathcal{Z}$ is discrete and finite can be analyzed since the quotient module $\mathcal{M}_{\rm q}$ is finite-dimensional. However, if $\mathcal{M}_0$ is taken to be the subspace of all functions vanishing on an infinite subset $\mathcal{Z}$ of Ω, then the situation is considerably more complex. We assume in what follows that $\mathcal{M}_0$ is the submodule of $\mathcal{M}$ consisting of all functions h that vanish on an analytic hypersurface $\mathcal{Z}$ of Ω. In general, it is not enough for $\mathcal{M}_0$ to be the closure of an ideal $\mathcal{I}$ with zero set $\mathcal{Z}$ unless we assume, among other assumptions, that $\mathcal{I}$ is the intersection of prime ideals. Some further complications arise in case $\mathcal{M}_0$ is the closure of an ideal $\mathcal{I}$ that is not but for the present we restrict attention to the largest submodule with zero set $\mathcal{Z}$. However, the approach developed here can be applied to analyze the case in which $\mathcal{M}_0$ is the closure of a principal ideal and we plan to take up this and related questions in a future paper.

We can view $\mathcal{M}_{\rm q}$ as a module over $\mathcal{A}(\Omega)$ by compressing the module action on $\mathcal{M}$ to $\mathcal{M}_{\rm q}$ (cf. [7, p. 41]). It follows from [1] that both $\mathcal{M}_0$ and $\mathcal{M}_{\rm q}$ are functional Hilbert spaces over Ω and $\mathcal{Z}$ respectively. We let $K(\cdot,w)$, $K_0(\cdot,w)$ and $K_{\rm q}(\cdot,w)$ denote the reproducing kernels for $\mathcal{M},\mathcal{M}_0$ and $\mathcal{M}_{\rm q}$, respectively.

It is easy to verify that

$$K(\cdot,w)=K_{\rm q}(\cdot,w)+K_0(\cdot,w).$$

We consider the Hilbert space $\mathcal{M}_{\rm res}$ obtained by restricting the functions in $\mathcal{M}$ to the set $\mathcal{Z}$, that is,

$$\mathcal{M}_{\rm res}=\Big\{h_0\colon\mathcal{Z}\to\mathbb{C},\text{holomorphic }\ \Big|\ h_0=h|_{\mathcal{Z}}\ \text{ for some } h\in\mathcal{M}\Big\}.$$

The norm of $h_0\in\mathcal{M}_{\rm res}$ is

$$\|h_0\|=\inf\{\|h\|:h|_{\mathcal{Z}}=h_0 \text{ for } h\in\mathcal{M}\}.$$

Aronszajn [1, p. 351] shows that the restriction map $R:\mathcal{M}_{\rm q}\to\mathcal{M}_{\rm res}$ is unitary as follows.

Let $P:\mathcal{M}\to\mathcal{M}_{\rm q}$ be the projecton. For $h\in\mathcal{M}$, we have

$$\begin{aligned}\|Ph\| &= \inf\{\|h+h_1\|:h_1\in\mathcal{M}_0\}\\ &= \inf\{\|\tilde{h}\|:\tilde{h}\,|_{\mathcal{Z}}=h|_{\mathcal{Z}}\}\end{aligned}$$

Since the functions Ph and h have the same restriction to $\mathcal{Z}$, it follows that the map $R:Ph\to h|_{\mathcal{Z}}$ is an isometry. If h_0 is in $\mathcal{M}_{\rm res}$ then there exists $h\in\mathcal{M}$ such that $h|_{\mathcal{Z}}=h_0$ and it follows that $R^*h_0=Ph$. Thus R is unitary. This can be used to show that the reproducing kernel $K_{\rm res}(\cdot,w)$ for $\mathcal{M}_{\rm res}$ is $K_{\rm res}(\cdot,w)=K(\cdot,w)|_{\mathcal{Z}}$, $w\in\mathcal{Z}$ [1, p. 351].

Restricting the module map $(f,h)\to f\cdot h$ in both arguments to $\mathcal{Z}$, we see that $\mathcal{M}_{\rm res}$ is a module over the algebra

$$\mathcal{A}_{\rm res}(\Omega)\stackrel{\rm def}{=}\{f|_{\mathcal{Z}}:f\in\mathcal{A}(\Omega)\}.$$

Although $\mathcal{A}_{\rm res}(\Omega)$ need not be a function algebra, its completion is and $\mathcal{M}_{\rm res}$ is a Hilbert module over it. Let $i:\mathcal{Z}\to\Omega$ be the inclusion map and $i^*:\mathcal{A}(\Omega)\to\mathcal{A}_{\rm res}(\Omega)$ be the map $i^*f=f\circ i$. We can push forward the module defined over $\mathcal{A}_{\rm res}(\Omega)$ to a module over $\mathcal{A}(\Omega)$ via the map

$$(f,h)\to(i^*f)\cdot h,\quad f\in\mathcal{A}(\Omega),\ h\in\mathcal{M}. \qquad (1.2)$$

Recall that two Hilbert modules $\mathcal{M}_1$ and $\mathcal{M}_2$ over the algebra $\mathcal{A}(\Omega)$ are said to be isomorphic if there is an unitary operator $T:\mathcal{M}_1\to\mathcal{M}_2$ intertwining the two module actions, that is, $f\cdot Th=Tf\cdot h$ for $f\in\mathcal{A}(\Omega)$ and $h\in\mathcal{M}_1$. Any operator satisfying the latter condition is said to be a module map.

THEOREM 1.1 *Let $\mathcal{M}$ be a Hilbert module over the algebra $\mathcal{A}(\Omega)$ and $\mathcal{M}_0$ be the submodule of functions vanishing on a fixed subset $\mathcal{Z}\subseteq\Omega$. The push forward $i_*\mathcal{M}_{\rm res}$ is isomorphic to the quotient module $\mathcal{M}_{\rm q}$.*

Proof: First, we reprove the fact that the restriction map is unitary in a way that gells well with the spirit of this paper. We have $K_{\rm q}(\cdot,w)=K(\cdot,w)$ for $w\in\mathcal{Z}$. The map

$$h\stackrel{R}{\to}\langle h,K_{\rm q}(\cdot,w)\rangle=\langle h,K(\cdot,w)\rangle=h|_{\mathcal{Z}}\,,\quad h\in\mathcal{M}_{\rm q},\ w\in\mathcal{Z}$$

is the restriction map on $\mathcal{M}_{\rm q}$. If we define an inner product on $R(\mathcal{M}_{\rm q})$ as $\langle Rh,Rh'\rangle\stackrel{\rm def}{=}\langle h,h'\rangle$, then it follows that

$$\langle Rh,\,K|_{\rm res}\,(\cdot,w)\rangle=\langle h,K(\cdot,w)\rangle=\langle h,K_{\rm q}(\cdot,h)\rangle=h(w),\quad w\in\mathcal{Z}.$$

Thus the reproducing kernel for the space $R(\mathcal{M}_{\rm q})$ is $K|_{\rm res}\,(\cdot,w),\ w\in\mathcal{Z}$. Therefore, $R(\mathcal{M}_{\rm q})$ is the Hilbert space $\mathcal{M}_{\rm res}$. By our construction R is an onto isometry.

We only need to verify that $R:\mathcal{M}_{\rm q}\to\mathcal{M}_{\rm res}$ is a module map, that is, $(f\circ i)\cdot(Rh)=RP(f\cdot h)$ for all $h\in\mathcal{M}_{\rm q}$. Note that for $w\in\mathcal{Z}$, we have $\langle h,M_f^*PK_{\rm q}(\cdot,w)\rangle=\langle h,M_f^*K(\cdot,w)\rangle$. This implies that $\langle PM_f,K_{\rm q}(\cdot,w)\rangle=f(w)\langle h,K_{\rm q}(\cdot,w)\rangle$ for $h\in\mathcal{M}_{\rm q}$, $w\in\mathcal{Z}$. Further,

$$\begin{aligned}\langle f\circ i\cdot Rh,K_{\rm res}(\cdot,w)\rangle &= \langle Rh,M_{f\circ i}^*K_{\rm res}(\cdot,w)\rangle\\ &= \langle h,\overline{f(w)}R^*K_{\rm res}(\cdot,w)\rangle\\ &= \langle h,\overline{f(w)}K_{\rm q}(\cdot,w)\rangle\\ &= \langle PM_fh,K_{\rm q}(\cdot,w)\rangle\\ &= \langle P(f\cdot h),R^*K_{\rm res}(\cdot,w)\rangle\\ &= \langle RP(f\cdot h),K_{\rm res}(\cdot,w)\rangle.\end{aligned}$$

This calculation verifies that R is a module map and the proof is complete.

We point out that if $\mathcal{Z}$ happens to be an open subset of Ω, then $\mathcal{M}_{\rm q}$ equals $\mathcal{M}$. In this case, $i_*\mathcal{M}_{\rm res}$ and $\mathcal{M}$ are isomorphic. Thus, we don't distinguish the modules $\mathcal{M}$ and $\mathcal{M}_{\rm res}$.

For our analysis, we will assume that $\mathcal{Z}$ is an analytic hypersurface in Ω in the sense of [10, Definition 8, p. 17]. Let $U\subseteq\Omega$ be a fixed open set containing a given point $z_0\in\mathcal{Z}$. We may choose local co–ordinates ([10, Theorem 9, p.17]), $\phi\stackrel{\rm def}{=}(\phi^1,\ldots,\phi^m):U\subseteq\Omega\to\mathbb{C}^m$ such that

$$\mathcal{Z}\cap U=\{z\in U:\phi^1(z)=0\}. \qquad (1.3)$$

In view of the remark preceeding Corollary 3 in [10, p. 34], if the second Cousin problem is solvable for Ω, then there exists a global defining function, which we will again denote by ϕ^1, for the hypersurface $\mathcal{Z}$. It is easy to see that, even though the function ϕ need not define global co-ordinates for Ω, it extends to a holomorphic function on Ω. We will assume throughout that the function ϕ is bounded and that the neighbourhood U of $z_0\in\mathcal{Z}$ has been chosen such that ϕ is bi-holomorphic on U. If $i:U\to\Omega$ is the inclusion map, then $\phi\circ i:U\to V\subseteq\mathbb{C}^m$, V open in $\mathbb{C}^m$, is a bi-holomorphic map.

Let $\Gamma x(v)=\langle x,K(\cdot,\phi^{-1}(v)\rangle$, $x\in\mathcal{M}$ and $0\in V$. Let $\mathcal{N}$ be the set of holomorphic functions $\{\Gamma x:x\in\mathcal{M}\}$ on V with the inner product $\langle\Gamma x,\Gamma y\rangle_{\mathcal{N}}\stackrel{\rm def}{=}\langle x,y\rangle_{\mathcal{M}}$. Thus Γ is an onto isometry. Obviously, the kernel function for $\mathcal{N}$ is

$$K_\phi(v_1,v_2)=K(\phi^{-1}(v_1),\phi^{-1}(v_2)),\quad v_1,v_2\in V. \qquad (1.4)$$

As remarked immediately after Theorem 1.1, the module $\mathcal{M}_{\rm res\ U}$ is isomorphic to $\mathcal{M}$ and that we are merely looking at a different realization of the functional Hilbert space, this time as holomorphic functions on U. We push forward the module $\mathcal{M}_{\rm res\ U}$ under the map ϕ so that the module action for $\phi_*\mathcal{M}_{\rm res\ U}$ is given by $(f,h)\to(f\circ\phi)\cdot h$ for $f\in\mathcal{A}(V)$ and $h\in\mathcal{M}_{\rm res\ U}$.

LEMMA 1.2 *The modules $\mathcal{N}$ and the push forward $\phi_*\mathcal{M}_{\text{res U}}$ are isomorphic.*

Proof: For the proof we need only to verify that Γ is a module map. This amounts to verifying $\Gamma(f\circ\phi)\cdot h = f\cdot\Gamma h$. However,

$$\begin{aligned}\langle f\cdot(\Gamma h), K_\phi(\cdot,v)\rangle &= \langle \Gamma h, M_f^* K_\phi(\cdot,v)\rangle\\ &= \overline{f(v)}\langle \Gamma h, K_\phi(\cdot,v)\rangle\\ &= \overline{f\circ\phi\circ\phi^{-1}(v)}\Gamma h(v)\\ &= \langle h, M_{f\circ\phi}^* K(\cdot,\phi^{-1}(v))\rangle\\ &= (\Gamma(f\circ\phi)\cdot h)(v)\\ &= \langle \Gamma(f\circ\phi)\cdot h, K_\phi(\cdot,v)\rangle\end{aligned}$$

This verification completes the proof.

Since $\frac{\partial\phi^1}{\partial z_1}(z_0)\neq 0, z_0\in\mathcal{Z}$, we may choose $\phi=(\phi^1,\ldots,\phi^m)$, where ϕ^ℓ $(\ell\neq 1)$ is the projection to z_ℓ, to be our new co-ordinate system.

We now relate this description of the various modules to complex geometry (cf. [4]). Let

$$0\longleftarrow \mathcal{M}_{\rm q}\overset{\rm Q}{\longleftarrow}\mathcal{M}\overset{X}{\longleftarrow}\mathcal{M}_0\longleftarrow 0 \tag{1.5}$$

be an exact sequence of Hilbert modules, where X is the inclusion map. We obtain a localisation by tensoring with the one-dimensional module $\mathbb{C}_w$. Then it is not hard to see that while $\dim\mathcal{M}\otimes_{\mathcal{A}(\Omega)}\mathbb{C}_w$ and $\dim\mathcal{M}_0\otimes_{\mathcal{A}(\Omega)}\mathbb{C}_w$ equal 1, the dimension of $\mathcal{M}_{\rm q}\otimes_{\mathcal{A}(\Omega)}\mathbb{C}_w$ is one or zero according as w is in $\mathcal{Z}$ or not. These localisations give rise to hermitian anti–holomorphic vector bundles. An anti–holomorphic frame determines such a bundle. In the following description s is an anti–holomorphic frame, which is described in terms of the reproducing kernel. We obtain two hermitian anti-holomorphic vector bundles

$$E\overset{\rm def}{=}\Big\{(\mathcal{M}\otimes_{\mathcal{A}(\Omega)}\mathbb{C}_w,w)\to w, s(w)=K(\cdot,w)\in\mathcal{M}\otimes_{\mathcal{A}(\Omega)}\mathbb{C}_w\Big\}$$

and

$$E_0\overset{\rm def}{=}\Big\{(\mathcal{M}_0\otimes_{\mathcal{A}(\Omega)}\mathbb{C}_w,w)\to w, s(w)=K_0(\cdot,w)\in\mathcal{M}_0\otimes_{\mathcal{A}(\Omega)}\mathbb{C}_w, w\notin\mathcal{Z}\Big\},$$

which live on Ω, while

$$E_{\rm q}\overset{\rm def}{=}\Big\{(\mathcal{M}_{\rm q}\otimes_{\mathcal{A}(\Omega)}\mathbb{C}_w,w)\to w, s(w)=K_{\rm q}(\cdot,w)\in\mathcal{M}_{\rm q}\otimes_{\mathcal{A}(\Omega)}\mathbb{C}_w, w\in\mathcal{Z}\Big\}$$

lives only on $\mathcal{Z}$.

From Theorem 1.1, it follows that this last bundle is equivalent to

$$E_{\rm res}\overset{\rm def}{=}\Big\{(\mathcal{M}_{\rm res}\otimes_{\mathcal{A}_{\rm res}(\Omega)}\mathbb{C}_w,w)\to w, s(w)=K_{\rm res}(\cdot,w)\in\mathcal{M}_{\rm res}\otimes_{\mathcal{A}_{\rm res}(\Omega)}\mathbb{C}_w, w\in\mathcal{Z}\Big\}.$$

Further, the metric for the bundle is obtained by restriction, that is, $K_{\rm res}(w,w)=K(w,w),\quad w\in\mathcal{Z}$. The following diagram captures our situation.

$$\begin{array}{ccccc}\mathcal{M}_{\rm res}\otimes_{\mathcal{A}_{\rm res}(\Omega)}\mathbb{C}_w\cong & i^*(\mathcal{M}\otimes_{\mathcal{A}(\Omega)}\mathbb{C}_w) & \overset{i^*}{\longleftarrow} & \mathcal{M}\otimes_{\mathcal{A}(\Omega)}\mathbb{C}_w & \\ & \tilde{s}\Big\uparrow\!\Big\downarrow & & \Big\uparrow\!\Big\downarrow s & , \\ & \mathcal{Z} & \overset{i}{\longrightarrow} & \Omega & \end{array}\qquad \begin{array}{rcl} s(w) &=& K(\cdot,w)\\ \tilde{s}(w) &=& K_{\rm res}(\cdot,w)\\ \langle\tilde{s}(w),\tilde{s}(w)\rangle &=& \langle s(w),s(w)\rangle|_{\mathcal{Z}}\,.\end{array}$$

Every anti–hermitian holomorphic vector bundle has a unique connection compatible with both the complex structure and the metric. (This construction is usually stated for hermitian holomorphic vector bundles but only formal changes are required to handle the anti–holomorphic case.) The curvature matrix of this connection is a hermitian matrix of $(1,1)$ – forms. In the case of a line bundle, there is a simple expression for the curvature [11, p. 184, (14)]

$$\mathcal{K}(w) = -\sum_{i,j=1}^{m} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \|s(w)\|^2 dw_i \wedge d\bar{w}_j.$$

There is a natural way in which we may restrict the curvature of a bundle on an open set V to a submanifold of the form $\{u \in V : u_1 = 0\}$, which is a submanifold with co-ordinates $\{u_2, \ldots, u_m\}$. The differential operators ∂ and $\bar{\partial}$ restrict naturally to this submanifold. Thus the restriction of the curvature is

$$\begin{aligned}
\mathcal{K}|_{\mathrm{res}\{\mathrm{u}_1=0\}}(u) &\stackrel{\text{def}}{=} \sum_{i,j=2}^{m} \Big(\tfrac{\partial^2}{\partial u_i \partial \bar{u}_j} \log \gamma\Big)(0, u_2, \ldots, u_m)\; du_i \wedge d\bar{u}_j \\
&= \sum_{i,j=2}^{m} \Big(\tfrac{\partial^2}{\partial u_i \partial \bar{u}_j} \log \gamma(0, u_2, \ldots, u_m)\Big)\; du_i \wedge d\bar{u}_j \\
&= \sum_{i,j=2}^{m} \Big(\tfrac{\partial^2}{\partial u_i \partial \bar{u}_j} \log \gamma|_{\mathrm{res}\{\mathrm{u}_1=0\}}\Big)\; du_i \wedge d\bar{u}_j,
\end{aligned}$$

where γ is the hermitian metric for the bundle on V.

The bundles E_{q} and E_{res} are equivalent by Theorem 1.1. This implies the corresponding curvatures are equal. However, since the hypersurface $\mathcal{Z}$ is an arbitrary analytic hypersurface, there is no obvious way to relate the curvature for E_{res} to that of E. For this, we must realise the hypersurface $\mathcal{Z}$ as $\{u \in \phi(\Omega) : u_1 = 0\}$. The pull–back of the bundle on $V = \phi(U)$ obtained from localising the module $\mathcal{N}$ is equivalent to the bundle E. This follows from Lemma 1.2, which says that the bundles obtained from localisations of $\phi_* \mathcal{M}_{\mathrm{res\ U}}$ and $\mathcal{N}$ are equivalent. This in turn means that the pull – back of the bundle E under the map ϕ^{-1} is equivalent to the bundle $E_{\mathcal{N}}$ obtained from the localisation of $\mathcal{N}$. Applying the pull-back operation once more, this time under the map ϕ, we see that $\phi^* E_{\mathcal{N}} \simeq \phi^*((\phi^{-1})^* E) \simeq E$. As it was pointed out earlier, the curvature for bundle $E_{\mathcal{N}}$ restricts naturally to the submanifold $\phi(\mathcal{Z} \cap U) = \{0, u_2, \ldots, u_m\}$; we will pull-back this restriction under the map $\phi|_{\mathrm{res}\ \mathcal{Z}}$ to obtain the curvature of the bundle E_{res}. Indeed, we have

$$\begin{aligned}
\phi|_{\mathrm{res}}^*\Big(\mathcal{K}_{\mathcal{N}}(u)\Big) &\stackrel{\text{def}}{=} \phi^*\Big(\sum_{i,j=2}^{n} (\tfrac{\partial^2}{\partial u_i \partial \bar{u}_j} \log(K_\phi|_{\mathrm{res}}))(u,u)\; du_i \wedge d\bar{u}_j\Big) \\
&= \sum_{i,j=2}^{m} \Big(\tfrac{\partial^2}{\partial u_i \partial \bar{u}_j} \log(K_\phi|_{\mathrm{res}})\Big)(\phi|_{\mathrm{res}}(z), \phi|_{\mathrm{res}}(z))\; \phi|_{\mathrm{res}}^*(du_i \wedge d\bar{u}_j) \\
&= \sum_{i,j=2}^{m} \tfrac{\partial^2}{\partial u_i \partial \bar{u}_j} \log K(\phi^{-1}(0, u_2, \ldots, u_m), \phi^{-1}(0, u_2, \ldots, u_m))\; \phi|_{\mathrm{res}}^*(du_i \wedge d\bar{u}_j) \qquad (1.6)
\end{aligned}$$

With the choice of co-ordinates we have made, $D\phi$ acts as the identity on the normal subspace to the zero set $\mathcal{Z} \subseteq \Omega$. Thus, if we first restrict the curvature $\mathcal{K}_{\mathcal{N}}$ to the zero set $\{u_1 = 0\}$, then the pull - back operation is redundant except for identifying the basis $\{\frac{\partial}{\partial u_2}, \ldots, \frac{\partial}{\partial u_m}\}$ with the basis $\{\frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_m}\}$ in the normal subspace of $T\Omega$ corresponding to $\mathcal{Z}$. If we take K_{res} to be the metric restricted to the zero set with respect to this particular co - ordinate system, then the above calculation proves:

THEOREM 1.3 *The curvature of the bundle* $E_{\rm res}$ *is the restriction* $\sum_{i,j=2}^{m}\left(\frac{\partial^2}{\partial z_i\partial\bar z_j}\log K|_{\rm res}\right) dz_i\wedge d\bar z_j$ *for a suitable choice of co-ordinates.*

The statement of this theorem is perhaps related to the Adjunction formula I [9, p. 146]. It is possible to state Theorem 1.3 in a co-ordinate free manner, that is, in a form that doesn't require $\mathcal{Z}$ to be expressed in special co-ordinates. The tangent bundle for $\mathcal{Z}$ is a subbundle of the tangent bundle on Ω with the induced hermitian metrics agreeing. This fact induces an orthogonal projection from the bundle with two-form sections over Ω to the corresponding bundle over $\mathcal{Z}$ which acts to restrict the curvature of E viewed only on $\mathcal{Z}$ to yield the curvature of $E_{\rm res}$. In the co-ordinates introduced above for $\mathcal{Z}$, this is the action described in the statement of the theorem.

The previous discussion identifies the quotient module as $\mathcal{M}_{\rm res}$. If $\mathcal{M}_{\rm q}$ is in $B_1(\mathcal{Z})$ (cf. [4]), then the curvature of the bundle $E_{\rm q}$ is a complete unitary invariant for the quotient module. Further, Theorem 1.3 shows how to calculate this curvature. In spite of this we look for simpler invariants for the quotient module. The reasons for this are twofold. First, the curvature of the bundle $E_{\rm res}$ is not always easy to calculate. Secondly, we suspect that the curvature calculation for the quotient module has some analogy with the earlier work of Bott and Chern [2]. They start out with a short exact sequence of complex hermitian vector bundles. One of their results relates the Chern classes associated with these bundles in a very simple manner. Even though our starting point is a short exact sequence of Hilbert modules and we can associate complex hermitian vector bundles via the localisation technique, the resulting vector bundles do not form a short exact sequence. We proceed somewhat differently to look for purely topological or geometric invariants in our situation.

In addition to the bundles E and E_0 which we have discussed, we will need to consider the bundle $\mathrm{Hom}(E_0, E)$. The localisation $X(w)$ of the inclusion map $X:\mathcal{M}_0\to\mathcal{M}$ from (1.5) provides a section for this bundle which is non zero off the set $\mathcal{Z}$. The maps $w\to X(w)^*X(w)$ and $w\to X(w)X(w)^*$ define metrics for E_0 and E respectively. The map $X(w)$ is in $\mathrm{Hom}(E_0,E)$ and mediates between these metrics and the ones defined by the hermitian structure on E.

The following Theorem attempts to expand on the discussion on page 119 in [7]. Let $\phi^1 = f_1^{r_1}\cdots f_k^{r_k}$ be the factorisation of ϕ^1 into irreducibles $f_1,\dots,f_k$. Then one may drop the multiplicities $r_1,\dots,r_k$ and take the product of $f_1,\dots,f_k$ to be the defining function, in the sense of [10, p. 33], for the hypersurface $\mathcal{Z}$. As pointed out earlier, we may choose a global defining function for the hypersurface $\mathcal{Z}$ as long as the second Cousin problem can be solved on Ω. We refer the reader to [9, chapter 3] for a discussion of currents.

THEOREM 1.4 *Let* Ω *be a domain in* $\mathbb{C}^m$ *for which the second Cousin problem is solvable. If* ϕ^1 *is the defining function for the hypersurface* $\mathcal{Z}$*, then*

$$\sum_{i,j=1}^{m}\frac{\partial^2}{\partial w_i\partial\bar w_j}\log X(w)^*X(w)dw_i\wedge d\bar w_j=\sum_{i,j=1}^{m}\frac{\partial^2}{\partial w_i\partial\bar w_j}|\phi^1|^2dw_i\wedge d\bar w_j-\mathcal{K}_0(w)+\mathcal{K}(w).$$

is valid as an equation for currents on Ω.

Proof: Let U be an open subset of Ω containing a point $z_0\in\mathcal{Z}$. Recall that $\mathcal{M}_0$ is the space of all functions in $\mathcal{M}$ which vanish on $\mathcal{Z}$. If ϕ^1 is a defining function for $\mathcal{Z}$, then ϕ^1 and all $h\in\mathcal{M}_0$ vanish on $\mathcal{Z}$. Hence $h=\phi^1\cdot g$, for some holomorphic function g defined on the open set U with $g\neq 0$ on U. Let e_n be an orthonormal basis for $\mathcal{M}_0$. The reproducing kernel has the expansion $K_0(z,w)=\sum_{n=0}^{\infty}e_n(z)\overline{e_n(w)}$. Since $e_n(z)=\phi^1(z)g_n(z)$ on the set U for each n, it follows that $K_0(z,w)=\phi^1(z)\overline{\phi^1(w)}\chi_U(z,w)$ on U, where $\chi_U(z,w)=\sum_{n=0}^{\infty}g_n(z)\overline{g_n(w)}$. The reproducing property of $K_0(\cdot,w)$ implies that $K_0(w,w)$ does not vanish on $\Omega\backslash\mathcal{Z}$. Since ϕ^1 is a defining function, it follows that $\chi_U(w,w) \neq 0$ off the set $\mathcal{Z} \cap U$. We point out that, in fact $\chi_U(w,w)$ is never zero on U. If $\chi_U(w,w) = 0$ for some $w \in U$, then $\sum_{n=0}^{\infty} |g_n(w)|^2 = 0$. It follows that $g_n(w) = 0$ for each n. This in turn would mean the order of the zero at w for each $f \in \mathcal{M}_0$ is strictly greater than 1. This contradiction proves our assertion.

Note that $K_0(w,w)$ differs from $\chi_U(w,w)$ by the absolute value of a nonvanishing holomorphic function on an open set which does not intersect $\mathcal{Z}$. Therefore,

$$\sum_{i,j=1}^{m} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log |\chi_U(w,w)|^2 dw_i \wedge d\bar{w}_j = -\mathcal{K}_0(w), \tag{1.7}$$

for w in any open subset of U disjoint from $\mathcal{Z}$. The real analytic nature of the curvature determines it everywhere once we know it on any open set. Since χ_U is not zero on U, it follows that (1.7) is valid on all of U

It is easy to see that $X(w)^*X(w) = K_0(w,w)/K(w,w)$. The following calculation is valid on the open set $U \subseteq \Omega$ in the distributional sense.

$$\begin{aligned}
&\sum_{i,j=1}^{m} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log X(w)^*X(w) dw_i \wedge d\bar{w}_j \\
&= \sum_{i,j=1}^{m} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \Big((|\phi^1(w)|^2 \chi_U(w,w))/K(w,w) \Big) dw_i \wedge d\bar{w}_j \\
&= \sum_{i,j=1}^{m} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \Big(\log |\phi^1(w)|^2 + \log |\chi_U(w,w)| \Big) dw_i \wedge d\bar{w}_j + \mathcal{K}(w) \\
&= \sum_{i,j=1}^{m} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log |\phi^1(w)|^2 dw_i \wedge d\bar{w}_j - \mathcal{K}_0(w) + \mathcal{K}(w)
\end{aligned} \tag{1.8}$$

This calculation for an arbitrary $z_0 \in U \cap \mathcal{Z}$ together with the fact that ϕ^1 is a global deifining function completes the proof.

The Poincare-Lelong equation [9, p. 388] relates the current

$$c(f) \stackrel{\text{def}}{=} \sum_{i,j=1}^{m} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log |f(w)|^2 dw_i \wedge d\bar{w}_j \tag{1.9}$$

to the fundamental class of the divisor associated with f.

Any two defining function for the hypersurface $\mathcal{Z}$ differ by a nonvanishing holomorphic function. The expression in (1.9) depends only on the hypersurface $\mathcal{Z}$ if $f = \phi^1$ is a defining function for the hypersurface $\mathcal{Z}$.

2 Examples

We have identified the curvature of the quotient module in Theorem 1.3. Further, the three term alternating sum $\sum_{i,j=1}^{m} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log (X(w)^*X(w)) dw_i \wedge d\bar{w}_j - \mathcal{K}_0 + \mathcal{K}$ is the current $c(\phi^1)$. This current depends only on $\mathcal{Z}$ since ϕ^1 is a defining function for $\mathcal{Z}$. In the following subsections we calculate the difference $\mathcal{K}_0|_{\mathcal{Z}} - \mathcal{K}|_{\mathcal{Z}}$.

2.1 EXAMPLES (n, ε)

Let $\mathcal{M}$ denote the functional Hilbert space on the bi–disk with the reproducing kernel

$$K(z,w) = (1 - z_1\bar{w}_1)^{-n}(1 - z_2\bar{w}_2)^{-n}, \quad z, w \in \mathbb{D}^2, \text{ for a fixed } n \in \mathbb{N}. \qquad (2.1)$$

Let $p_\varepsilon = z_1 - \varepsilon z_2$ and $\mathcal{Z}_\varepsilon$ be the associated zero set.

We know that the vectors $\{z_1^k z_2^\ell : k, \ell \geq 0\}$ form a complete orthogonal set in $\mathcal{H}$. The norm of these vectors is obtained from the power series expansion

$$K(z,z) = \frac{1}{(1-|z_1|^2)^n}\frac{1}{(1-|z_2|^2)^n} = 1 + n(|z_1|^2 + |z_2|^2) + \frac{n(n+1)}{2}|z_1|^4 + n^2|z_1|^2|z_2|^2 + \frac{n(n+1)}{2}|z_2|^4 + \cdots,$$

where the co–efficient of $|z_1|^{2\ell}|z_2|^{2k}$ is $\|z_1^\ell z_2^k\|^{-2}$. In the following calculation, we have fixed an arbitrary pair n and ε. To describe the subspace $\mathcal{M}_q$, consider the homogeneous polynomial $\sum_{\ell=0}^k a_\ell z_1^{k-\ell} z_2^\ell$, and note that

$$\begin{aligned}\Big\langle \sum_{\ell=0}^k a_\ell z_1^{k-\ell} z_2^\ell, z_1^i z_2^j (z_1 - \varepsilon z_2)\Big\rangle &= \Big\langle \sum_{\ell=0}^k a_\ell z_1^{k-\ell} z_2^\ell, \Big(z_1^{i+1} z_2^j - \varepsilon z_1^i z_2^{j+1}\Big)\Big\rangle \\ &= \sum_{\ell=0}^k \Big(a_\ell \langle z_1^{k-\ell} z_2^\ell, z_1^{i+1} z_2^j\rangle - \varepsilon\langle z_1^{k-\ell} z_2^\ell, z_2^i z_2^{j+1}\rangle\Big).\end{aligned}$$

Since

$$\langle z_1^{k-\ell} z_2^\ell, z_1^{i+1} z_2^j\rangle = \begin{cases} 0 & i+1 \neq k-\ell \\ & \text{or } j \neq \ell \\ 1 & i+1 = k-\ell \\ & \text{and } j = \ell \end{cases},$$

it follows that if we put $a_0 = \frac{1}{\|z_1^k\|^2}$, then $\sum_{\ell=0}^k a_\ell z_1^{k-\ell} z_2^\ell$ is orthogonal to $\mathcal{M}_0$ if and only if

$$a_1 = \frac{1}{\varepsilon\|z_1^{k-1} z_2\|^2}, \; a_2 = \frac{1}{\varepsilon^2\|z_1^{k-2} z_2^2\|^2}, \; \ldots, \; a_k = \frac{1}{\varepsilon^k\|z_2^k\|^2}. \qquad (2.2)$$

Let $\mathcal{P}_k$ denote the space of homogeneous polynomials of degree k on $\mathbb{D}^2$. Then $\mathcal{M}_\mathrm{q} = \bigoplus_{k=0}^\infty (\mathcal{P}_k \cap \mathcal{M}_\mathrm{q})$. It is easily seen that the dimension of $\mathcal{M}_q \cap \mathcal{P}_k = 1$. Hence $\{e_k : k \geq 0\}$ is an orthogonal spanning set for $\mathcal{M}_q$, where

$$\begin{aligned} e_k(z_1, z_2) &= \sum_{\ell=0}^k \frac{z_1^{k-\ell} z_2^\ell}{\varepsilon^\ell \|z_1^{k-\ell} z_2^\ell\|^2} \\ \|e_k(z_1, z_2)\|^2 &= \sum_{\ell=0}^k \frac{\|z_1^{k-\ell} z_2^\ell\|^2}{\varepsilon^{2\ell}\|z_1^{k-\ell} z_2^\ell\|^4} = \sum_{\ell=0}^k \|z_1^{k-\ell} z_2^\ell\|^{-2}\varepsilon^{-2\ell}.\end{aligned}$$

We calculate the module action for the Hardy space. Note that, in this case, $\|z_1^{k-\ell} z_2^\ell\| = 1$. Hence $\|e_k\|^2 = \sum_{\ell=0}^k \varepsilon^{-2\ell}$.

$$PM_1 \frac{e_k}{\|e_k\|} = P\frac{\Big(z_1^{k+1} + \cdots + z_2^k z_1 \varepsilon^{-k} + z_2^{k+1}\varepsilon^{-(k+1)} - z_2^{k+1}\varepsilon^{-(k+1)}\Big)}{\|e_{k+1}\|\,\|e_k\|}\|e_{k+1}\|$$

$$\begin{aligned}
&= \frac{1}{\|e_k\|}\left\{\|e_{k+1}\|\frac{e_{k+1}}{\|e_{k+1}\|} - \langle \frac{e_{k+1}}{\|e_{k+1}\|}, \frac{z_2^{k+1}}{\varepsilon^{k+1}}\rangle \frac{e_{k+1}}{\|e_{k+1}\|}\right\}\\
&= \frac{1}{\|e_k\|}\frac{e_{k+1}}{\|e_{k+1}\|}\left\{\|e_{k+1}\| - \frac{1}{\|e_{k+1}\|\varepsilon^{2k+2}}\right\},\\
\frac{\varepsilon^{2k+2}\|e_{k+1}\|^2-1}{\varepsilon^{2k+2}\|e_{k+1}\|} &= \frac{\varepsilon^{2k+2}+\cdots+\varepsilon^2}{\varepsilon^{2k+2}\|e_{k+1}\|} = \frac{\left(1+\cdots+\varepsilon^{-2k}\right)}{\|e_{k+1}\|} = \frac{\|e_k\|^2}{\|e_{k+1}\|},
\end{aligned}$$

it follows that

$$PM_1\frac{e_k}{\|e_k\|} = \frac{e_{k+1}}{\|e_k\|\,\|e_{k+1}\|}\frac{\varepsilon^{2k+2}\|e_{k+1}\|^2-1}{\varepsilon^{2k+2}\|e_{k+1}\|} = \frac{\|e_k\|^2}{\|e_{k+1}\|}\frac{e_{k+1}}{\|e_k\|\,\|e_{k+1}\|} = \frac{\|e_k\|}{\|e_{k+1}\|}\frac{e_{k+1}}{\|e_{k+1}\|}.$$

Similarly,

$$\begin{aligned}
P(\varepsilon^{-1})M_2\frac{e_k}{\|e_k\|} &= P\frac{\frac{z_1^k}{\varepsilon}z_2+\cdots+\frac{z_2^{k+1}}{\varepsilon^{k+1}}+z_1^{k+1}-z_1^{k+1}}{\|e_k(z)\|}\\
&= \frac{1}{\|e_k(z)\|}\left\{\frac{\|e_{k+1}\|e_{k+1}}{\|e_{k+1}\|} - \langle \frac{e_{k+1}}{\|e_{k+1}}, z_1^{k+1}\rangle \frac{e_{k+1}}{\|e_{k+1}\|}\right\}\\
&= \frac{e_{k+1}}{\|e_{k+1}\|\|e_k\|}\left\{\|e_{k+1}\| - \frac{1}{\|e_{k+1}\|}\right\}\\
&= \frac{1}{\varepsilon^2}\frac{\left(\frac{1}{\varepsilon^{2k}}+\cdots+1\right)}{\|e_{k+1}\|}\frac{e_{k+1}}{\|e_k\|\,\|e_{k+1}\|}\\
&= \frac{1}{\varepsilon^2}\frac{\|e_k\|}{\|e_{k+1}\|}\frac{e_{k+1}}{\|e_{k+1}\|}.
\end{aligned}$$

Thus, we have

$$PM_1\frac{e_k}{\|e_k\|} = \varepsilon(PM_2)\frac{e_k}{\|e_k\|} = \frac{\|e_k\|}{\|e_{k+1}\|}e_{k+1}.$$

We are now ready to calculate the kernel function for $\mathcal{M}_0$. We know that

$$\begin{aligned}
e_0(z_1,z_2) &= 1\\
e_1(z_1,z_2) &= \frac{z_1}{\|z_1\|^2}+\frac{z_2}{\varepsilon\|z_2\|^2} = \frac{n(z_1+\frac{z_2}{\varepsilon})}{\sqrt{n(1+\varepsilon^{-2})}}\\
e_2(z_1,z_2) &= \left(\frac{z_1^2}{\|z_1^2\|^2}+\frac{z_1z_2}{\varepsilon\|z_1z_2\|^2}+\frac{z_1^2}{\varepsilon^2\|z_2\|^2}\right)\left(\frac{n(n+1)}{2}+\frac{n^2}{\varepsilon^2}+\frac{n(n+1)}{2\varepsilon^4}\right)^{-1/2}.
\end{aligned}$$

The kernel function $K_q(z,z)$ admits an expansion of the form $\sum_{k=0}^{\infty}|e_k|^2\|e_k\|^{-2}$. We will write $K(z,z) = \sum_{\ell=0}^{\infty} f_\ell(z_1,z_2)$, where each f_ℓ is a homogeneous real analytic polynomial of degree ℓ. Of course,

$$K_0 = K - K_q = \sum_{k=0}^{\infty}(f_k - |e_k|^2\|e_k\|^{-2}).$$

It is not hard to see that $|p_\varepsilon|^2$ is a factor of $f_k - |e_k|^2\|e_k\|^{-2}$ for $k\ge 0$. We will carry out this factorisation for $k\le 2$ since it will carry all the information relevant to the curvature calculation.

Since $f_0 - |e_0|^2\|e_0\|^{-2} = 0$, we start with $k = 1$:

$$\begin{aligned} n(|z_1|^2+|z_2|^2) - |e_1(z_1,z_2)|^2\|e_1\|^{-2} &= n(|z_1|^2+|z_2|^2) - n\frac{|z_1+\frac{z_2}{\varepsilon}|^2}{1+\varepsilon^{-2}} \\ &= n\{|z_1|^2+|z_2|^2 - \frac{|z_1\varepsilon+z_2|^2}{1+\varepsilon^2}\} \\ &= n\{(1+\varepsilon^2)(|z_1|^2+|z_2|^2) - (|z_1|^2\varepsilon^2+|z_2|^2+\varepsilon z_1\overline{z}_2+\varepsilon\overline{z}_1 z_2)\} \\ &= \frac{n}{1+\varepsilon^2}\{\varepsilon^2|z_2|^2+|z_1|^2+\varepsilon z_1\overline{z}_2+\varepsilon\overline{z}_1 z_2\} \\ &= \frac{n}{1+\varepsilon^2}|z_1-\varepsilon z_2|^2. \end{aligned}$$

Now we calculate the case with $k = 2$:

$$\begin{aligned} &\left(\frac{n(n+1)}{2}|z_1|^4 + n^2|z_1|^2|z_2|^2 + \frac{n(n+1)}{2}|z_2|^4\right) - |e_2(z_1,z_2)|^2\|e_2\|^{-2} \\ &= \frac{1}{2}\Big(n((1+\varepsilon^2)n+(1+\varepsilon^4))\Big)\Bigg(\left(\frac{n(n+1)}{2}|z_1|^4 + n^2|z_1|^2|z_2|^2 + \frac{n(n+1)}{2}|z_2|^4\right) \\ &\qquad - \left|\varepsilon^2\frac{n(n+1)}{2}z_1^2 + \varepsilon n^2 z_1 z_2 + \frac{n(n+1)}{2}z_2^2\right|^2\Bigg)\frac{2}{n((1+\varepsilon^2)^2n+(1+\varepsilon^4))} \\ &= \Bigg[\frac{n^2(n+1)}{4}((1+\varepsilon^2)^2n+(1+\varepsilon^4))|z_1|^4 + \frac{n^3((1+\varepsilon^2)^2n+(1+\varepsilon^4))}{2}|z_1|^2|z_2|^2 \\ &\qquad + \frac{n^2(n+1)}{4}((1+\varepsilon^2)^2n+(1+\varepsilon^4))|z_2|^4 \\ &\quad - \Bigg\{\frac{\varepsilon^4n^2(n+1)^2}{4}|z_1|^4 + \frac{\varepsilon^3n^3(n+1)}{2}z_1z_2\overline{z}_1^2 + \frac{\varepsilon^2n^2(n+1)^2}{4}z_2^2\overline{z}_1\overline{z}_1^2 \\ &\qquad + \frac{\varepsilon^3n^3(n+1)}{2}z_1^2\overline{z}_1\overline{z}_2 + \varepsilon^2n^4|z_1|^2|z_2|^2 + \frac{\varepsilon n^3(n+1)}{2}z_2^2\overline{z}_1\overline{z}_2 \\ &\quad + \frac{\varepsilon^2n^2(n+1)^2}{4}z_1^2\overline{z}_2^2 + \frac{\varepsilon n^3(n+1)}{2}z_1z_2\overline{z}_2^2 + \frac{n^2(n+1)^2}{4}|z_2|^4\Bigg\}\Bigg]\frac{2}{n((1+\varepsilon^2)^2n+(1+\varepsilon^4))} \\ &= (z_1-\varepsilon z_2)\Bigg\{\frac{\varepsilon^2n^3(n+1)}{2}|z_1|^2(\overline{z}_1-\varepsilon\overline{z}_2) + \frac{n^3(n+1)}{2}|z_2|^2(\overline{z}_1-\varepsilon\overline{z}_2) \\ &\qquad + \frac{n^2(n+1)^2}{4}(\overline{z}_1^2-\varepsilon^2\overline{z}_1^2)(z_1+\varepsilon z_2)\Bigg\}\frac{2}{n((1+\varepsilon^2)^2n+(1+\varepsilon^4))} \\ &= |z_1-\varepsilon z_2|^2\left\{\frac{\varepsilon^2n^3(n+1)}{2}|z_1|^2 + \frac{n^3(n+1)}{2}|z_2|^2 + \frac{n^2(n+1)^2}{4}|z_1+\varepsilon z_2|^2\right\}\frac{2}{n((1+\varepsilon^2)^2n+(1+\varepsilon^4))}. \end{aligned}$$

Consequently, we have

$$\begin{aligned} &K_0(z,z) = |z_1-\varepsilon z_2|^2 \\ &\quad \Big\{\frac{n}{1+\varepsilon^2} + \frac{2}{n((1+\varepsilon^2)^2n+(1+\varepsilon^4))}\{\frac{\varepsilon^2n^3(n+1)}{2}|z_1|^2 + \frac{n^3(n+1)}{2}|z_2|^2 + \frac{n^2(n+1)^2}{4}|z_1+\varepsilon z_2|^2\} + \cdots\Big\}. \end{aligned}$$

Thus we have determined the norm $K_0(w,w) = |w_1-w_2|^2\chi(w,w)$. As pointed out earlier, we may use $\chi(w,w)$ to calculate the curvature $\mathcal{K}_0$ for the bundle E_0. In view of Lemma 2.3 [13], at least we can read off the curvature of E_0 at 0 from this calculation. To compute the curvature of the restriction we use Theorem 1.3 and find that

$$\mathcal{K}_0|_{\mathcal{Z}}(0) = -\frac{2(\frac{n^3(n+1)}{2}(1+\varepsilon^4)+n^2(n+1)^2\varepsilon^2)}{n((1+\varepsilon^2)^2 n+(1+\varepsilon^4))}\,\frac{1+\varepsilon^2}{n} \tag{2.3}$$

A similar application of [13, Lemma 2.3] to the expansion for the kernel function K shows that $\mathcal{K}|_{\mathcal{Z}}(0) = -n(1+\varepsilon^2)$. This proves :

PROPOSITION 2.1 *Let $\mathcal{M}$ denote the functional Hilbert space on the bi–disk with the reproducing kernel given in (2.1). Then $\mathcal{M}$ is a module over the algebra $\mathcal{A}(\mathbb{D}^2)$. Let $\mathcal{M}_0$ be the submodule of functions vanishing on $\mathcal{Z} = \{(z_1, z_2) \in \mathbb{D}^2 : z_1 - \varepsilon z_2 = 0\}$. Then*

$$\mathcal{K}_0|_{\mathcal{Z}}(0) - \mathcal{K}|_{\mathcal{Z}}(0) = -\frac{2(2n+1)\varepsilon^2}{((1+\varepsilon^2)^2 n+(1+\varepsilon^4))}(1+\varepsilon^2).$$

We find that the difference $\mathcal{K}_0|_{\mathcal{Z}}(0) - \mathcal{K}|_{\mathcal{Z}}(0)$ is not independent of n for arbitrary ε. This shows that the difference of the curvatures for E and E_0 does not depend on just the geometry of $\mathcal{Z}$ and its embedding of $\mathcal{Z}$ in Ω.

2.2 EXAMPLES $(n, 1)$

We discuss the case $\varepsilon = 1$ at some length. In this case, $\mathcal{K}_0|_{\mathcal{Z}}(0) = 2n+2$. For $w \in \mathbb{D}$, let $\varphi(z) = \frac{z-w}{1-\bar{w}z}$ be the Möbius map of the unit disk.

PROPOSITION 2.2 *For $\varepsilon = 1$ and any n, we have $\mathcal{K}_0|_{\mathcal{Z}}(w) = |\varphi'(w)|^2\mathcal{K}_0(0)$. Similarly, $\mathcal{K}|_{\mathcal{Z}}(w) = |\varphi'(w)|^2\mathcal{K}(0)$.*

Proof: Clearly, a theorem similar to the one in [12, p. 457] is valid for the module $\mathcal{M}$. Explicitly, the module maps $h \to f \cdot h$ and $h \to (f \circ (\phi, \phi)) \cdot h$ for $f \in \mathcal{A}(\mathbb{D}^2)$ are unitarily equivalent. In other words, $\varphi_*\mathcal{M}$ and $\mathcal{M}$ are isomorphic. The unitary inducing the unitary equivalence is given by the formula

$$U_\varphi : h \to \Big(\det D(\varphi, \varphi)\Big)^{n/2}\Big(h \circ (\varphi, \varphi)\Big), \quad h \in \mathcal{M}. \tag{2.4}$$

It is clear that each of these unitary operator leaves $\mathcal{M}_0$ invariant. As pointed out in [3, p. 115], in this case, the submodule $\mathcal{M}_0 \simeq \varphi_*\mathcal{M}_0$ and the quotient module $\mathcal{M}_{\rm q} \simeq \varphi_*\mathcal{M}_{\rm q}$. We can now calculate the curvature for these modules at any point $(\varphi(0), \varphi(0))$ using the equivalence under the Möbius map and a change of variable formula (cf. [12]). This calculation applied to the module $\mathcal{M}$ will give us the second part of the theorem. In view of Theorem 1.3, the restriction of the curvature $\mathcal{K}_0$ corresponds to the curvature of the module $\mathcal{M}_0|_{\text{res }\mathcal{Z}}$. But the unitary U_φ leaves the module $\mathcal{M}_0|_{\text{res }\mathcal{Z}}$ invariant. Thus the previous discussion applies to this last module and completes the proof of the first part.

This Proposition enables us to calculate the difference of the restrictions of the two curvature at an arbitrary point if the zero set is of the form $\{(z_1, z_2) \in \mathbb{D}^2 : z_1 = z_2\}$.

PROPOSITION 2.3 *Let $\mathcal{M}$ denote the functional Hilbert space on the bi–disk with the reproducing kernel given in (2.1). Then $\mathcal{M}$ is a module over the algebra $\mathcal{A}(\mathbb{D}^2)$. Let $\mathcal{M}_0$ be the submodule of functions vanishing on $\mathcal{Z} = \{(z_1, z_2) \in \mathbb{D}^2 : z_1 - z_2 = 0\}$. Then*

$$(\mathcal{K}_0 - \mathcal{K})|_{\mathcal{Z}}(z_2) = -2(1-|z_2|^2)^{-2}dz_2 \wedge d\bar{z}_2.$$

Proof: It follows from [13, Lemma 2.3] that

$$\mathcal{K}_0|_{\mathcal{Z}}(0) = -(2n+2)dz_2 \wedge d\bar{z}_2.$$

Proposition 2.2 implies that at an arbitrary point z in the unit disk we must have

$$\mathcal{K}_0|_{\mathcal{Z}}(z_2) = -(2n+2)(1-|z_2|^2)^{-2}dz_2 \wedge d\bar{z}_2.$$

Since $\mathcal{K}|_{\mathcal{Z}}(z_2) = -2n(1-|z_2|^2)^{-2}dz_2 \wedge d\bar{z}_2$, it follows that the difference of the curvatures restricted to the zero set $\mathcal{Z}$ is given by

$$(\mathcal{K}_0 - \mathcal{K})|_{\mathcal{Z}}(z_2) = -2(1-|z_2|^2)^{-2}dz_2 \wedge d\bar{z}_2.$$

This completes the proof.

All our calculations so far are for an arbitrary n. However, the final expression for the differnece of the curvatures is independent of n. As we pointed out at the end of the previous section, this is not always the case. However, if $\varepsilon = 0$, then the difference of the two curvatures is again independent of n. We will discuss the case $\varepsilon = 0$ in the next section in slightly greater generality.

2.3 The case of a product domain

PROPOSITION 2.4 *Let $\Omega = \Omega_1 \times \Omega_2$ be a product domain in $\mathbb{C}^2$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be functional Hilbert spaces over Ω_1 and Ω_2, respectively. Suppose that both $\mathcal{M}_i$ are Hilbert modules over the algebras $\mathcal{A}(\Omega_i)$, $i = 1, 2$. Then the Hilbert space tensor product $\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2$ is a Hilbert module over the algebra $\mathcal{A}(\Omega_1 \times \Omega_2)$. Let $\mathcal{M}_0$ be the submodule of functions in $\mathcal{M}$ which vanish on $\mathcal{Z} = \{(w_1, w_2) \in \Omega: w_1 = 0\}$. Then*

$$\Big(\mathcal{K}_0(w) - \mathcal{K}(w)\Big)\Big|_{\mathcal{Z}} = \mathcal{K}_{\mathrm{q}}(w) - \mathcal{K}|_{\mathcal{Z}}(w) = 0.$$

Proof: If $\mathcal{M}_1$ and $\mathcal{M}_2$ are functional Hilbert spaces over Ω_1 and Ω_2, respectively, then the Hilbert space tensor product $\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2$ is a functional Hilbert space over Ω. The reproducing kernel $K(\cdot, w)$ for $\mathcal{M}$ is simply the product of the two reproducing kernels for $\mathcal{M}_1$ and $\mathcal{M}_2$ (cf. [1]). Since $\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2$, it follows that $\mathcal{M}_0 = z_1\mathcal{M}_1 \otimes \mathcal{M}_2$. Let $w = (w_1, w_2) \in \Omega$. We denote the kernel functions for $\mathcal{M}_1$ and $\mathcal{M}_2$ by $K_1(\cdot, w)$ and $K_2(\cdot, w)$, respectively. To calculate the kernel function for the module $z_1\mathcal{M}_1$, we assume, without loss of generality, that $K_1(\cdot, 0) = 1$, that is,

$$K_1(z_1, w_1) = \sum_{n,m=0}^{\infty} a_{nm} z_1^n \bar{w}_1^m, \quad a_{00} = 1,\ a_{10} = 0 = a_{01}.$$

The subspace $z_1\mathcal{M}_1$ is the ortho-complement of the 1 - dimensional space of scalars in $\mathcal{M}_1$. Consequently, the reproducing kernel for this subspace is

$$\begin{aligned} K_1(z_1, w_1) - 1 &= z_1\bar{w}_1 \left(\sum_{n,m=1}^{\infty} a_{nm} z_1^{n-1} \bar{w}_1^{m-1} \right) \\ &= z_1\bar{w}_1 f(z_1, w_1), \end{aligned}$$

where $f(z_1, w_1) = \sum_{n,m=1}^{\infty} a_n z_1^{n-1} \bar{w}_1^{m-1}$. Thus the reproducing kernel $K_0(z, w)$ for the subspace $\mathcal{M}_0$ is

$$K_0(z, w) = (z_1\bar{w}_1) f(z_1, w_1) K_2(z_2, w_2).$$

The reproducing kernel $K_{\rm q}(\cdot, w)$ is the difference

$$\begin{aligned} K_{\rm q}(z,w) &= K(z,w) - K_0(z,w) \\ &= K(z,w) - (z_1\bar{w}_1)f(z_1,w_1)K_2(z_2,w_2) \\ &= \Big(K_1(z_1,w_1) - (z_1\bar{w}_1)f(z_1,w_1)\Big)K_2(z_2,w_2) \\ &= K_2(z_2,w_2). \end{aligned} \tag{2.5}$$

We can now calculate the curvatures of these three modules.

$$\begin{aligned} \mathcal{K}(w) &= -\sum_{i,j=1}^{2} \tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} \log K_1(w_1,w_1)K_2(w_2,w_2)\, dw_i \wedge d\bar{w}_j \\ &= -\sum_{i=1}^{2} \tfrac{\partial^2}{\partial w_i \partial \bar{w}_i} \log K_i(w_i,w_i)\, dw_i \wedge d\bar{w}_j. \\ \mathcal{K}_{\rm q}(w) &= -\tfrac{\partial^2}{\partial w_2 \partial \bar{w}_2} \log K_2(w_2,w_2)\, dw_2 \wedge d\bar{w}_2. \\ \mathcal{K}_0(w) &= -\sum_{i,j=1}^{2} \tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} \log (w_1\bar{w}_1)f(w_1,w_1)K_2(w_2,w_2)\, dw_i \wedge d\bar{w}_j \\ &= -\Big(1 + \tfrac{\partial^2}{\partial w_1 \partial \bar{w}_1} \log f(w_1,w_1)\Big)\, dw_1 \wedge d\bar{w}_1 - \tfrac{\partial^2}{\partial w_2 \partial \bar{w}_2} \log K_2(w_2,w_2) dw_2 \wedge d\bar{w}_2. \end{aligned}$$

Now restricting the curvatures, we find that

$$(\mathcal{K}_0(w) - \mathcal{K}(w))|_{\mathcal{Z}} = \mathcal{K}_{\rm q}(w) - \mathcal{K}|_{\mathcal{Z}}\,(w) = 0.$$

This completes the proof.

2.4 Example $(1,1)$

In this subsection, we make more detailed calculations for the special case of the Hardy module. The reproducing kernel has an expansion of the form

$$\begin{aligned} K_{\rm q}(w,w) &= \sum_{n=0}^{\infty} |e_n(w_1,w_2)|^2/(n+1) \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} |w_1^{n-j} w_2^j|^2 \right) /(n+1) \\ &= \sum_{n=0}^{\infty} \frac{|w_1^{n+1} - w_2^{n+1}|^2}{(|w_1 - w_2|^2)(n+1)}. \end{aligned}$$

Since this series is absolutely convergent, we can sum term by term to obtain

$$\begin{aligned} &K_{\rm q}(w,w) \\ &= \frac{1}{|w_1-w_2|^2}\Big(-\log(1-|w_1|^2) + \log(1-\overline{w}_1 w_2) + \log(1 - w_2\overline{w}_1) - \log(1-|w_2|^2\Big) \\ &= \frac{1}{|w_1-w_2|^2} \log\left\{ \frac{|1-w_1\overline{w}_2|^2}{(1-|w_1|^2)(1-|w_2|^2)} \right\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|w_1 - w_2|^2} \log\left(1 + \frac{|(w_1-w_2)|^2}{(1-|w_1|^2)(1-|w_2|^2)}\right)\\
&= \frac{1}{|w_1 - w_2|^2} \left(\frac{|(w_1-w_2)|^2}{(1-|w_1|^2)(1-|w_2|^2)} - \frac{1}{2}\frac{|w_1-w_2|^4}{((1-|w_1|^2)^2(1-|w_2|^2))^2} + \cdots\right)\\
&= K(w,w)\left(1 - \frac{1}{2}K(w,w)|w_1-w_2|^2 + \cdots\right).
\end{aligned}
$$

The reproducing kernel $K_0(\cdot, w)$ is obtained by taking the difference

$$
\begin{aligned}
K_0(w,w) &= K(w,w) - K_{\mathrm{q}}(w,w)\\
&= K(w,w) - K(w,w)\left(1 - \frac{1}{2}K(w,w)|w_1-w_2|^2 + \cdots\right)\\
&= |w_1-w_2|^2 (K(w,w))^2 \left(\frac{1}{2} - \frac{1}{3}K(w,w)|w_1-w_2|^2 + \cdots\right).
\end{aligned}
$$

Recalling (1.8), we calculate

$$
\begin{aligned}
\mathcal{K}(X(w)X(w)^*) &\overset{\text{def}}{=} -\sum_{i,j=1}^{2} \tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \frac{K_0(w,w)}{K(w,w)} dw_i \wedge d\bar{w}_j\\
&= -\sum_{i,j=1}^{2} \tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} \log\left\{|w_1-w_2|^2 K(w,w)\left(\frac{1}{2} - \frac{1}{3}|w_1-w_2|^2 K(w,w) + \cdots\right)\right\} dw_i \wedge d\bar{w}_j\\
&= -\sum_{i,j=1}^{2} \tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} \log|w_1-w_2|^2 \; dw_i \wedge d\bar{w}_j\\
&\quad -\sum_{i,j=1}^{2} \tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} \log K(w,w)\; dw_i \wedge d\bar{w}_j\\
&\quad -\sum_{i,j=1}^{2} \tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} \log\left(\frac{1}{2} - \frac{1}{3}|w_1-w_2|^2 K(w,w) + \cdots\right) dw_i \wedge d\bar{w}_j.
\end{aligned}
$$

The following calculation occurs frequently in what follows.

$$
\begin{aligned}
&\tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} |w_1-w_2|^{2n} K(w,w)\; dw_i \wedge d\bar{w}_j\\
&= \frac{\partial}{\partial w_i}\left((w_1-w_2)^n\left((-1)^{j+1} n(\bar{w}_1-\bar{w}_2)^{n-1} + \Big(\frac{\partial}{\partial \bar{w}_j} K(w,w)^n\Big)(\bar{w}_1-\bar{w}_2)^n\right)\right) dw_i \wedge d\bar{w}_j\\
&= (-1)^{j+1} n(\bar{w}_1-\bar{w}_2)^{n-1}\Big\{ (-1)^{i+1}(w_1-w_2)^{n-1} K(w,w)\\
&\qquad\qquad \Big(\frac{\partial}{\partial w_i} K(w,w)^n\Big)(w_1-w_2)^n\Big\}\; dw_i \wedge d\bar{w}_j\\
&\quad + (\bar{w}_1-\bar{w}_j)^n \Big\{(-1)^i n (w_1-w_2)^{n-1} K(w,w)\\
&\qquad\qquad \Big(\tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} K(w,w)^n\Big)(w_1-w_2)^n\Big\}\; dw_i \wedge d\bar{w}_j.
\end{aligned}
$$

Thus,

$$
\tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} |w_1-w_2|^{2n} K(w,w)\; dw_i \wedge d\bar{w}_j\Big|_{\mathcal{Z}} = \begin{cases} 0 & \text{if } n > 1\\ (-1)^{i+j} K(w,w) dw_i \wedge d\bar{w}_j & \text{otherwise}. \end{cases} \qquad (2.6)
$$

We now to restrict these curvatures to the set $\mathcal{Z}$. By the calculation (2.6),

$$\sum_{i,j=1}^{n} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \left(\frac{1}{2} - \frac{1}{3}|w_1 - w_2|^2 K(w,w) + \cdots \right) \, dw_i \wedge d\bar{w}_j = 0.$$

Also the singular support of the current $T_{\mathcal{Z}} = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log |w_1 - w_2|^2 \, dw_i \wedge d\bar{w}_j$ is $\mathcal{Z}$. We may restrict $T_{\mathcal{Z}}$ to its singular support. It now follows that

$$\begin{aligned} \mathcal{K}(X(w)X(w)^*)|_{\mathcal{Z}} &= -\sum_{i,j=1}^{2} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log K(w,w) \, dw_i \wedge d\bar{w}_j \Big|_{\mathcal{Z}} - T_{\mathcal{Z}}|_{\mathcal{Z}} \\ &= -2(1 - |w_2|^2)^{-2} \, dw_2 \wedge d\bar{w}_2 - T_{\mathcal{Z}}|_{\mathcal{Z}}. \end{aligned} \tag{2.7}$$

Thus we have proved the following :

Proposition 2.5 *Let $\mathcal{M}$ be the Hardy module and $\mathcal{M}_0$ be the submodule of functions vanishing on the diagonal set $\mathcal{Z} = \{(w_1, w_2) \in \mathbb{D}^2 : w_1 = w_2\}$. Then*

$$\mathcal{K}(X(w)X(w)^*)|_{\mathcal{Z}} = -2(1 - |w_2|^2)^{-2} \, dw_2 \wedge d\bar{w}_2 - T_{\mathcal{Z}}\Big|_{\mathcal{Z}}.$$

2.5 Example $(2,1)$

Throughout this subsection $\mathcal{M}$ will stand for the Bergman module $\mathrm{B}^2(\mathbb{D}^2)$. Calculations similar to the ones in the previous subsection show that the kernel function $K_{\mathrm{q}}(\cdot, w)$ is obtained from the expansion

$$\begin{aligned} K_{\mathrm{q}}(w,w) &= \sum_{k=0}^{\infty} |e_k(w)|^2 \|e_k\|^{-2} \\ &= \sum_{k=0}^{\infty} \frac{6}{(k+1)(k+2)(k+3)} |w_1 - w_2|^{-6} \Big\{ \Big| (k+1)(w_1^{k+3} - w_2^{k+3}) \\ &\qquad - (k+3) w_1 w_2 (w_1^{k+1} - w_2^{k+1}) \Big|^2 \Big\} \\ &= 6|w_1 - w_2|^{-6} \Big\{ -2|w_1|^2 + 2\bar{w}_1 w_2 + 2 w_1 \bar{w}_2 - 2|w_2|^2 \\ &\qquad + (2 - (|w_1|^2 + \bar{w}_1 w_2 + w_1 \bar{w}_2 + |w_2|^2) + 2|w_1 \bar{w}_2|^2) \log \frac{|1 - w_1 \bar{w}_2|^2}{(1 - |w_1|^2)(1 - |w_2|^2)} \Big\} \\ &= 6|w_1 - w_2|^{-6} \Big\{ -2|w_1 - w_2|^2 + (|w_1 - w_2|^2 + 2(1 - |w_1|^2)(1 - |w_2|^2)) \\ &\qquad \log \Big(1 + \frac{|w_1 - w_2|^2}{(1 - |w_1|^2)(1 - |w_2|^2)} \Big) \Big\} \\ &= \frac{1}{(1 - |w_1|^2)^2 (1 - |w_2|^2)^2} - \frac{|w_1 - w_2|^2}{(1 - |w_1|^2)^3 (1 - |w_2|^2)^3} + \frac{9}{10} \frac{|w_1 - w_2|^4}{(1 - |w_1|^2)^4 (1 - |w_2|^2)^4} - \cdots \\ &= K(w,w) \Big(1 - \frac{|w_1 - w_2|^2}{(1 - |w_1|^2)(1 - |w_2|^2)} + \frac{9}{10} \frac{|w_1 - w_2|^4}{(1 - |w_1|^2)^2 (1 - |w_2|^2)^2} - \cdots \Big). \end{aligned} \tag{2.8}$$

The kernel function for the submodule $\mathcal{M}_0$ is now calculated as the difference

$$\begin{aligned} K_0(w,w) &= K(w,w) - K_{\mathrm{q}}(w,w) \\ &= K(w,w) \frac{|w_1 - w_2|^2}{(1 - |w_1|^2)(1 - |w_2|^2)} \Big(1 - \frac{9}{10} \frac{|w_1 - w_2|^2}{(1 - |w_1|^2)(1 - |w_2|^2)} + \cdots \Big). \end{aligned} \tag{2.9}$$

Now, we can calculate the curvature

$$\begin{aligned}
\mathcal{K}(X(w)X(w)^*) &\stackrel{\text{def}}{=} -\sum_{i,j=1}^{2} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \frac{K_0(w,w)}{K(w,w)}\, dw_i \wedge d\bar{w}_j \\
&= -\sum_{i,j=1}^{2} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log\Big(\frac{|w_1-w_2|^2}{(1-|w_1|^2)(1-|w_2|^2)}(1-\frac{9}{10}\frac{|w_1-w_2|^2}{(1-|w_1|^2)(1-|w_2|^2)}+\cdots)\Big)\, dw_i \wedge d\bar{w}_j \\
&= -\sum_{i,j=1}^{2} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log |w_1-w_2|^2\, dw_i \wedge d\bar{w}_j \\
&\quad -\sum_{i,j=1}^{2} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \frac{1}{(1-|w_1|^2)(1-|w_2|^2)}\, dw_i \wedge d\bar{w}_j \\
&\quad -\sum_{i,j=1}^{2} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \left(1-\frac{9}{10}\frac{|w_1-w_2|^2}{(1-|w_1|^2)(1-|w_2|^2)}+\cdots\right) dw_i \wedge d\bar{w}_j.
\end{aligned}$$

Using the calculation from (2.6), we have proved :

PROPOSITION 2.6 *Let $\mathcal{M}$ be the Bergman module and $\mathcal{M}_0$ be the submodule of functions vanishing on the diagonal set $\mathcal{Z} = \{(w_1,w_2) \in \mathbb{D}^2 : w_1 = w_2\}$. Then*

$$\mathcal{K}(X(w)X(w)^*)|_{\mathcal{Z}} = -2\,(1-|w_2|^2)^{-2}\, dw_2 \wedge d\bar{w}_2 - T_{\mathcal{Z}}\Big|_{\mathcal{Z}}.$$

3 Higher Multiplicity Localisations

Let $\Omega \subseteq \mathbb{C}^m$ be a bounded domain and $\mathcal{A}(\Omega)$ be the algebra consisting of all functions which are holomorphic in some open set U containing the closure $\bar{\Omega}$ of the set Ω. Let $\mathcal{M}$ be a functional Hilbert space consisting of holomorphic functions on Ω. The map $(f,h) \to f \cdot h$, $f \in \mathcal{A}(\Omega), h \in \mathcal{M}$, where $f \cdot h$ is the pointwise product of complex functions, turns $\mathcal{M}$ into a module over the algebra $\mathcal{A}(\Omega)$. One possibility is that $\mathcal{M}$ is the closure of an ideal in $\mathcal{A}(\Omega)$. The problem of characterizing such modules and, in particular, deciding when two are equivalent, was solved in [8]. A rigidity phenomenon intervenes and is detected by the use of higher multiplicity localization. We recall now the definition of second-order localization and its calculation for modules that are functional Hilbert spaces.

For $w \in \Omega$ and $\alpha \in \mathbb{C}$ possibly depending on w, let us fix a two dimensional module $\mathbb{C}^2_{\omega,\alpha}$ over the algebra $\mathcal{A}(\Omega)$ via the action :

$$\left(f, \begin{pmatrix}\lambda\\ \mu\end{pmatrix}\right) \to \begin{pmatrix} f(w) & (\partial_\alpha f)(w) \\ 0 & f(w)\end{pmatrix}\begin{pmatrix}\lambda\\ \mu\end{pmatrix},$$

where $\partial_\alpha f(w) = \alpha_1 \frac{\partial f}{\partial z_1}(w) + \cdots + \alpha_m \frac{\partial f}{\partial z_m}(w)$. The module tensor product $\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^2_{\omega,\alpha}$ is the orthogonal complement of the following subspace $\mathcal{N}$ in the Hilbert space $\mathcal{M} \otimes \mathbb{C}^2_{\omega,\alpha}$.

$$\mathcal{N} = \Big\{ f\cdot h \otimes \begin{pmatrix}a\\0\end{pmatrix} + f\cdot k \otimes \begin{pmatrix}0\\b\end{pmatrix} - h \otimes f\cdot \begin{pmatrix}a\\0\end{pmatrix} - k \otimes f \cdot \begin{pmatrix}0\\b\end{pmatrix} : \\ h \text{ and } k \text{ are in } \mathcal{M},\ \begin{pmatrix}a\\b\end{pmatrix} \text{ is in } \mathbb{C}^2 \text{ and } f \text{ is in } \mathcal{A}(\Omega) \Big\}$$

$$= \Big\{a(f - f(w)) \cdot h \otimes \begin{pmatrix}1\\0\end{pmatrix} + b((f - f(w)) \cdot k) \otimes \begin{pmatrix}0\\1\end{pmatrix} + b\partial_\alpha f(w) k \otimes \begin{pmatrix}1\\0\end{pmatrix}:$$
$$h \text{ and } k \text{ are in } \mathcal{M}, \begin{pmatrix}a\\b\end{pmatrix} \text{ is in } \mathbb{C}^2 \text{ and } f \text{ is in } \mathcal{A}(\Omega)\Big\}.$$

Since $\mathcal{M}$ is a functional Hilbert space it admits a kernel function $K(\cdot, w)$; that is, *(i)* the function $K(\cdot, w)\colon \Omega \to \mathbb{C}$ is anti-holomorphic for each $w \in \Omega$, *(ii)* $\overline{K(z,w)} = K(w,z)$, and *(iii)* $\langle h, K(\cdot, w)\rangle = h(w)$ for $h \in \mathcal{M}$. It is easy to see, as a consequence of the reproducing property, that

$$\langle h, \partial_\alpha K(\cdot, w)\rangle = \partial_\alpha h(w),$$

where $\partial_\alpha K(\cdot, w) = \alpha_1 \frac{\partial}{\partial w_1} K(\cdot, w) + \cdots + \alpha_m \frac{\partial}{\partial w_m} K(\cdot, w)$. Using these properties of the reproducing kernel, we easily verify that *(i)* $u(w) = K(\cdot, w) \otimes \binom{0}{1} \perp \mathcal{N}$, *(ii)* $v(w) = (K(\cdot, w) \otimes \binom{1}{0} + \partial_\alpha K(\cdot, w) \otimes \binom{0}{1}) \perp \mathcal{N}$, and *(iii)* $\{u(w), v(w)\}$ span $\mathcal{N}^\perp$. We infer that the set $\{u(w), v(w)\}$ is a basis for $\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^2_{\omega,\alpha}$, and $\dim \mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^2_{\omega,\alpha} = 2$. We will now obtain an orthonormal basis for the localisation $\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^2_{\omega,\alpha}$.

The vector $\mu(w) = \frac{u(w)}{\|u(w)\|}$ is a unit vector in $\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^2_{\omega,\alpha}$. To obtain another unit vector orthogonal to $\mu(w)$ in $\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^2_{\omega,\alpha}$, we set

$$\begin{aligned} e(w) &= v(w) - \langle v(w), \mu(w)\rangle \mu(w) \\ &= K(\cdot, w) \otimes \begin{pmatrix}1\\0\end{pmatrix} + \left(\partial_\alpha K(\cdot, w) - \frac{\langle \partial_\alpha K(\cdot, w), K(\cdot, w)\rangle}{\|K(\cdot, w)\|^2} K(\cdot, w)\right) \otimes \begin{pmatrix}0\\1\end{pmatrix}. \end{aligned}$$

The unit vector $\eta(w) = \frac{e(w)}{\|e(w)\|} \in \mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^2_{\omega,\alpha}$ is orthogonal to $\mu(w)$. Thus $\{\mu(w), \eta(w)\}$ is an orthonormal basis for $\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^2_{\omega,\alpha}$. We will need the following expression for the norm $\|e(w)\|$.

$$\begin{aligned} \|e(w)\|^2 &= \|K(\cdot, w)\|^2 + \left(\|\partial_\alpha K(\cdot, w)\|^2 - \frac{|\langle \partial_\alpha K(\cdot, w), K(\cdot, w)\rangle|^2}{\|K(\cdot, w)\|^2}\right) \\ &= \|K(\cdot, w)\|^2 - \|K(\cdot, w)\|^2 \left(\frac{|\langle \partial_\alpha K(\cdot, w), K(\cdot, w)\rangle|^2 - \|\partial_\alpha K(\cdot, w)\|^2 \|K(\cdot, w)\|^2}{\|K(\cdot, w)\|^4}\right) \\ &= \|K(\cdot, w)\|^2 \left\{1 - \left(\frac{|\langle \partial_\alpha K(\cdot, w), K(\cdot, w)\rangle|^2 - \|\partial_\alpha K(\cdot, w)\|^2 \|K(\cdot, w)\|^2}{\|K(\cdot, w)\|^4}\right)\right\}. \end{aligned}$$

Let $\pi\colon E \stackrel{\text{def}}{=} \{\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^2_{\omega,\alpha} : w \in \Omega\} \to \Omega$, $\pi(\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^2_{\omega,\alpha}, w) = w$. The map $w \to \{u(w), v(w)\}$ provides an anti–holomorphic frame for the bundle E over Ω. Further, each fibre is an inner product space and we see that the Grammian matrix $H(w)$ is

$$H(w) = \begin{pmatrix} \|u(w)\|^2 & \langle u(w), v(w)\rangle \\ \overline{\langle u(w), v(w)\rangle} & \|v(w)\|^2 \end{pmatrix},$$

where

$$\begin{aligned} \|u(w)\|^2 &= \|K(\cdot, w)\|^2, \\ \langle u(w), v(w)\rangle &= \langle K(\cdot, w), \partial_\alpha K(\cdot, w)\rangle, \\ \|v(w)\|^2 &= \|K(\cdot, w)\|^2 + \|\partial_\alpha K(\cdot, w)\|^2. \end{aligned}$$

The determinant of the Grammian

$$\begin{aligned}\det H(w) &= \|K(\cdot,w)\|^2\left(\|K(\cdot,w)\|^2+\|\partial_\alpha K(\cdot,w)\|^2\right)+|\langle K(\cdot,w),\partial_\alpha K(\cdot,w)\rangle| \\ &= \|K(\cdot,w)\|^4+\|K(\cdot,w)\|^2\|\partial_\alpha K(\cdot,w)\|^2-|\langle K(\cdot,w),\partial_\alpha K(\cdot,w)\rangle| \\ &= \|K(\cdot,w)\|^2\|e(w)\|^2.\end{aligned}$$

A factorisation for the Gram matrix $H(w)$ is obtained via the matrix $\Gamma(w)$ as follows. Let

$$\Gamma(w)=\begin{pmatrix}\|K(\cdot,w)\| & \dfrac{\langle K(\cdot,w),\partial_\alpha K(\cdot,w)\rangle}{\|K(\cdot,w)\|} \\ 0 & \|e(w)\|\end{pmatrix}.$$

The factorisation

$$\Gamma(w)^*\Gamma(w)=H(w), \tag{3.1}$$

follows from

$$\frac{|\langle K(\cdot,w),\partial_\alpha K(\cdot,w)\rangle|^2}{\|K(\cdot,w)\|}+\|e(w)\|^2=\|K(\cdot,w))\|^2+\|\partial_\alpha K(\cdot,w)\|^2.$$

We now obtain a projection formula for $\mathrm{pr}:\mathcal{M}\otimes\mathbb{C}^2_{\omega,\alpha}\longrightarrow\mathcal{M}\otimes_{\mathcal{A}(\Omega)}\mathbb{C}^2_{\omega,\alpha}$,

$$\begin{aligned}\mathrm{pr}: h\otimes\begin{pmatrix}a\\0\end{pmatrix}+k\otimes\begin{pmatrix}0\\b\end{pmatrix} &\longrightarrow (\langle h\otimes\begin{pmatrix}a\\0\end{pmatrix}+k\otimes\begin{pmatrix}0\\b\end{pmatrix},\mu(w)\rangle)\mu(w) \\ &\qquad +(\langle h\otimes\begin{pmatrix}a\\0\end{pmatrix}+k\otimes\begin{pmatrix}0\\b\end{pmatrix},\eta(w)\rangle)\eta(w) \\ &= \left\{a\langle h,K(\cdot,w)\rangle+\left(\langle k,\partial_\alpha K(\cdot,w)\rangle-\frac{\langle K(\cdot,w),\partial_\alpha K(\cdot,w)\rangle}{\|K(\cdot,w)\|^2}\langle k,K(\cdot,w)\rangle\right)b\right\}\frac{\eta(w)}{\|e(w)\|} \\ &\qquad +\frac{b}{\|K(\cdot,w)\|}\langle k,K(\cdot,w)\rangle\mu(w).\end{aligned}$$

Set $a=\dfrac{\|e(w)\|}{\langle h,K(\cdot,w)\rangle}$ and $b=0$. From the preceeding calculation, it follows that

$$\mathrm{pr}(h\otimes\begin{pmatrix}a\\0\end{pmatrix})=a(\frac{\langle h,K(\cdot,w)\rangle}{\|e(w)\|})=\eta(w). \tag{3.2}$$

Similarly, set

$$\begin{aligned}a &= -\langle h,K(\cdot,w)\rangle^{-1}\left(\langle k,\partial_\alpha K(\cdot,w)\rangle-\frac{\langle K(\cdot,w),\partial_\alpha K(\cdot,w)\rangle}{\|K(\cdot,w)\|^2}\langle k,K(\cdot,w)\rangle\right)\frac{\|K(\cdot,w)\|}{\langle k,K(\cdot,w)\rangle}. \\ b &= \frac{\|K(\cdot,w)\|}{\langle k,K(\cdot,w)\rangle}.\end{aligned}$$

It now follows that

$$\mathrm{pr}(h\otimes\begin{pmatrix}a\\0\end{pmatrix}+k\otimes\begin{pmatrix}0\\b\end{pmatrix})=\mu(w). \tag{3.3}$$

Let $\mathcal{M}_0$ be a submodule of $\mathcal{M}$. We let $K_0(\cdot,w)$ denote the kernel function for $\mathcal{M}_0$. Similarly, let $\{\mu_0(w),\eta_0(w)\}$ be the orthonormal basis for $\mathcal{M}_0\otimes_{\mathcal{A}(\Omega)}\mathbb{C}^2_{\omega,\alpha}$. Let $X\colon\mathcal{M}_0\to\mathcal{M}$ be the inclusion map. Then $X^*\colon \mathcal{M} \to \mathcal{M}_0$ is the projection and $X^*K(\cdot,w) = K_0(\cdot,w)$. Again, there is a short exact sequence of modules as in previous sections with the initial module defined as the "quotient" $\mathcal{M}_{\mathrm{q}}$ of $\mathcal{M}$ by the range of X. If $\mathcal{M}_0$ is not prime-like, then ordinary localization will not determine sufficiently many geometric invariants for $\mathcal{M}_{\mathrm{q}}$ and we must allow higher multiplicity localizations. The following calculations show some of the invariants obtained and their relation to those obtained from ordinary localization.

We now see that

$$\mu_0(w) \stackrel{\mathrm{pr}^{-1}}{\longrightarrow} h \otimes \begin{pmatrix} a \\ 0 \end{pmatrix} + k \otimes \begin{pmatrix} 0 \\ b \end{pmatrix} \stackrel{X\otimes \mathrm{Id}}{\longrightarrow} Xh \otimes \begin{pmatrix} a \\ 0 \end{pmatrix} + Xk \otimes \begin{pmatrix} 0 \\ b \end{pmatrix} \stackrel{\mathrm{pr}}{\longrightarrow}$$
$$b\frac{\langle Xk, K(\cdot,w)\rangle}{\|K(\cdot,w)\|}\mu(w) + \Big\{a\langle Xh, K(\cdot,w)\rangle + b\Big(\langle Xk, \partial_\alpha K(\cdot,w)\rangle - \frac{\langle K(\cdot,w), \partial_\alpha K(\cdot,w)\rangle}{\|K(\cdot,w)\|^2}\langle Xk, K(\cdot,w)\rangle\Big)\Big\}\frac{\eta(w)}{\|e(w)\|}.$$

In view of the calculations

$$\begin{aligned} b\langle Xk, K(\cdot,w)\rangle\frac{\mu(w)}{\|K(\cdot,w)\|} &= b\langle Xk, K(\cdot,w)\rangle\frac{\mu(w)}{\|K(\cdot,w)\|} \\ &= \Big\langle k, \frac{K_0(\cdot,w)}{\|K_0(\cdot,w)\|}\Big\rangle^{-1}\langle Xk, K(\cdot,w)\rangle\frac{\mu(w)}{\|K(\cdot,w)\|} \\ &= \frac{\|K_0(\cdot,w)\|}{\|K(\cdot,w)\|}\mu(w) \end{aligned}$$

and

$$\begin{aligned} & a\langle Xh, K(\cdot,w)\rangle + b\Big(\langle Xk, \partial_\alpha K(\cdot,w)\rangle - \frac{\langle K(\cdot,w), \partial_\alpha K(\cdot,w)\rangle}{\|K(\cdot,w)\|^2}\langle Xk, K(\cdot,w)\rangle\Big) \\ =\ & b\Big\{\Big(-\langle k, \partial_\alpha K_0(\cdot,w)\rangle - \frac{\langle K_0(\cdot,w), \partial_\alpha K_0(\cdot,w)\rangle}{\|K_0(\cdot,w)\|^2}\langle k, K_0(\cdot,w)\rangle\Big) + \\ & \qquad \Big(\langle Xk, \partial_\alpha K(\cdot,w)\rangle - \frac{\langle K(\cdot,w), \partial_\alpha K(\cdot,w)\rangle}{\|K(\cdot,w)\|^2}\langle Xk, K(\cdot,w)\rangle\Big)\Big\} \\ =\ & \|K_0(\cdot,w)\|\Big(\frac{\langle K_0(\cdot,w), \partial_\alpha K_0(\cdot,w)\rangle}{\|K_0(\cdot,w)\|^2} - \frac{\langle K(\cdot,w), \partial_\alpha K(\cdot,w)\rangle}{\|K(\cdot,w)\|^2}\Big), \end{aligned}$$

we obtain the projection formula

$$\mu_0(w) \stackrel{X\otimes_{\mathcal{A}(\Omega)}\mathrm{Id}}{\longrightarrow} \frac{K_0(w,w)}{K(w,w)}\mu(w) + \frac{\|K_0(\cdot,w)\|}{\|e(w)\|}\Big(\frac{\langle K_0(\cdot,w), \partial_\alpha K_0(\cdot,w)\rangle}{\|K_0(\cdot,w)\|^2} - \frac{\langle K(\cdot,w), \partial_\alpha K(\cdot,w)\rangle}{\|K(\cdot,w)\|^2}\Big)\eta(w). \tag{3.4}$$

Now we obtain the other projection formula

$$\begin{aligned} \eta_0(w) &\stackrel{\mathrm{pr}^{-1}}{\longrightarrow} h \otimes \begin{pmatrix} a \\ 0 \end{pmatrix} \stackrel{X\otimes \mathrm{Id}}{\longrightarrow} Xh \otimes \begin{pmatrix} a \\ 0 \end{pmatrix} \\ &\stackrel{\mathrm{pr}}{\longrightarrow} a\frac{\langle Xh, K(\cdot,w)\rangle}{\|e(w)\|}\eta(w) \\ &= \frac{\|e_0(w)\|}{\|e(w)\|}\eta(w). \end{aligned} \tag{3.5}$$

Now we can calculate the matrix for the operator

$$X(w) \stackrel{\text{def}}{=} X \otimes_{\mathcal{A}(\Omega)} \text{Id}\colon \mathcal{M}_0 \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^2_{\omega,\alpha} \to \mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}^2_{\omega,\alpha} \qquad (3.6)$$

with respect to the two orthonormal bases $\{\mu(w), \eta(w)\}$ and $\{\mu_0(w), \eta_0(w)\}$. From the equations (3.4) and (3.5), we see that

$$X(w) = \begin{pmatrix} \frac{\|K_0(\cdot,w)\|}{\|K(\cdot,w)\|} & \frac{\|K_0(\cdot,w)\|}{\|e(w)\|}\Big(\frac{\langle K_0(\cdot,w), \partial_\alpha K_0(\cdot,w)\rangle}{\|K_0(\cdot,w)\|^2} - \frac{\langle K(\cdot,w), \partial_\alpha K(\cdot,w)\rangle}{\|K(\cdot,w)\|^2}\Big) \\ 0 & \frac{\|e_0(w)\|}{\|e(w)\|} \end{pmatrix}.$$

Let $H_0(w)$ be the Grammian for the localisation $\mathcal{M}_0 \otimes_{\mathcal{A}(.)} \mathbb{C}^2_{\omega,\alpha}$ We obtain a factorisation $H_0(w) = \Gamma_0(w)^*\Gamma_0(w)$ similar to (3.1). Calculating the inverse of Γ,

$$\Gamma(w)^{-1} = \begin{pmatrix} \|K(\cdot,w)\|^{-1} & \frac{\langle K(\cdot,w), \partial_\alpha K(\cdot,w)\rangle}{\|K(\cdot,w)\|^2\|e(w)\|} \\ 0 & \|e(w)\|^{-1} \end{pmatrix}$$

verifies that $X(w) = \Gamma_0(w)\Gamma(w)^{-1}$. Consequently,

$$\begin{aligned} X(w)X(w)^* &= \Gamma_0(w)\Gamma(w)^{-1}\Gamma(w)^{*\,-1}\Gamma_0(w)^* \\ &= \Gamma_0(w)\Big(\Gamma(w)^*\Gamma(w)\Big)^{-1}\Gamma_0(w)^* \\ &= \Gamma_0(w)H(w)^{-1}\Gamma_0(w)^* \qquad (3.7) \end{aligned}$$

Similarly,

$$X(w)^*X(w) = \Gamma(w)^{-1*}H_0(w)\Gamma(w)^{-1}. \qquad (3.8)$$

With this explicit calculation of the operator $X^*(w)X(w)$, we can calculate some curvatures.

PROPOSITION 3.1 *Let $\mathcal{M}$ be a functional Hilbert space on the domain Ω. Assume that $\mathcal{M}$ is also a Hilbert module over the algebra $\mathcal{A}(\Omega)$. Let $\mathcal{M}_0$ be a submodule of $\mathcal{M}$. Let $X : \mathcal{M}_0 \to \mathcal{M}$ be the inclusion map and $X(w)$ be the map in (3.6). Then*

1. $\sum_{i,j=1}^m \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log\det(X(w)^*X(w)) = \mathcal{K}_0(w) - \mathcal{K}(w) - \sum_{i,j=1}^m \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \frac{\|e_0(w)\|}{\|e(w)\|}$ *and*

2. $\sum_{i,j=1}^m \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log\operatorname{tr}(X(w)^*X(w)) = \sum_{i,j=1}^m \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log\operatorname{tr}(H_0(w)H(w)^{-1})$.

Proof: Since $\det(X(w)X(w)^*) = \frac{\det H_0(w)}{\det H(w)}$ by (3.7), the first statement in the proposition follows from :

$$\begin{aligned} \sum_{i,j=1}^m \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log\det(X(w)^*X(w)) &= \sum_{i,j=1}^m \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log\Big(\frac{\det H_0(w)}{\det H(w)}\Big) \\ &= -\sum_{i,j=1}^m \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \frac{\|K_0(\cdot,w)\|^2\,\|e_0(w)\|^2}{\|K(\cdot,w)\|^2\|e(w)\|^2} \\ &= -\sum_{i,j=1}^m \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log\|K_0(\cdot,w)\|^2 + \bar\partial\partial \log\|K(\cdot,w)\|^2 \\ &\quad -\sum_{i,j=1}^m \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log\|e_0(w)\|^2 + \sum_{i,j=1}^m \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log\|e(w)\|^2. \end{aligned}$$

Since

$$\begin{aligned}\operatorname{tr}\Big(X(w)^*X(w)\Big) &= \operatorname{tr}\Big(\Gamma(w)^{-1*}H_0(w)\Gamma(w)^{-1}\Big)\\ &= \operatorname{tr}\Big(H_0(w)\Gamma(w)^{-1}(\Gamma(w))^{-1*}\Big)\\ &= \operatorname{tr}\Big(H_0(w)H(w)^{-1}\Big),\end{aligned}$$

it follows that

$$\sum_{i,j=1}^{m} \tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \operatorname{tr}(X(w)^*X(w)) = \sum_{i,j=1}^{m} \tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} \log(\operatorname{tr}(H_0(w)H(w)^{-1})).$$

This completes the proof of the second statement.

The first two terms in the first statement of the Proposition would have been obtained from the rank one localisation. So, the new information, if any, is contained in the last two terms i.e. $-\partial\bar{\partial}\log\frac{\|e_0(w)\|}{\|e(w)\|}$.

Here is another expression for $X(w)^*X(w)$:

$$\begin{aligned}X(w)^*X(w) &= \Gamma(w)^{-1*}H_0(w)\Gamma(w)^{-1}\\ &= \Gamma(w)\Big(\Gamma(w)^*\Gamma(w)\Big)^{-1}H_0(w)\Gamma(w)^{-1}\\ &= \Gamma(w)H(w)^{-1}H_0(w)\Gamma(w)^{-1}\\ &= \Big(\Gamma(w)H(w)^{-\frac{1}{2}}\Big)H(w)^{-\frac{1}{2}}H_0(w)H(w)^{-\frac{1}{2}}\Big(\Gamma(w)H(w)^{-\frac{1}{2}}\Big)^{-1}.\end{aligned}$$

This last expression may relate well to the following definition of curvature for $X(w)^*X(w)$ thought of as a metric on a rank 2 bundle on Ω.

$$\mathcal{K}(X(w)^*X(w)) \stackrel{\text{def}}{=} \bar{\partial}\Big((X(w)^*X(w))^{-1}\partial(X(w)^*X(w))\Big).$$

However, this formula is valid only if the curvature is calculated with respect to the unique connection compatible with the complex structure as well as the hermitian metric. Since we did not use a holomorphic frame in computing the matrix for $X(w)^*X(w)$, the above consideration does not apply. It is not clear how we should get back to our complex geometric set up.

REFERENCES

[1] N. Aronszajn, *Theory of Reproducing kernels*, Trans. Amer.Math. Soc., 68(1950), 337-404.

[2] R. Bott and S. S. Chern, Hermitian vector bundles and the equidistribution of the zeros of their holomorphic sections, Acta Math., 114(1968), 71-112.

[3] D. N. Clark and G. Misra, *On homogeneous contractions and unitary representations of* $SU(1,1)$, J. Operator Th., 30(1993), 109-122.

[4] M. J. Cowen and R. G. Douglas, *Complex geometry and operator theory*, Acta Math., 141 (1978), 187-261.

[5] R. G. Douglas, *On Šilov resolutions of Hilbert modules*, in Operator Theory : Advances and Applications, V. 28, Birkhauser Verlag, Basel (1988) 51-60.

[6] R. G. Douglas, *Models and resolutions for Hilbert modules*, Contemporay Mathematics, to appear.

[7] R. G. Douglas and V. I. Paulsen, *Hilbert Modules over Function Algebras*, Longman Research Notes, 217, 1989.

[8] R. G. Douglas, V. I. Paulsen, H. Sah and K. Yan, *Algebraic reduction and rigidity for Hilbert modules*, Amer. J. Math, 117(1995), 75-92.

[9] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, 1978.

[10] R. C. Gunning and H. Rossi, *Analytic functions of Several Complex Variables*, Prentice Hall, 1965.

[11] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Volume II, John Wiley & Sons, 1969.

[12] G. Misra, *Curvature and discrete series representations of* $SL_2(R)$, J. Int. Eqn. Operator Th., 9(1986), 452-459.

[13] R. O. Wells, Jr., *Differential Analysis on Complex Manifolds*, Springer Verlag, 1973.

SUNY at Stony Brook
Stony Brook
NY 11794

Indian Statistical Institute
R. V. Colleg Post
Bangalore 560 059