Convolution operators defined by singular measures on the motion group

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Abstract

This paper contains an L^p improving result for convolution operators defined by singular measures associated to hypersurfaces on the motion group. This needs only mild geometric properties of the surfaces, and it extends earlier results on Radon type transforms on $\mathbb{R}^n$. The proof relies on the harmonic analysis on the motion group.

1 Introduction

The classical Radon transforms satisfy L^p improving properties (see [7]) and they are closely related to certain convolution operators associated to singular measures (see e.g. [13]). The above results have been extended in many ways, not necessarily related to convolution structures, see e.g. [5], [12], [15] and the references therein.

Our starting point is the following result, proved in [10].

Theorem 1 *Let Γ be a convex compact curve in the plane and let γ be the arc-length measure of Γ. We identify $\theta \in [0, 2\pi]$ with $e^{i\theta} \in S^1$ (the unit circle). Let γ_θ be the rotated measure, i.e. $\int_{\mathbb{R}^2} f(x)\, d\gamma_\theta(x) = \int_{\mathbb{R}^2} f\left(e^{i\theta}x\right) d\gamma(x)$. Consider the operator T defined by*

$$Tf(x, \theta) = \left(f *_{\mathbb{R}^2} \gamma_\theta\right)(x) \ ,$$

where $x \in \mathbb{R}^2$ and $*_{\mathbb{R}^2}$ denotes the convolution in $\mathbb{R}^2$. Then

$$\|Tf\|_{L^3(\mathbb{R}^2 \times S^1)} \le c \, \|f\|_{L^{3/2}(\mathbb{R}^2)} \ .$$

The proof of this theorem relies on an estimate for the average decay of the Fourier transform $\widehat{\gamma}$ proved by A.N. Podkorytov in [8] (see also [3]), which has been extended to several variables in [2]. The following statement is different from the one in [2], but it can be proved by a mild variation of the original argument.

Theorem 2 *Let Γ be a compact convex submanifold of codimension 1 in $\mathbb{R}^n$ (i.e. Γ can be seen as the graph of a convex function defined in a convex domain in $\mathbb{R}^{n-1}$). Let $\gamma = \chi\sigma$ where σ is the surface measure on Γ and χ is a smooth cutoff supported in the interior of Γ. Then*

$$\int_{S^{n-1}} |\widehat{\gamma}(R\omega)|^2 \, d\omega \leq cR^{-(n-1)},$$

where $d\omega$ is the normalized measure on the unit sphere S^{n-1}. Moreover the constant c depends only on χ and the diameter of Γ.

The above theorem easily implies the following extension of Theorem 1 (see [1]). For $k \in SO(n)$ and γ a measure on $\mathbb{R}^n$, let γ_k be defined by $\int_{\mathbb{R}^n} f d\gamma_k = \int_{\mathbb{R}^n} f(ky) d\gamma(y)$, so that $\widehat{\gamma}_k(\xi) = \widehat{\gamma}(k^{-1}\xi)$.

Theorem 3 *Let Γ be a compact convex submanifold of codimension 1 in $\mathbb{R}^n$ and let $\gamma = \chi\sigma$ where σ is the surface measure on Γ and χ is a smooth cutoff function supported in the interior of Γ. Consider the operator T defined by*

$$Tf(x,k) = (f *_{\mathbb{R}^n} \gamma_k)(x) \ ,$$

where $x \in \mathbb{R}^n$, $k \in SO(n)$ and $*_{\mathbb{R}^n}$ denotes the convolution in $\mathbb{R}^n$. Then

$$\|Tf\|_{L^{n+1}(\mathbb{R}^n \times SO(n))} \leq c \|f\|_{L^{(n+1)/n}(\mathbb{R}^n)} \ .$$

The operator T in Theorem 3 can be seen as a convolution operator on the motion group M_n, which is $\mathbb{R}^n \times SO(n)$ equipped with the group product $(x,k)(y,h) = (x + ky, kh)$ and unit $(0,e)$. Indeed the convolution of two functions F and G on M_n is defined by

$$(F *_{M_n} G)(x,k) = \int_{M_n} F\left(x - kh^{-1}y, kh^{-1}\right) G(y,h) \, dydh,$$

where dh is the Haar measure on $SO(n)$.

Note that if $F(x,k) = f(x)$ and μ denotes the measure on M_n defined by

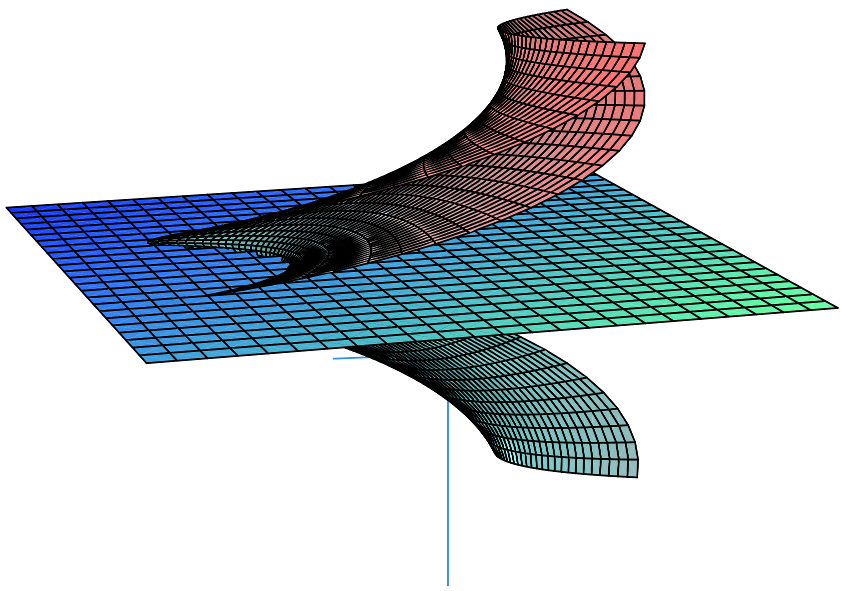
$$\int_{M_n} G(x,k) \, d\mu(x,k) = \int_{M_n} G(x,k) \, d\gamma_k(x) \, dk$$

we have

$$F *_{M_n} \mu(x,k) = \int_{M_n} F\left(x - kh^{-1}y, kh^{-1}\right) d\mu(y,h) \tag{1}$$
$$= \int_{M_n} f\left(x - kh^{-1}y\right) d\gamma_h(y)\, dh$$
$$= \int_{M_n} f(x - ky)\, d\gamma(y)\, dh$$
$$= \int_{\mathbb{R}^n} f(x - ky)\, d\gamma(y) = (f *_{\mathbb{R}^n} \gamma_k)(x).$$

The above family $\{\gamma_k\}$ of hypersurfaces in $\mathbb{R}^n$ turns out to be a manifold in $\mathbb{R}^n \times SO(n)$. Indeed for any $k_0 \in SO(n)$ the coset $\{(x, k_0) : x \in \mathbb{R}^n\}$ contains the $n-1$ dimensional manifold Γ_{k_0}, i.e. the manifold Γ rotated by k_0. The union of the manifolds Γ_{k_0} is a hypersurface X in $\mathbb{R}^n \times SO(n)$.

When $n = 2$, the Γ's are convex curves and their union can be seen as a 2-dimensional surface in $\mathbb{R}^2 \times S^1$; the picture shows this surface in the particular case $\Gamma(t) = (t, t^2 + 1)$, together with the plane $\theta = \pi$:



In this paper we want to replace the above manifold X with a more general manifold Y in $\mathbb{R}^n \times SO(n)$, so that the action of Γ as a convolution operator on $\mathbb{R}^n$ is averaged not only on rotations, but on a wider family of transformations. In order to deal with this more general setting it is natural to work in the Euclidean motion group M_n rather than in $\mathbb{R}^n \times SO(n)$ and take advantage of the representation theory of M_n.

2 Main result

The following is our main result. By (1) it is an extension of Theorem 3.

Theorem 4 *Let* $n \geq 2$ *and let* Y *be a* $\mathcal{C}^1$ *submanifold of codimension* 1 *in* M_n. *Assume that* Y *can be locally represented as* $F(x,k) = 0$ *with* $\nabla_x F(x,k) \neq 0$. *Assume furthermore that for every* $k_0 \in SO(n)$ *the intersection* $Y \cap \{(x, k_0) : x \in \mathbb{R}^n\}$ *is a convex hypersurface*[1] *in* $\mathbb{R}^n$. *Choose* $\chi \in \mathcal{C}_c^1(M_n)$ *and let* μ *be the measure on* M_n *given by* $\int_{M_n} f d\mu = \int_Y f \chi d\sigma$, *where* σ *is the surface measure on* Y. *Then, if* $Tf(x,k) = (f *_{M_n} d\mu)(x,k)$, *we have*

$$\|Tf\|_{L^{n+1}(M_n)} \leq c_n \|f\|_{L^{(n+1)/n}(M_n)} \tag{2}$$

Proof. Without loss of generality, we may assume that Y is the graph of the function

$$x_1 = \Phi(x', k),$$

where we use the notation $x' = (x_2, \ldots, x_n)$. Thus

$$\int_{M_n} f d\mu = \int_{\mathbb{R}^{n-1}} \int_{SO(n)} f(\Phi(x',k), x', k)\, \nu(x',k)\, dk dx',$$

where ν is the product of χ by a Jacobian term. For every $z \in \mathbb{C}$, let i_z be the distribution on $\mathbb{R}$ defined by

$$\langle i_z, \eta \rangle = \frac{1}{\Gamma(z)} \int_0^{+\infty} \eta(t)\, t^{z-1} dt\ .$$

We define the family of distributions μ^z by

$$\mu^z = \mu *_{M_n} I_z,$$

where I_z is the distribution defined by

$$I_z(x,k) = i_z(x_1) \otimes \delta_0(x_2) \otimes \cdots \otimes \delta_0(x_n) \otimes \delta_e(k)\,.$$

For any $k \in SO(n)$ define the measure μ_k on $\mathbb{R}^n$ by

$$\int_{\mathbb{R}^n} g d\mu_k = \int_{\mathbb{R}^{n-1}} g(\Phi(x',k), x')\, \nu(x',k)\, dx'.$$

Then define the distribution E_z on $\mathbb{R}^n$ by

$$E_z(x) = i_z(x_1) \otimes \delta_0(x_2) \otimes \cdots \otimes \delta_0(x_n)$$

[1]This means that the above intersection is the graph of a convex function on a convex set (after choosing suitable coordinates in $\mathbb{R}^{n-1}$).

and let $\mu_k^z = \mu_k *_{\mathbb{R}^n} E_z$. Then it can be easily shown that

$$\int_{M_n} f(x,k)\, d\mu(x,k) = \int_{SO(n)} \int_{\mathbb{R}^n} f(x,k)\, d\mu_k(x)\, dk$$

$$\langle \mu^z, f \rangle_{M_n} = \int_{SO(n)} \langle \mu_k^z, f(\cdot,k) \rangle_{\mathbb{R}^n} dk.$$

We introduce the analytic family of operators

$$T^z f = f * \mu^z.$$

Then the proof follows from Stein's complex interpolation theorem and the following result.

Lemma 5 *For every real s we have*

$$T^{1+is} : L^1(M_n) \longrightarrow L^\infty(M_n)\ , \tag{3}$$

$$T^{-(n-1)/2+is} : L^2(M_n) \longrightarrow L^2(M_n)\ . \tag{4}$$

Proof of the Lemma. Let us prove (3) first. Indeed for $g \in L^1(M_n)$

$$\left\langle \mu^{1+is}, g \right\rangle_{M_n} = \langle \mu *_{M_n} I_{1+is}, g \rangle_{M_n} = \langle I_{1+is}, g *_{M_n} \widetilde{\mu} \rangle_{M_n}$$

$$= \frac{1}{\Gamma(1+is)} \int_0^{+\infty} (g *_{M_n} \widetilde{\mu})(x_1, 0, \ldots, 0, e)\ x_1^{is} dx_1$$

$$= \frac{1}{\Gamma(1+is)} \int_0^{+\infty} x_1^{is} \int_{M_n} g\left((x_1, 0, \ldots, 0, e)(y_1, \ldots, y_n, k)^{-1}\right)$$

$$d\widetilde{\mu}(y_1, \ldots, y_n, k)\, dx_1$$

$$= \frac{1}{\Gamma(1+is)} \int_0^{+\infty} x_1^{is} \int_{M_n} g((x_1, 0, \ldots, 0, e)(y_1, \ldots, y_n, k))$$

$$d\mu(y_1, \ldots, y_n, k)\, dx_1$$

$$= \frac{1}{\Gamma(1+is)} \int_0^{+\infty} x_1^{is} \int_{M_n} g(x_1 + y_1, y_2, \ldots, y_n, k)\, d\mu(y_1, \ldots, y_n, k)\, dx_1$$

$$= \frac{1}{\Gamma(1+is)} \int_0^{+\infty} \int_{\mathbb{R}^{n-1}} \int_{SO(n)} g(x_1 + \Phi(y', k), y', k)\, x_1^{is} \nu(y', k)\, dk dy' dx_1$$

(where $\widetilde{\mu}$ is defined by $\int_{M_n} f(y,k)\, d\widetilde{\mu}(y,k) = \int_{M_n} f\left((y,k)^{-1}\right) d\mu(y,k)$).

The substitution $y_1 = x_1 + \Phi(y', k)$, along with the boundedness of ν, immediately gives

$$\left| \left\langle \mu^{1+is}, g \right\rangle_{M_n} \right| \leq c \left\| g \right\|_{L^1(M_n)},$$

so that $\mu^{1+is} \in L^\infty(M_n)$. This proves (3).

Now we turn to the proof of (4). We need first to recall a few facts from the representation theory of M_n.

The unitary dual $\widehat{M_n}$ $(n \geq 2)$ can be described in the following way (here [11] is a reference for the representation theory of M_n, see also [14]). Let $L = SO(n-1)$, considered as a subgroup of $SO(n)$. For each $\sigma \in \widehat{L}$ realised on a Hilbert space V_σ of dimension d_σ consider the space $L^2(SO(n), \sigma)$ consisting of functions φ on $SO(n)$ taking values in $\mathbb{C}^{d_\sigma \times d_\sigma}$, the space of $d_\sigma \times d_\sigma$ complex matrices, satisfying the condition

$$\varphi(\ell k) = \sigma(\ell)\varphi(k) \ , \qquad \ell \in L \ , \quad k \in SO(n)$$

which are also square integrable on $SO(n)$:

$$\int_{SO(n)} \|\varphi\|^2 \, dk = \int_{SO(n)} \operatorname{tr}\left(\varphi(k)^* \varphi(k)\right) dk < \infty \ .$$

Note that $L^2(SO(n), \sigma)$ is a Hilbert space under the inner product

$$(\varphi, \psi) = \int_{SO(n)} \operatorname{tr}\left(\varphi(k)^* \psi(k)\right) dk \ .$$

For each $\lambda > 0$ and $\sigma \in \widehat{L}$ we define a representation $\pi_{\lambda,\sigma}$ of M_n on $L^2(SO(n), \sigma)$ as follows. For $\varphi \in L^2(SO(n), \sigma)$ and $(x, k) \in M_n$ let

$$\pi_{\lambda,\sigma}(x, k)\varphi(\ell) = \exp\left(2\pi i \lambda \ell^{-1} e_1 \cdot x\right) \varphi(\ell k) \ ,$$

where $e_1 = (1, 0, \ldots, 0)$ and $\ell \in SO(n)$. If $\varphi_j(k)$ are the column vectors of $\varphi \in L^2(SO(n), \sigma)$ then $\varphi_j(\ell k) = \sigma(\ell)\varphi_j(k)$ for all $\ell \in L$. Therefore $L^2(SO(n), \sigma)$ can be written as a direct sum of d_σ copies of $H(SO(n), \sigma)$ which is defined to be the space of square integrable $\varphi : SO(n) \to \mathbb{C}^{d_\sigma}$ satisfying

$$\varphi(\ell k) = \sigma(\ell)\varphi(k) \, , \qquad \ell \in L \ .$$

It can be shown that $\pi_{\lambda,\sigma}$ restricted to $H(SO(n), \sigma)$ is an irreducible representation of M_n. Moreover, any infinite dimensional irreducible unitary representation of M_n is unitarily equivalent to one and only one $\pi_{\lambda,\sigma}$. Finite dimensional irreducible unitary representations of $SO(n)$ also yield irreducible unitary representations of M_n. As they do not appear in the Plancherel formula we neglect them. We remark that when $n = 2$ the unitary dual $\widehat{L}$ contains only the trivial representation.

Given $f \in L^1(M_n) \cap L^2(M_n)$ we define the group Fourier transform of f by

$$\pi_{\lambda,\sigma}(f) = \int_{M_n} f(x, k)\, \pi_{\lambda,\sigma}\left((x, k)^{-1}\right) dx dk \ .$$

It can be shown that $\pi_{\lambda,\sigma}(f)$ is a Hilbert-Schmidt operator on $H(SO(n),\sigma)$ and we have the Plancherel formula

$$\sum_{\sigma\in\widehat{L}}\int_0^{+\infty}\|\pi_{\lambda,\sigma}(f)\|_{HS}^2\lambda^{n-1}d\lambda=\omega_n\int_{M_n}|f(x,k)|^2\,dxdk\ ,$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm.

Applying Plancherel formula to $T^{-(n-1)/2+is}f$ we get

$$\begin{aligned}
&\left\|T^{-(n-1)/2+is}f\right\|_{L^2(M_n)}^2\\
&=\left\|f*_{M_n}\mu^{-(n-1)/2+is}\right\|_{L^2(M_n)}^2\\
&=\omega_n\sum_{\sigma\in\widehat{L}}\int_0^{+\infty}\left\|\pi_{\lambda,\sigma}\left(\mu^{-(n-1)/2+is}\right)\pi_{\lambda,\sigma}(f)\right\|_{HS}^2\lambda^{n-1}d\lambda\\
&\leq\omega_n\sum_{\sigma\in\widehat{L}}\int_0^{+\infty}\left\|\pi_{\lambda,\sigma}\left(\mu^{-(n-1)/2+is}\right)\right\|_{OP}^2\ \ \|\pi_{\lambda,\sigma}(f)\|_{HS}^2\lambda^{n-1}d\lambda\ ,
\end{aligned}$$

where $\|\cdot\|_{OP}$ is the operator norm on $H(SO(n),\sigma)$. We shall show below that

$$\left\|\pi_{\lambda,\sigma}\left(\mu^{-(n-1)/2+is}\right)\right\|_{OP}\leq c_n \tag{5}$$

uniformly in λ and σ, so that

$$\left\|T^{-(n-1)/2+is}f\right\|_{L^2(M_n)}^2\leq c_n\omega_n\sum_{\sigma\in\widehat{L}}\int_0^{+\infty}\|\pi_{\lambda,\sigma}(f)\|_{HS}^2\lambda^{n-1}d\lambda=c_n\|f\|_{L^2(M_n)}^2.$$

We now prove (5). For $\varphi,\psi\in H(SO(n),\sigma)$ we have

$$\pi_{\lambda,\sigma}\left((x,k)^{-1}\right)\varphi(u)=\exp\left(-2\pi i\lambda u^{-1}e_1\cdot k^{-1}x\right)\varphi\left(uk^{-1}\right).$$

Assume for a moment $\operatorname{Re}z>0$, then μ^z is a measure and

$$\begin{aligned}
&\pi_{\lambda,\sigma}(\mu^z)\varphi(u)\\
&=\int_{M_n}\pi_{\lambda,\sigma}\left((x,k)^{-1}\right)\varphi(u)\,d\mu^z(x,k)\\
&=\int_{M_n}\exp\left(-2\pi i\lambda u^{-1}e_1\cdot k^{-1}x\right)\varphi\left(uk^{-1}\right)d\mu^z(x,k)\,.
\end{aligned}$$

Therefore

$$
\begin{aligned}
&\left\langle \pi_{\lambda,\sigma}\left(\mu^{z}\right)\varphi,\psi\right\rangle_{H(SO(n),\sigma)} \\
&= \int_{SO(n)} \left\langle \pi_{\lambda,\sigma}\left(\mu^{z}\right)\varphi\left(u\right),\psi\left(u\right)\right\rangle_{\mathbb{C}^{d_\sigma}}\ du \\
&= \int_{SO(n)}\int_{M_n} \exp\left(-2\pi i\lambda u^{-1}e_1\cdot k^{-1}x\right)\left\langle \varphi\left(uk^{-1}\right),\psi\left(u\right)\right\rangle_{\mathbb{C}^{d_\sigma}} d\mu^{z}\left(x,k\right)du \\
&= \int_{SO(n)}\int_{SO(n)}\int_{\mathbb{R}^n} \exp\left(-2\pi i\lambda\, k\, u^{-1}e_1\cdot x\right)\left\langle \varphi\left(uk^{-1}\right),\psi\left(u\right)\right\rangle_{\mathbb{C}^{d_\sigma}} \\
&\times d\mu_k^{z}\left(x\right)du dk \\
&= \int_{SO(n)}\int_{SO(n)} \widehat{\mu_k^{z}}\left(\lambda k\, u^{-1}e_1\right)\left\langle \varphi\left(uk^{-1}\right),\psi\left(u\right)\right\rangle_{\mathbb{C}^{d_\sigma}} du dk \\
&= \int_{SO(n)}\int_{SO(n)} \widehat{\mu_k}\left(\lambda k\, u^{-1}e_1\right)\ \widehat{E_z}\left(\lambda k\, u^{-1}e_1\right)\left\langle \varphi\left(uk^{-1}\right),\psi\left(u\right)\right\rangle_{\mathbb{C}^{d_\sigma}} du dk.
\end{aligned}
$$

By analytic continuation, the equality

$$
\begin{aligned}
&\left\langle \pi_{\lambda,\sigma}\left(\mu^{z}\right)\varphi,\psi\right\rangle_{H(SO(n),\sigma)} \\
&= \int_{SO(n)}\int_{SO(n)} \widehat{\mu_k}\left(\lambda k\, u^{-1}e_1\right)\ \widehat{E_z}\left(\lambda k\, u^{-1}e_1\right)\left\langle \varphi\left(uk^{-1}\right),\psi\left(u\right)\right\rangle_{\mathbb{C}^{d_\sigma}} du dk
\end{aligned}
$$

holds also for $z = -\left(n-1\right)/2 + is$. By Cauchy-Schwarz inequality

$$
\begin{aligned}
&\left|\int_{SO(n)} \widehat{\mu_k}\left(\lambda k\, u^{-1}e_1\right)\ \widehat{E}_{-(n-1)/2+is}\left(\lambda k\, u^{-1}e_1\right)\left\langle \varphi\left(uk^{-1}\right),\psi\left(u\right)\right\rangle_{\mathbb{C}^{d_\sigma}} du\right| \\
&\le \left\|\left\langle \varphi\left(uk^{-1}\right),\psi\left(u\right)\right\rangle_{\mathbb{C}^{d_\sigma}}\right\|_{L^2(SO(n),du)} \\
&\times \left\|\widehat{\mu_k}\left(\lambda k\, u^{-1}e_1\right)\ \widehat{E}_{-(n-1)/2+is}\left(\lambda k\, u^{-1}e_1\right)\right\|_{L^2(SO(n),du)}
\end{aligned}
$$

By [6, Ch. 2] we know that

$$
\left|\widehat{E}_{-(n-1)/2+is}\left(\lambda k\, u^{-1}e_1\right)\right| \le C\lambda^{(n-1)/2}.
$$

Now

$$\begin{aligned}
\widehat{\mu_k}(\xi) &= \int_{\mathbb{R}^n} \exp(-2\pi i \xi \cdot x)\, d\mu_k(x) \\
&= \int_{\mathbb{R}^{n-1}} \exp(-2\pi i \xi \cdot (\Phi(x',k), x'))\, \nu(x',k)\, dx' \\
&= \int_{\mathbb{R}^{n-1}} \exp(-2\pi i \xi \cdot (\Phi(x',k), x')) \frac{\nu(x',k)}{\sqrt{1+|\nabla_{x'}\Phi(x',k)|^2}} \\
&\times \sqrt{1+|\nabla_{x'}\Phi(x',k)|^2}\, dx' \\
&= \int_{\mathbb{R}^n} \exp(-2\pi i \xi \cdot x) \frac{\nu(x',k)}{\sqrt{1+|\nabla_{x'}\Phi(x',k)|^2}} d\zeta_k(x),
\end{aligned}$$

where $d\zeta_k$ is the surface measure of the convex hypersurface in $\mathbb{R}^n$ given by the intersection $Y \cap \{(x,k) : x \in \mathbb{R}^n\}$. By Theorem 2 we get

$$\begin{aligned}
&\left\| \widehat{\mu_k}\left(\lambda k\, u^{-1} e_1\right)\ \widehat{E}_{-(n-1)/2+is}\left(\lambda k\, u^{-1} e_1\right) \right\|^2_{L^2(SO(n),du)} \\
&\leq c\lambda^{n-1} \int_{SO(n)} \left|\widehat{\mu_k}\left(\lambda k u^{-1} e_1\right)\right|^2 du \leq c.
\end{aligned}$$

To end the proof we observe that

$$\begin{aligned}
&\int_{SO(n)} \|\langle \varphi(uk), \psi(u)\rangle_{\mathbb{C}^{d_\sigma}}\|_{L^2(SO(n),du)}\, dk \\
&= \int_{SO(n)} \left\{ \int_{SO(n)} |\langle \varphi(uk), \psi(u)\rangle_{\mathbb{C}^{d_\sigma}}|^2\, du \right\}^{1/2} dk \\
&\leq \left\{ \int_{SO(n)} \int_{SO(n)} |\langle \varphi(uk), \psi(u)\rangle_{\mathbb{C}^{d_\sigma}}|^2\, du dk \right\}^{1/2} \\
&\leq \left\{ \int_{SO(n)} \int_{SO(n)} |\varphi(uk)|^2 |\psi(u)|^2\, du dk \right\}^{1/2}.
\end{aligned}$$

By Fubini's theorem and the invariance of the Haar measure on $SO(n)$ we get

$$\int_{SO(n)} \|\langle \varphi(uk), \psi(u)\rangle_{\mathbb{C}^{d_\sigma}}\|_{L^2(SO(n),du)}\, dk \leq \|\varphi\|_{H(SO(n),\sigma)} \|\psi\|_{H(SO(n),\sigma)}.$$

This ends the proof of the Lemma. Hence Theorem 4 is proved.

Remark 6 *For functions on M_n which are independent of the rotational variable, i.e. for functions F such that $F(x,k) = f(x)$, Theorem 4 can be obtained from Theorem 3. Indeed*

$$\left\{\int_{SO(n)}\int_{\mathbb{R}^n} |F *_{M_n} \mu(x,k)|^{n+1}\, dxdk\right\}^{1/(n+1)}$$
$$= \left\{\int_{SO(n)}\int_{\mathbb{R}^n} \left|\int_{SO(n)}\int_{\mathbb{R}^n} f\left(x - k\tau^{-1}y\right) d\mu_\tau(y)\, d\tau\right|^{n+1} dxdk\right\}^{1/(n+1)}$$
$$\leq \int_{SO(n)} \left\{\int_{SO(n)}\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} f\left(x - k\tau^{-1}y\right) d\mu_\tau(y)\right|^{n+1} dxdk\right\}^{1/(n+1)} d\tau$$
$$= \int_{SO(n)} \left\{\int_{SO(n)}\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} f(x - \sigma y)\, d\mu_\tau(y)\right|^{n+1} dxd\sigma\right\}^{1/(n+1)} d\tau$$
$$= \int_{SO(n)} \left\{\int_{SO(n)}\int_{\mathbb{R}^n} |f *_{\mathbb{R}^n} \mu_{\tau,\sigma}(x)|^{n+1}\, dxd\sigma\right\}^{1/(n+1)} d\tau$$
$$\leq \int_{SO(n)} c\, \|f\|_{L^{\frac{n+1}{n}}(\mathbb{R}^n)}\, d\tau = c\, \|F\|_{L^{\frac{n+1}{n}}(M_n)},$$

where $\mu_{\tau,\sigma}$ denotes the measure μ_τ rotated by σ. This yields the following weaker version of Theorem 4. For a general F let $\widetilde{F}(x,k) = \sup_{\tau\in SO(n)} |F(x,k\tau)|$ then

$$\|F * \mu\|_{L^{n+1}(M_n)} \leq \left\|\widetilde{F} * \mu\right\|_{L^{n+1}(M_n)} \leq c\left\|\widetilde{F}\right\|_{L^{\frac{n+1}{n}}(M_n)} = c\, \|F\|_{L_x^{\frac{n+1}{n}}(L_\tau^\infty(M_n))}.$$

The above seems to be the best we can get by using earlier results such as the ones in [1].

Remark 7 *A familiar example (the characteristic function of a small ball) and the previous remark can be used to show that the indices in (2) cannot be improved.*

Remark 8 *It is interesting to compare Theorem 4 with Theorem 1.1 in [9] where it is shown that the L^p improving property of a measure is related to the fact that the supporting manifold generates the full group.*

Remark 9 *The techniques in our paper are L^2 in nature and they seem to provide only $L^p - L^{p'}$ results. We do not know how to get mixed norm estimates similar to the ones which have been proved in [10] through certain L^r estimates for the average decay of Fourier transforms (note that in general these L^r estimates cannot be obtained by interpolation between L^2 and L^∞, see e.g. [4]).*

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