ON THEOREMS OF BEURLING AND HARDY FOR CERTAIN STEP TWO NILPOTENT GROUPS

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ABSTRACT. In this paper we prove Hardy's theorem and a version of Cowling-Price theorem for all step two nilpotent Lie groups using ... urling's theorem on Euclidean space ... urling's theorem is proved for all step two nilpotent Lie groups ... .

1. INTRODUCTION

A c l asinal of H l rthrans ar eo a dy foyr sehat fo r rmFou o nir

$$|f(x)| \leq Ce^{-a|x|^2}, \ \ |\hat{f}(\xi)| \leq Ce^{-b|\xi|^2}$$

wh $\hat{f}$ is h r e Eu r trans c fl defi i de n fned rmFou y of ri e

$$\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx$$

th n $f = 0$ whe re $ab > \frac{1}{4}$ ... $f(x) = Ce^{-a|x|^2}$ fo $= \frac{1}{4}$. Re nt ab ... pai d ana t o d ... [1 9]. T (case $\frac{1}{4}$) follows ... $ab > \frac{1}{4}$ ... e t a 2, nam l fo e c r o ... $d \geq n$ ... d iti h o n

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)||\hat{f}(y)|e^{|(x,y)|}dx\ dy < \infty$$

impl $f = 0$ e . s

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Fo e h casis He g gr p H^n, of which is ... nipo nt t l eps. In ana i e gr s th m has b ... of H [1 o 9]. r Ley th ar $\mathcal{L}$ b e pl th acian H^n ... $q_a(z,t)$. Le $\hat{f}(\lambda)$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$ ... Fou e r trans of H^n ... that

$$|f(z,t)| \leq Cq_a(z,t), \ \ \hat{f}(\lambda)^*\hat{f}(\lambda) \leq C\hat{q}_{2b}(\lambda)$$

impl $f = 0$ when ... $a < b$. In [1 9] r a t has b ee m is tr n ... ch a th fied ... nipo nt t l eps L . i e gr ou

A r efo ul o n ain of H^n po ... of Fo ... Weyl ... trans 2n d 5 fo $\hat{f}(\lambda, (\xi,\eta))$, $\xi, \eta \in \mathbb{R}^n$ ... -Weyl ... H^n fo o T his has b in pr ov ... H ... [1 ... 8]. If ... $\hat{f}(\lambda)^*\hat{f}(\lambda) \leq C\hat{q}_{2b}(\lambda)$ th n it is sh wn in s o n 2 that ... $\hat{f}(\lambda, (\xi,\eta))\phi_\lambda, \phi_\lambda)$ ... wh r $\phi_\lambda = \Phi_0^\lambda$, o n

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is a Gaussian on $\mathbb{R}^n$ has a Gaussian decay as a function of (ξ, η). This observation allows us to restate Hardy's theorem for H^n as follows :

If $|f(z,t)| \leq C q_a(z,t)$ and $|\langle \hat{f}(\lambda, (\xi,\eta))\phi_\lambda, \phi_\lambda\rangle| \leq Ce^{-\frac{1}{4}|\lambda|\left(1-e^{-4b|\lambda|}\right)(|\xi|^2+|\eta|^2)}$ for all $\lambda \in \mathbb{R}\setminus\{0\}$ then $f = 0$ a.e for $a < b$.

We also have the following version of Beurling's theorem :

$$\int_{H^n}\int_{\mathbb{C}^n} |f(x+iy,t)||(\hat{f}(\lambda,(\xi,\eta))\phi_\lambda,\phi_\lambda)|e^{|\lambda||(x\cdot\eta-y\cdot\xi)|}\, d\xi\, d\eta\, dx\, dy\, dt < \infty$$

for all $\lambda \in \mathbb{R}\setminus\{0\}$ implies $f = 0$.

The following version of Cowling-Price theorem is also proved : if

$$f(q_a)^{-1} \in L^p(H^n), 1 \leq p \leq \infty \text{ and } \hat{f}(\lambda)e^{bH(\lambda)} \in \mathcal{S}_q$$

for all $\lambda \in \mathbb{R}\setminus\{0\},\ 1 \leq q \leq \infty$ where $\mathcal{S}_q$ denotes the set of Schatten q-class operators and $H(\lambda) = -\Delta + \lambda^2|x|^2$ is the scaled Hermite operator, then $f = 0$ whenever $a < b$. These results are proved in section 2. Analogous results are proved for all step two stratified groups, general step two groups in section 4, 5 respectively.

For various formulations of Hardy's theorem on nilpotent Lie groups we refer to Ray [15] , Astengo et al [1] and Kaniuth-Kumar [13]. As we have mentioned earlier a source for other references on Hardy's theorem is the monograph [19]. In a private conversation Prof. M. Cowling has indicated to the second author another proof of Hardy's theorem for step two stratified groups [5]

2. Heisenberg group revisited

In this section we give a new proof of Hardy's theorem for the Fourier transform on H^n using Beurling's theorem. The proof given in [17] uses Gelfand pairs associated to H^n and properties of the metaplectic representations. In view of this the earlier proof is not suitable for generalizing to other nilpotent Lie groups. Here we give a proof which works for non-isotropic Heisenberg groups also. Let π_λ be the Schrödinger representations of H^n with parameter λ. Explicitly $\pi_\lambda(z,t) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the unitary operator given by

$$\pi_\lambda(z,t)\phi(\xi) = e^{i\lambda t}e^{i\lambda(x\cdot\xi+\frac{1}{2}x\cdot y)}\phi(\xi+y)$$

where $\phi \in L^2(\mathbb{R}^n)$, $\xi \in \mathbb{R}^n$ and $z = x+iy$. We define $\pi_\lambda(z) = \pi_\lambda(z,0)$ so that $\pi_\lambda(z,t) = e^{i\lambda t}\pi_\lambda(z)$. For $f \in L^1(\mathbb{C}^n)$ its Weyl transform $W_\lambda(f)$ is the bounded operator on $L^2(\mathbb{R}^n)$ given by

$$W_\lambda(f)\phi = \int_{\mathbb{C}^n} f(z)\pi_\lambda(z)\phi\, dz\ .$$

It is clear that $\|W_\lambda(f)\| \leq \|f\|_1$ and for $f \in L^1\bigcap L^2(\mathbb{C}^n)$, $W_\lambda(f)$ is Hilbert-Schmidt and we have the Plancherel theorem

$$\|W_\lambda(f)\|^2_{\text{HS}} = (2\pi)^n|\lambda|^{-n}\int_{\mathbb{C}^n}|f(z)|^2\, dz.$$

Thus W_λ is an isometric isomorphism between $L^2(\mathbb{C}^n)$ and $\mathcal{S}_2$. For $f \in L^1(H^n)$ set

$$f^\lambda(z) = \int_{-\infty}^{\infty} e^{i\lambda t}f(z,t)\, dt$$

to be the inverse Fourier transform of f in the t-variable. Then from the definition of $\hat{f}(\lambda)$ it follows that $\hat{f}(\lambda) = W_\lambda(f^\lambda)$. For $\lambda = 1$ we define $W(z) = \pi_1(z)$.

For $\mu \in \mathbb{N}$ let Φ_μ be the normalized Hermite functions on $\mathbb{R}^n$. For $\mu, \nu \in \mathbb{N}$ the special Hermite functions $\Phi_{\mu\nu}$ are defined by

$$\Phi_{\mu\nu}(z) = (2\pi)^{-\frac{n}{2}} \left(W(z)\Phi_\mu, \Phi_\nu\right).$$

These functions form an orthonormal basis for $L^2(\mathbb{C}^n)$ and they are expressible in terms of Laguerre functions. For our purposes we only require the formula

$$\Phi_{\mu,0}(z) = (2\pi)^{-\frac{n}{2}} \left(\frac{1}{\mu!}\right)^{\frac{1}{2}} \left(\frac{i\bar{z}}{\sqrt{2}}\right)^{\mu} e^{-\frac{1}{4}|z|^2}. \tag{2.1}$$

We refer to [21] for these and more on special Hermite functions.

Given $d = (d_1, d_2, \cdots, d_n), d_j > 0$ the non-isotropic Heisenberg group H_d^n is just $\mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(z,t)(w,s) = (z+w, t+s+\frac{1}{2}\sum_{j=1}^{n} d_j \Im(z_j \bar{w}_j)).$$

For each $\lambda \in \mathbb{R} \setminus \{0\}$ there exists an irreducible unitary representation π_λ realized on $L^2(\mathbb{R}^n)$ given by

$$\pi_\lambda(z,t)\phi(\xi) = e^{i\lambda t} e^{i\lambda(\sum_{j=1}^{n} d_j(x_j\xi_j + \frac{1}{2}x_j y_j))} \phi(\xi + y).$$

These are all the infinite dimensional irreducible unitary representations of H_d^n up to unitary equivalence. The group Fourier transform of $f \in L^1(H_d^n)$ is defined by

$$\hat{f}(\lambda) = \int_{H_d^n} f(z,t)\pi_\lambda(z,t)\, dz\, dt.$$

The sublaplacian $\mathcal{L}$ for this group is

$$\mathcal{L} = -\sum_{j=1}^{n}(X_j(d)^2 + Y_j(d)^2)$$

where

$$X_j(d) = \frac{\partial}{\partial x_j} + \frac{1}{2}d_j y_j \frac{\partial}{\partial t},\ Y_j(d) = \frac{\partial}{\partial y_j} - \frac{1}{2}d_j x_j \frac{\partial}{\partial t}.$$

Let $q_{a,d}$ be the heat kernel corresponding to this sublaplacian. Then it can be proved as in the case of H^n that it satisfies the estimate

$$|q_{a,d}(z,t)| \le Ce^{-\frac{A}{a}(|z|^2 + |t|)}$$

for some $C, A > 0$. Also the explicit expression for $q_{a,d}^\lambda$ is given by

$$q_{a,d}^\lambda(z) = C_n \prod_{j=1}^{n} \left(\frac{d_j\lambda}{\sinh d_j\lambda a}\right) e^{-\frac{1}{4}d_j\lambda(\coth d_j\lambda a)|z_j|^2}.$$

Using the explicit formula for representations we can calculate that

$$\pi_\lambda(\mathcal{L}) = \sum_{j=1}^{n}\left(-\frac{\partial^2}{\partial\xi^2} + \lambda^2 d_j^2 \xi_j^2\right) = H(|\lambda|d),$$

$$\pi_\lambda(q_{a,d}) = e^{-aH(|\lambda|d)}.$$

Given $r = (r_1, r_2, \cdots, r_n),\ r_j > 0$ we define $U(r) : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ by

$$U(r)\phi(\xi) = \prod_{j=1}^{n} r^{\frac{1}{4}}\phi(\sqrt{r}_1\xi_1, \sqrt{r}_2\xi_2 \cdots, \sqrt{r}_n\xi_n).$$

Then $U(r)$ is a unitary operator on $L^2(\mathbb{R}^n)$ and

$$H(|\lambda|d)U(|\lambda|d)\Phi_\mu = \left(\sum_{j=1}^{n}(|\lambda|(2\mu_j+1)d_j)\right)U(|\lambda|d)\Phi_\mu.$$

For $\xi = (\xi', \xi'')$ where $\xi', \xi'' \in \mathbb{R}^n$ we define

$$\hat{f}(\lambda,\xi) = \pi_\lambda(\xi' + i\xi'', 0)\hat{f}(\lambda)\pi_\lambda(\xi' + i\xi'', 0)^*$$

and call it the Fourier-Weyl transform of f on H_d^n. It can be easily checked that

$$\hat{f}(\lambda,\xi) = \int_{\mathbb{R}^{2n}} e^{i\lambda\sum_{j=1}^{n} d_j(x_j\xi_j'' - y_j\xi_j')} f^\lambda(x,y)\pi_\lambda(x+iy,0)dx\,dy$$

where we have written $z = x + iy$ and $f^\lambda(x, y)$ stands for $f^\lambda(z)$.

With this preparation we have the following version of Beurling's theorem for H_d^n.

Theorem 2.1. *Let $f \in L^1(H_d^n)$ be such that for every $\lambda \in \mathbb{R}\setminus\{0\}$*

$$\int_{\mathbb{R}^{2n}}\int_{H_d^n} |f(x+iy,t)||\langle\hat{f}(\lambda,(\xi',\xi''))\phi_\lambda,\phi_\lambda\rangle|e^{|\lambda||\sum_{j=1}^{n} d_j(x_j\xi_j''-y_j\xi_j')|}\,dx\,dy\,dt\,d\xi'\,d\xi'' < \infty,$$

where $\phi_\lambda(x) = U(|\lambda|d)\Phi_0(x)$. Then $f = 0$ a.e.

Proof. Let $g_\lambda(x,y) = f^\lambda(x,y)\langle\pi_\lambda(x+iy,0)\phi_\lambda,\phi_\lambda\rangle$. Then $\hat{f}(\lambda,(\xi',\xi''))$ is the Fourier transform of g_λ at $|\lambda|(-d_1\xi_1'',\cdots,-d_n\xi_n'',d_1\xi_1',\cdots,d_n\xi_n')$.

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} |g_\lambda(x,y)|\,|\widehat{g}_\lambda(\xi',\xi'')|e^{|x\cdot\xi'+y\cdot\xi''|}\,dx\,dy\,d\xi'\,d\xi''\\
\leq\ &\int_{\mathbb{R}^{2n}}\int_{H_d^n} |f(x+iy,t)||\langle\hat{f}(\lambda,(\xi',\xi''))\phi_\lambda,\phi_\lambda\rangle|e^{|\lambda||\sum_{j=1}^{n} d_j(x_j\xi_j''-y_j\xi_j')|}dx\,dy\,dt\,d\xi'\,d\xi''\\
<\ &\infty
\end{aligned}$$

by our hypothesis. Now applying Beurling's theorem on $\mathbb{R}^{2n}$ to the function g_λ we get $g_\lambda = 0$ a.e. Since $\langle\pi_\lambda(x+iy,0)\phi_\lambda,\phi_\lambda\rangle = \prod_{j=1}^{n} e^{-\frac{1}{4}|\lambda|\ d_j(x_j^2+y_j^2)}$ is non vanishing everywhere, $f^\lambda = 0$ a.e. for all λ and hence $f = 0$ a.e. □

Applying Beurling's theorem on $\mathbb{R}^{2n}$ we can also prove the following version of Cowling-Price theorem.

Theorem 2.2. *Let f be a function on H_d^n such that $f(q_{a,d})^{-1} \in L^p(H_d^n)$ and $\widehat{f}(\lambda)e^{bH(|\lambda|d)} \in \mathcal{S}_q$, $1 \leq p, q \leq \infty$. Then $f = 0$ a.e whenever $a < b$.*

Proof. Define $g(z,t) = (q_{a,d})^{-1}(z,t)f(z,t)$. From the hypothesis $g \in L^p(H_d^n)$ and using the estimate $|q_{a,d}(z,t)| \leq Ce^{-\frac{A}{a}(|z|^2+|t|)}$ it is easy to see that f^λ can be extended as a holomorphic function in the strip $|\Im\lambda| < \frac{A}{a}$ of the complex plane.

For $p = \infty$,

$$\begin{aligned}
|f^\lambda(z)| &\leq \|g(z,\cdot)\|_\infty \int_{-\infty}^{\infty} |q_{a,d}(z,t)|\,dt\\
&\leq \|g(z,\cdot)\|_\infty e^{-\frac{1}{4a}|z|^2}.
\end{aligned}$$

For $1 \leq p \leq 2$,

$$\begin{aligned} |f^{\lambda}(z)| &\leq \int_{-\infty}^{\infty} |q_{a,d}(z,t)||g(z,t)|\, dt \\ &\leq \left(\int_{-\infty}^{\infty} |q_{a,d}(z,t)|^{p'}\, dt\right)^{\frac{1}{p'}} \left(\int_{-\infty}^{\infty} |g(z,t)|^{p}\, dt\right)^{\frac{1}{p}}. \end{aligned}$$

Now applying Hausdorff-Young inequality to the first integral

$$\begin{aligned} |f^{\lambda}(z)| &\leq \left(\int_{-\infty}^{\infty} |q^{\lambda}_{a,d}(z)|^{p}\, d\lambda\right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |g(z,t)|^{p}\, dt\right)^{\frac{1}{p}} \\ &\leq (4\pi)^{-n} \left(\int_{-\infty}^{\infty} \prod_{j=1}^{n} \left(\frac{d_j\lambda}{\sinh d_j\lambda a}\right)^{p} e^{-\frac{p}{4}(d_j\lambda \coth d_j\lambda a)|z_j|^2} d\lambda\right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |g(z,t)|^{p}\, dt\right)^{\frac{1}{p}} \\ &\leq Ce^{-\frac{1}{4a}|z|^2} \left(\int_{-\infty}^{\infty} |g(z,t)|^{p}\, dt\right)^{\frac{1}{p}}. \end{aligned}$$

When $2 < p < \infty$, $1 < p' < 2$ write $\frac{1}{p'} = \frac{\nu}{1} + \frac{1-\nu}{2}$ for some $0 < \nu < 1$. Since $\|q_{a,d}(z,\cdot)\|_1 \leq e^{-\frac{1}{4a}|z|^2}$ and $\|q_{a,d}(z,\cdot)\|_2 \leq e^{-\frac{1}{4a}|z|^2}$, applying Hölder's inequality with the pair of conjugate exponents $\frac{1}{\nu p'}$ and $\frac{2}{(1-\nu)p'}$, we get

$$\begin{aligned} \int_{-\infty}^{\infty} |q_{a,d}(z,t)|^{p'}\, dt &= \int_{-\infty}^{\infty} |q_{a,d}(z,t)|^{\nu p'} |q_a(z,t)|^{(1-\nu)p'}\, dt \\ &\leq \|q_{a,d}(z,\cdot)\|_1^{\nu p'} \|q_{a,d}(z,\cdot)\|_2^{(1-\nu)p'} \end{aligned}$$

which gives $\|q_{a,d}(\cdot,t)\|_{p'} \leq e^{-\frac{1}{4a}|z|^2}$. Therefore $f^{\lambda}e^{\frac{1}{4a}|z|^2} \in L^p(\mathbb{C}^n)$ for $1 \leq p \leq \infty$. Since every member of $\mathcal{S}_q$ is a bounded operator there exits $C > 0$ such that $\|\hat{f}(\lambda)e^{bH(|\lambda|d)}\phi\|_2 \leq C$ for all $\phi \in L^2(\mathbb{R}^n)$. As

$$H(|\lambda|d)U(|\lambda|d)\Phi_\alpha = \left(|\lambda| \sum_{j=1}^{n} (2\alpha_j + 1)d_j\right) U(|\lambda|d)\Phi_\alpha$$

for all $\alpha \in \mathbb{N}^n$, we have

$$\|\hat{f}(\lambda)U(|\lambda|d)\Phi_\alpha\|_2 \leq C \left(\prod_{j=1}^{n} e^{-b|\lambda|(2\alpha_j+1)d_j}\right) \tag{2.2}$$

for all $\alpha \in \mathbb{N}^n$.

For $\xi = (\xi', \xi'') \in \mathbb{R}^n \times \mathbb{R}^n$ and $r = (r_1, \cdots, r_n) \in \mathbb{R}^n$ let us denote the point $(r_1\xi'_1, \cdots r_n\xi'_n)$ by $r\xi$ and $(r_1\xi'_1, \cdots r_n\xi'_n, r_1\xi'', \cdots, r_n\xi''_n)$ by $(r\xi', r\xi'')$ and write $r(\xi' + i\xi'')$ for the point $r\xi' + ir\xi'' \in \mathbb{C}^n$. Using the fact that $\{U(|\lambda|d)\Phi_\alpha : \alpha \in \mathbb{N}^n\}$ form an orthonormal basis for $L^2(\mathbb{R}^n)$ we compute

$$\begin{aligned}
& \pi_\lambda(\xi' + i\xi'', 0)^* U(|\lambda|d)\Phi_0 \\
= & \sum_\alpha \langle \pi_\lambda(\xi' + i\xi'', 0)^* U(|\lambda|d)\Phi_0, U(|\lambda|d)\Phi_\alpha\rangle U(|\lambda|d)\Phi_\alpha \\
= & \sum_\alpha \langle U(|\lambda|d)^* \pi_\lambda(\xi' + i\xi'', 0)^* U(|\lambda|d)\Phi_0, \Phi_\alpha\rangle U(|\lambda|d)\Phi_\alpha \\
= & \sum_\alpha \langle W(-\sqrt{|\lambda|d}(\xi' + i\xi''))\Phi_0, \Phi_\alpha\rangle U(|\lambda|d)\Phi_\alpha \\
= & \sum_\alpha \bar{\Phi}_{\alpha,0}(\sqrt{|\lambda|d}(\xi' + i\xi''))U(|\lambda|d)\Phi_\alpha.
\end{aligned}$$

Using (2.2) and the above expression for $\pi_\lambda(\xi' + i\xi'', 0)^* U(|\lambda|d)\Phi_0$ we have

$$\begin{aligned}
& |\langle \hat{f}(\lambda, (\xi', \xi''))U(|\lambda|d)\Phi_0, U(|\lambda|d)\Phi_0\rangle| \\
= & |\langle \pi_\lambda(\xi' + i\xi'', 0)\hat{f}(\lambda)\pi_\lambda(\xi' + i\xi'', 0)^* U(|\lambda|d)\Phi_0, U(|\lambda|d)\Phi_0\rangle| \\
\leq & \|\hat{f}(\lambda)\pi_\lambda(\xi' + i\xi'', 0)^* U(|\lambda|d)\Phi_0\|_2 \\
\leq & C\sum_\alpha |\Phi_{\alpha,0}(\sqrt{|\lambda|d}(\xi' + i\xi''))| \, \|\hat{f}(\lambda)U(|\lambda|d)\Phi_\alpha\|_2 \\
\leq & C\sum_\alpha |\Phi_{\alpha,0}(\sqrt{|\lambda|d}(\xi' + i\xi''))| \, e^{-b|\lambda|(\sum_{j=1}^n (2\alpha_j+1)d_j)}.
\end{aligned}$$

Applying the Cauchy-Schwarz inequality and using formula for $\Phi_{\alpha,0}$ in the above estimate , we have for $a < b' < b$

$$\begin{aligned}
& |\langle \hat{f}(\lambda, (\xi', \xi''))U(|\lambda|d)\Phi_0, U(|\lambda|d)\Phi_0\rangle| \\
\leq & C\sum_\alpha |\Phi_{\alpha,0}(\sqrt{|\lambda|d}(\xi' + i\xi''))| e^{-(b-b')|\lambda|(\sum_{j=1}^n (2\alpha_j+1)d_j)} e^{-b'|\lambda|(\sum_{j=1}^n (2\alpha_j+1)d_j)} \\
\leq & C\left(\sum_\alpha |\Phi_{\alpha,0}(\sqrt{|\lambda|d}(\xi' + i\xi''))|^2 e^{-2b'|\lambda|(\sum_{j=1}^n (2\alpha_j+1)d_j)}\right)^{\frac{1}{2}} \\
\leq & C\left(\prod_{j=1}^n \sum_{\alpha_j=0}^\infty \left(\frac{1}{\alpha_j!}\, e^{-2b'|\lambda|(2\alpha_j+1)d_j} \left(\frac{1}{2}|\lambda|d_j({\xi'_j}^2 + {\xi''_j}^2)\right)^{\alpha_j} e^{-\frac{1}{2}|\lambda|d_j({\xi'_j}^2+{\xi''_j}^2)}\right)\right)^{\frac{1}{2}} \\
\leq & C\prod_{j=1}^n \left(e^{-\frac{1}{4}|\lambda|d_j({\xi'_j}^2+{\xi''_j}^2)} \left(\sum_{\alpha_j=0}^\infty \frac{1}{\alpha_j!}\left(\frac{1}{2}|\lambda|d_j({\xi'_j}^2 + {\xi''_j}^2)e^{-4b'|\lambda|d_j}\right)^{\alpha_j}\right)^{\frac{1}{2}}\right) \\
\leq & C\prod_{j=1}^n e^{-\frac{1}{4}|\lambda|d_j\left(1-e^{-4b'|\lambda|d_j}\right)({\xi'_j}^2+{\xi''_j}^2)}.
\end{aligned}$$

Let $g_\lambda(x, y) = f^\lambda(x, y)\langle \pi_\lambda(x + iy, 0)\phi_\lambda, \phi_\lambda\rangle$.

Then

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} |g_\lambda(x,y)|\ |\widehat{g}_\lambda(\xi',\xi'')| e^{|x\cdot\xi'+y\cdot\xi''|} dx\ dy\ d\xi'\ d\xi'' \\
\leq\ & \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} |f^\lambda(x,y)|\ |\langle \hat{f}(\lambda,(\xi',\xi''))\phi_\lambda,\phi_\lambda\rangle| e^{|\lambda|\sum_{j=1}^{n} d_j(|x_j\xi_j''|+|y_j\xi_j'|)} dx\ dy\ d\xi'\ d\xi'' \\
\leq\ & C\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} |f^\lambda(x,y)|\ e^{\frac{1}{4a}(|x|^2+|y|^2)} e^{-\frac{1}{4a}|((x,y)-2a|\lambda|d(\xi'',\xi'))|^2} \times \\
& \prod_{j=1}^{n} e^{-|\lambda|d_j\left(\frac{1}{4}\left(1-e^{-4b'|\lambda|d_j}\right)-a|\lambda|d_j\right)(\xi_j'^2+\xi_j''^2)} dx\ dy\ d\xi'\ d\xi''
\end{aligned}$$

Since $f^\lambda e^{\frac{1}{4a}(|x|^2+|y|^2)} \in L^p(\mathbb{R}^{2n})$ the above integral will be finite if $a|\lambda|d_j < \frac{1}{4}\left(1-e^{-4b'|\lambda|d_j}\right)$. Therefore applying Beurling's theorem to the function g_λ we conclude $g_\lambda = 0$ and hence $f^\lambda = 0$ whenever $a|\lambda|d_j < \frac{1}{4}\left(1-e^{-4b'|\lambda|d_j}\right)$. Since $a < b' < b$ we can choose $\delta > 0$ such that $a < b'e^{-2b'|\lambda|d_j} < b'$ for all j with $0 < |\lambda| < \delta$. Now

$$\begin{aligned}
1-e^{-4b'|\lambda|d_j} &= e^{-2b'|\lambda|d_j}(e^{2b'|\lambda|d_j}-e^{-2b'|\lambda|d_j}) \\
&> 4b'|\lambda|d_j e^{-2b'|\lambda|d_j} \\
&> 4a|\lambda|d_j
\end{aligned}$$

for all j and λ with $0 < |\lambda| < \delta$. Since f^λ can be extended to a holomorphic function in a strip of the complex plane we conclude $f^\lambda = 0$ for all λ whenever $a < b$ and hence $f = 0$. □

If $\hat{f}(\lambda)^*\hat{f}(\lambda) \leq Ce^{-2bH(|\lambda|d)}$ we have the estimate

$$\|\hat{f}(\lambda)U(|\lambda|d)\Phi_\alpha\|_2 \leq C\prod_{j=1}^{n} e^{-b|\lambda|(2\alpha_j+1)d_j}$$

for all $\alpha \in \mathbb{N}^n$ and hence $\langle \hat{f}(\lambda,(\xi',\xi''))U(|\lambda|d)\Phi_0, U(|\lambda|d)\Phi_0\rangle$ can be estimated as before. Thus the following version of Hardy's theorem for H_d^n is true.

Theorem 2.3. *If $|f(z,t)| \leq Cq_a(z,t)$, $\hat{f}(\lambda)^*\hat{f}(\lambda) \leq Ce^{-2bH(|\lambda|d)}$ for some $a, b > 0$ then $f = 0$ whenever $a < b$.*

Remark 2.4. *If $d_j = 1$ for all j then $H_d^n = H^n$ and hence all the results mentioned in section 1 for H^n are proved.*

3. Preliminaries on Step two nilpotent Lie groups

Let G be a step two nilpotent Lie group so that its Lie algebra $\mathfrak{g}$ has the decomposition $\mathfrak{g} = \mathfrak{v}\bigoplus\mathfrak{z}$, where $\mathfrak{z}$ is the centre of $\mathfrak{g}$ and $\mathfrak{v}$ is any subspace of $\mathfrak{g}$ complementary to $\mathfrak{z}$. We choose an inner product on $\mathfrak{g}$ such that $\mathfrak{v}$ and $\mathfrak{z}$ are orthogonal. Fix an orthonormal basis $\{e_1, e_2\cdots, e_m, T_1,\cdots,T_k\}$ so that $\mathfrak{v} = \mathbb{R}\ \mathrm{span}\{e_1, e_2\cdots, e_m\}$ and $\mathfrak{z} = \mathbb{R}\ \mathrm{span}\{T_1,\cdots,T_k\}$. Since $\mathfrak{g}$ is nilpotent the exponential map is surjective. We can identify G with $\mathfrak{v}\bigoplus\mathfrak{z}$ and write $(X+T)$ for $\exp(X+T)$ and denote it by (X,T) where $X \in \mathfrak{v}$ and $T \in \mathfrak{z}$. The product law on G is given by the Baker-Campbell-Hausdorff formula :

$$(X,T)(X',T') = (X+X', T+T'+\frac{1}{2}[X,X'])$$

for all $X, X' \in \mathfrak{v}$ and $T, T' \in \mathfrak{z}$.

For each $\lambda \in \mathfrak{z}^*$ real dual of $\mathfrak{z}$ consider the bilinear form B_λ on $\mathfrak{v}$ defined by

$$B_\lambda(X,Y) = \lambda([X,Y]) \text{ for all } X,Y \in \mathfrak{v}.$$

Let $\mathfrak{r}_\lambda = \{X : B_\lambda(X,Y) = 0, \forall\, Y \in \mathfrak{v}\}$ and $\mathfrak{m}_\lambda$ denote the orthogonal complement of $\mathfrak{r}_\lambda$ in $\mathfrak{v}$. Let $\Lambda = \{\lambda \in \mathfrak{z}^* : \dim \mathfrak{m}_\lambda \text{ is maximum}\}$. If $r_\lambda = \{0\}$ for each $\lambda \in \Lambda$ the Lie algebra is called an MW algebra after Moore and Wolf and the corresponding Lie group is called an MW group. For any orthonormal basis $\{X_j : 1 \leq j \leq m\}$ of $\mathfrak{v}$ define

$$\mathcal{L} = -\sum_{j=1}^{m} X_j^2$$

and call it the sublaplacian of G.

3.1. **Step two groups without MW- condition.** In this case $r_\lambda \neq \{0\}$ for each $\lambda \in \Lambda$ and $B_\lambda|_{\mathfrak{m}_\lambda}$ is nondegenerate and hence dim m_λ is even say $2n$. Then there exists an orthonormal basis $\{X_1(\lambda), Y_1(\lambda), \cdots, X_n(\lambda), Y_n(\lambda), Z_1(\lambda), \cdots Z_r(\lambda)\}$ of $\mathfrak{v}$ and positive numbers $d_i(\lambda) > 0$ such that

(i) $\mathfrak{r}_\lambda = \mathbb{R}$ span $\{Z_1(\lambda), \cdots, Z_r(\lambda)\}$,

(ii) $\lambda([X_i(\lambda), Y_j(\lambda)]) = \delta_{i,j} d_j(\lambda), 1 \leq i,j \leq n$ and
$\lambda([X_i(\lambda), X_j(\lambda)]) = 0,\ \lambda([Y_i(\lambda), Y_j(\lambda)]) = 0$ for $1 \leq i,j \leq n$,

(iii) $\mathbb{R}$ span $\{X_1(\lambda) \cdots, X_n(\lambda), Z_1(\lambda), \cdots, Z_r(\lambda), T_1, \cdots, T_k\} = \mathfrak{h}_\lambda$ is a polarization for λ.

We call the basis $\{X_1(\lambda), \cdots, X_n(\lambda), Y_1(\lambda), \cdots, Y_n(\lambda), Z_1(\lambda), \cdots, Z_r(\lambda), T_1, \cdots, T_k\}$ almost symplectic basis. Let $\xi_\lambda = \mathbb{R}$ span$\{X_1(\lambda) \cdots, X_n(\lambda)\}$ and $\eta_\lambda = \mathbb{R}$ span $\{Y_1(\lambda), \cdots, Y_n(\lambda)\}$. Then we have the decomposition $\mathfrak{g} = \xi_\lambda \bigoplus \eta_\lambda \bigoplus \mathfrak{r}_\lambda \bigoplus \mathfrak{z}$. We denote the element $\exp(X+Y+Z+T)$ of G by (X,Y,Z,T) for $X \in \xi_\lambda, Y \in \eta_\lambda, Z \in r_\lambda, T \in \mathfrak{z}$. Further we can write

$$(X,Y,Z,T) = \sum_{j=1}^{n} x_j(\lambda) X_j(\lambda) + \sum_{j=1}^{n} y_j(\lambda) Y_j(\lambda) + \sum_{j=1}^{r} z_j(\lambda) Z_j(\lambda) + \sum_{j=1}^{k} t_j T_j$$

and denote it by (x,y,z,t) suppressing the dependence of λ which will be understood from the context.

Since $\lambda|[\mathfrak{h}_\lambda, \mathfrak{h}_\lambda] = 0$ for $\mu \in \mathfrak{r}_\lambda^*$ we define character $\sigma_{\lambda,\mu}$ of $H_\lambda = exp(\mathfrak{h}_\lambda)$ by

$$\sigma_{\lambda,\mu}(X,Z,T) = e^{i\mu(Z)+i\lambda(T)} \text{ for all } (X,Z,T) \in H_\lambda.$$

Now proceeding as [1] the irreducible unitary representations $\pi_{\lambda,\mu}$ of G realized on $L^2(\eta_\lambda)$ can be described as follows :

$$(\pi_{\lambda,\mu}(X,Y,Z,T)\phi)\,(Y') = e^{i\lambda(T+[Y'+\frac{1}{2}Y, X-Y'+Z])} e^{i\mu(z)} \phi(Y+Y')$$

for all $\phi \in L^2(\eta_\lambda)$. Using the almost symplectic basis we have the following description :

$$(\pi_{\lambda,\mu}(x,y,z,t)\phi)\,(\xi) = e^{i\sum_{j=1}^{k} \lambda_j t_j + i\sum_{j=1}^{r} \mu_j z_j + i\sum_{j=1}^{n} d_j(\lambda)(x_j\xi_j + \frac{1}{2}x_j y_j)} \phi(\xi+y)$$

for all $\phi \in L^2(\eta_\lambda)$.

Define the Fourier transform of $f \in L^1(G)$ by

$$\hat{f}(\lambda,\mu) = \int_{\mathfrak{z}}\int_{\mathfrak{r}_\lambda}\int_{\eta_\lambda}\int_{\xi_\lambda} f(x,y,z,t)\pi_{\lambda,\mu}(x,y,z,t)\ dx\ dy\ dz\ dt$$

for all $\lambda \in \Lambda$ and $\mu \in r_\lambda^*$. We let

$$f^\lambda(x,y,z) = \int_{\mathfrak{z}} e^{i\sum_{j=1}^{k} \lambda_j t_j} f(x,y,z,t)\ dt,$$

$$f^{\lambda,\mu}(x,y) = \int\limits_{r_\lambda}\int\limits_{\mathfrak{z}} e^{i\sum_{j=1}^{k}\lambda_j t_j + i\sum_{j=1}^{r}\mu_j z_j} f(x,y,z,t)\, dt\, dz.$$

for all $\lambda \in \Lambda$ and $\mu \in r_\lambda^*$.

For suitable functions f and g on $\mathfrak{v}$ the λ-twisted convolution is defined by

$$f \times_\lambda g(v) = \int\limits_{\mathfrak{v}} f(v-v')g(v')e^{\frac{i}{2}\lambda([v,v'])}dv'.$$

An easy calculation shows that

$$(f * g)^\lambda(v) = f^\lambda \times_\lambda g^\lambda(v)$$

for $f, g \in L^1(G)$. For $\lambda \in \Lambda$ we define $\mathrm{Pf}(\lambda) = \prod_{j=1}^{n} d_j(\lambda)$ called the *Pfaffian* of λ. $\hat{f}(\lambda,\mu)$ is a Hilbert-Schmidt operator for $f \in L^1 \bigcap L^2(G)$ and its Hilbert-Schmidt norm is given by

$$\mathrm{Pf}(\lambda)\ \|\hat{f}(\lambda,\mu)\|_{\mathrm{HS}}^2 = (2\pi)^n \int\limits_{\eta_\lambda}\int\limits_{\xi_\lambda} |f^{\lambda,\mu}(x,y)|^2\ dx\ dy.$$

Polarizing this identity

$$(2\pi)^{-(n+r)}\ \mathrm{Pf}\ (\lambda) \int\limits_{r_\lambda^*} \mathrm{tr}\ \left(\hat{f}(\lambda,\mu)\hat{g}(\lambda,\mu)^*\right)\ d\mu = \int\limits_{\mathfrak{r}_\lambda}\int\limits_{\eta_\lambda}\int\limits_{\xi_\lambda} f^\lambda(x,y,z)\overline{g^\lambda(x,y,z)}\ dx\ dy\ dz.$$

For each $\lambda \in \Lambda, \mu \in r_\lambda^*$ and $g \in L^1 \bigcap L^2(\xi_\lambda \bigoplus \eta_\lambda \bigoplus r_\lambda)$ define the operator $W_{\lambda,\mu}(g)$ by

$$W_{\lambda,\mu}(g) = \int\limits_{r_\lambda}\int\limits_{\eta_\lambda}\int\limits_{\xi_\lambda} g(x,y,z)\pi_{\lambda,\mu}(x,y,z,0)\ dx\ dy\ dz.$$

With this notation for all $g, h \in L^2(\xi_\lambda \bigoplus \eta_\lambda \bigoplus r_\lambda)$ we have

$$(2\pi)^{-n-r}\mathrm{Pf}(\lambda) \int\limits_{r_\lambda^*} \mathrm{tr}\,(W_{\lambda,\mu}(g)W_{\lambda,\mu}(h)^*)\, d\mu = \int\limits_{r_\lambda}\int\limits_{\eta_\lambda}\int\limits_{\xi_\lambda} g(x,y,z)\bar{h}(x,y,z)\ dx\ dy\ dz.$$

If g is a Schwartz function then it has been shown in [1] $\mathrm{tr}\,(|W_{\lambda,\mu}(g)|)$ can be estimated in terms of Schwartz semi norms of g. In fact $(1+|\mu|)^k\ \mathrm{tr}\,(|W_{\lambda,\mu}(g)|) \leq Cr(\lambda)^l\|g\|_*$ for some $l, k > 0$ where $r(\lambda) = \sum_{j=1}^{n}(d_j(\lambda)^2 + d_j(\lambda)^{-2})$ and $\|\cdot\|_*$ is suitable sum of Schwartz semi norms.

Also we have the following inversion formula :

$$f^\lambda(x,y,z) = (2\pi)^{-(n+r)}\mathrm{Pf}(\lambda) \int\limits_{r_\lambda^*} \mathrm{tr}\left(\pi_{\lambda,\mu}(x,y,z,0)^*\hat{f}(\lambda,\mu)\right) d\mu.$$

With respect to almost symplectic basis the sublaplacian takes the normal form

$$\mathcal{L} = -\sum_{j=1}^{n}\left(X_j^2(\lambda) + Y_j^2(\lambda)\right) - \sum_{j=1}^{r} Z_j^2(\lambda).$$

Using the explicit form of the representations we can calculate that

$$\pi_{\lambda,\mu}(\mathcal{L}) = \sum_{j=1}^{n}\left(-\frac{\partial^2}{\partial\xi_j^2} + d_j^2(\lambda)\xi_j^2\right) + |\mu|^2 = H(\mu, d(\lambda))$$

where $d(\lambda) = (d_1(\lambda), \cdots, d_n(\lambda))$. Also we have

$$H(\mu, d(\lambda))U(d(\lambda))\Phi_\alpha = \left(|\mu|^2 + \sum_{j=1}^{n}(2\alpha_j+1)d_j(\lambda)\right)U(d(\lambda))\Phi_\alpha$$

for all $\alpha \in \mathbb{N}^n$.

3.2. **Step two groups with MW- condition.** In this case $r_\lambda = \{0\}$ and the irreducible unitary representations are parameterized by $\lambda \in \Lambda$ and given by

$$(\pi_\lambda(x,y,t)\phi)(\xi) = e^{i\sum_{j=1}^{k}\lambda_j t_j + i\sum_{j=1}^{n} d_j(\lambda)(x_j\xi_j + \frac{1}{2}x_j y_j)}\phi(\xi + y)$$

for all $\phi \in L^2(\eta_\lambda)$. Define the Fourier transform of $f \in L^1(G)$ by

$$\hat{f}(\lambda) = \int_{\mathfrak{z}}\int_{\eta_\lambda}\int_{\xi_\lambda} f(x,y,t)\pi_\lambda(x,y,t)\ dx\ dy\ dt$$

for all $\lambda \in \Lambda$. Also define

$$f^\lambda(x,y) = \int_{\mathfrak{z}} e^{i\sum_{j=1}^{k}\lambda_j t_j} f(x,y,t)\ dx\ dy\ dt$$

for all $\lambda \in \Lambda$.

For each $\lambda \in \Lambda$ and $g \in L^1 \bigcap L^2(\xi_\lambda \bigoplus \eta_\lambda)$ define

$$W'_\lambda(g) = \int_{\eta_\lambda}\int_{\xi_\lambda} g(x,y)\pi_\lambda(x,y,0)\ dx\ dy$$

and call the Weyl transform of g. Then $W'_\lambda(g)$ is kernel operator with kernel

$$K_\lambda(\xi,y) = \int_{\xi_\lambda} e^{i\frac{1}{2}\sum_{j=1}^{n} d_j(\lambda)(\xi_j + y_j)x_j} g(x, y-\xi)\ dx.$$

Moreover if $g \in L^2(\xi_\lambda \bigoplus \eta_\lambda)$ then $W'_\lambda(g)$ is a Hilbert-Schmidt operator and we have the Plancherel theorem

$$\mathrm{Pf}\ (\lambda)\ \|W'_\lambda(g)\|^2_{\mathrm{HS}} = (2\pi)^n \int_{\eta_\lambda}\int_{\xi_\lambda} |g(x,y)|^2\ dx\ dy.$$

Polarising this identity we obtain

$$\mathrm{Pf}\ (\lambda)\ \mathrm{tr}\ (W'_\lambda(g)^* W'_\lambda(h)) = (2\pi)^n \int_{\eta_\lambda}\int_{\xi_\lambda} g(x,y)\bar{h}(x,y)\ dx\ dy.$$

If g is from Schwartz class function then K_λ is so and it has been shown in [1] $\mathrm{tr}(|W'_\lambda(g)|)$ can be estimated in terms of sums of Schwartz semi norms of K_λ. Also it is easy to see that $\|K_\lambda\|_* \le r(\lambda)^k \|g\|_{**}$ for some $k > 0$ where $\|\cdot\|_*, \|\cdot\|_{**}$ are suitable sum of Schwartz semi norms. The inversion formula takes the form:

$$f^\lambda(x,y,z) = (2\pi)^{-n}\mathrm{Pf}(\lambda)\ \mathrm{tr}\ \left(\pi_\lambda(x,y,0)^*\hat{f}(\lambda)\right).$$

The sublaplacian takes the normal form

$$\mathcal{L} = -\sum_{j=1}^{n}(X_j(\lambda)^2 + Y_j(\lambda)^2)$$

with respect to almost symplectic basis and

$$\pi_\lambda(\mathcal{L}) = \sum_{j=1}^{n}(-\frac{d^2}{d\xi_j^2} + d_j(\lambda)^2\xi_j^2) = H(d(\lambda)).$$

4. Step two stratified groups

Two step Lie algebra $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$ is called stratified if it is generated by $\mathfrak{v}$ and the corresponding group is called stratified group. If the group G is stratified then there exists a smooth function $p_a(v,t)$ on $G \times (0,\infty)$ such that $f * p_a(v,t)$ solves the heat equation associated with the sublaplacian $\mathcal{L}$ with initial condition f, see Folland-Stein [9]. This p_a is called the heat kernel associated with $\mathcal{L}$. In this section we will prove Hardy's and Cowling-Price theorems for all step two stratified groups.

4.1. **Step two stratified group without MW- condition.** Using the inversion formula stated in section 3.1 it can be proved as in the case of Heisenberg group that

$$\hat{p}_a(\lambda,\mu) = e^{-aH(\mu,d(\lambda))}$$

and

$$p_a^\lambda(x,y,z) = (4\pi)^{-n}(4\pi a)^{-\frac{r}{2}} e^{-\frac{1}{4a}|z|^2} \prod_{j=1}^{n} \left(\frac{d_j(\lambda)}{\sinh a d_j(\lambda)}\right) e^{-\frac{1}{4}d_j(\lambda)\coth a d_j(\lambda)(x_j^2+y_j^2)}.$$

Now we want to study the behavior of $d_j(\lambda)$ as $\lambda \to 0$. We show that $d_j(\lambda) \to 0$ as $\lambda \to 0$. Note that $B_{r\omega} = rB_\omega$ for $r \in (0,\infty)$ and $\omega \in S^{k-1}$. Then it follows that $X_j(r\omega) = X_j(\omega), Y_j(r\omega) = Y_j(\omega), Z_j(r\omega) = Z_j(\omega)$ and $d_j(r\omega) = rd_j(\omega)$ for all j. The entries of the matrix $(B_\lambda(e_i,e_j))_{i,j}$ are continuous functions in λ and $\pm i d_j(\lambda)$ being eigenvalues of the matrix, d_j are also continuous in λ. Using the fact that $d_j(r\omega) = rd_j(\omega)$ we conclude $d_j(\lambda) \to 0$ as $\lambda \to 0$. $p_a^\lambda(v)$ is a continuous function in λ for each $v \in \mathfrak{v}$ and is given by

$$\begin{aligned} p_a^\lambda(v) &= (4\pi)^{-n}(4\pi a)^{-\frac{r}{2}} e^{-\frac{1}{4a}\sum_{j=1}^{r}\langle v,Z_j(\lambda)\rangle^2} \\ &\quad \times \prod_{j=1}^{n}\left(\frac{d_j(\lambda)}{\sinh a d_j(\lambda)}\right) e^{-\frac{1}{4}d_j(\lambda)\coth a d_j(\lambda)\left(\langle v,X_j(\lambda)\rangle^2+\langle v,Y_j(\lambda)\rangle^2\right)}. \end{aligned}$$

Write $\lambda = r\omega$ for $r \in (0,\infty)$ and $\omega \in S^{k-1}$ and compute $\lim_{r\to 0} p_a^{r\omega}(v)$ remembering the fact that $X_j(r\omega) = X_j(\omega), Y_j(r\omega) = Y_j(\omega), Z_j(r\omega) = Z_j(\omega)$ and $d_j(r\omega) = rd_j(\omega)$ for all j. We get

$$\begin{aligned} p_a^0(v) &= (4\pi)^{-n}(4\pi a)^{-\frac{r}{2}} a^{-n} e^{-\frac{1}{4a}\left(\sum_{i=1}^{r}\langle v,Z_j(\omega)\rangle^2+\sum_{j=1}^{n}\langle v,X_j(\omega)\rangle^2+\sum_{j=1}^{n}\langle v,Y_j(\omega)\rangle^2\right)} \\ &= (4\pi)^{-n}(4\pi a)^{-\frac{r}{2}} a^{-n} e^{-\frac{1}{4a}(\sum_{j=1}^{2n+r}\langle v,e_j\rangle^2)} \\ &= (4\pi)^{-n}(4\pi a)^{-\frac{r}{2}} a^{-n} e^{-\frac{1}{4a}|v|^2}. \end{aligned} \tag{4.1}$$

Writing $v = (x(\lambda),y(\lambda),z(\lambda)) \in \xi_\lambda \bigoplus \eta_\lambda \bigoplus r_\lambda = \mathfrak{v}$

$$p_a^0(x(\lambda),y(\lambda),z(\lambda)) = (4\pi)^{-n}(4\pi a)^{-\frac{r}{2}} a^{-n} e^{-\frac{1}{4a}(|x(\lambda)|^2+|y(\lambda)|^2+|z(\lambda)|^2)}.$$

With this preparation we have following versions of Hardy and Cowling-Price theorems.

Theorem 4.1. *Let G be a step two stratified group without MW condition. Let f be a function on G such that $|f(v,t)| \le Cp_a(v,t)$ and $\hat{f}(\lambda,\mu)^*\hat{f}(\lambda,\mu) \le Ce^{-2bH(\mu,d(\lambda))}$ for every $\mu \in r_\lambda^*$, and $\lambda \in \Lambda$. Then $f = 0$ whenever $a < b$.*

Proof. Let $f^*(x,y,z,t) = \bar{f}(-x,-y,-z,-t)$ and consider the function

$$h_\lambda(z) = \int_{\xi_\lambda}\int_{\eta_\lambda} f^\lambda *_3 f^{*^\lambda}(x,y,z)\ dx\ dy$$

where $*_3$ means convolution is taken in third variable. Since $|f^\lambda(x,y,z)| \le C\ p_a^0(x,y,z)$ we have the following estimate on h :

$$|h_\lambda(z)| \le Ce^{-\frac{1}{8a}|z|^2}.$$

Also

$$\hat{f}(\lambda,\mu)^*\hat{f}(\lambda,\mu) \le Ce^{-2bH(\mu,d(\lambda))}$$

for every $\mu \in r_\lambda^*$ and $\lambda \in \Lambda$ gives the estimate

$$\begin{aligned}\|\hat{f}(\lambda,\mu)\|_{\mathrm{HS}}^2 &\le C\ e^{-2b|\mu|^2}\prod_{j=1}^{n}\sum_{\alpha_j\in\mathbb{N}} e^{-2b(2\alpha_j+1)d_j(\lambda)}\\ &\le C_\lambda\ e^{-2b|\mu|^2}.\end{aligned}$$

Therefore

$$\begin{aligned}\hat{h}_\lambda(\mu) &= \int_{\xi_\lambda}\int_{\eta_\lambda} |f^{\lambda,\mu}(x,y)|^2\ dx\ dy\\ &= (2\pi)^{-n}\mathrm{Pf}(\lambda)^{-1}\|\hat{f}(\lambda,\mu)\|_{\mathrm{HS}}^2\\ &\le C_\lambda\ e^{-2b|\mu|^2}.\end{aligned}$$

Now applying Hardy's theorem on r_λ we conclude that $h_\lambda = 0$ and hence $\hat{f}(\lambda,\mu) = 0$ for all $\mu \in r_\lambda^*$ and $\lambda \in \Lambda$ whenever $a < b$. Therefore $f = 0$ for $a < b$. □

Theorem 4.2. *Let G be a step two stratified group without MW condition. Let f be a function on G such that $f(p_a)^{-1} \in L^p(G)$, $1 \le p \le \infty$ and $\hat{f}(\lambda,\mu)(\hat{p}_b(\lambda,\mu))^{-1} \in \mathcal{S}_q$, $1 \le q \le \infty$ for every $\mu \in r_\lambda^*$ and $\lambda \in \Lambda$. Then $f = 0$ whenever $a < b$.*

Proof. Using the explicit formula for $p_a^\lambda(v)$ it can be proved as in the case of H_d^n that $e^{\frac{1}{4a}|v|^2}f^\lambda \in L^p(\mathfrak{v})$. Let $g(v,t) = e^{-\alpha|t|^2}h(v)$, where $\alpha > 0$ and h is a smooth function with supp $h \subset \{v : |v| < \delta\}$. Choose a' such that $a < a' < b$. Then for all $v \in \mathfrak{v}$ with $|v| > \frac{\delta\sqrt{a'}}{\sqrt{a'}-\sqrt{a}}$ and $v' \in$ supp h we have $|v-v'| \ge |v|-|v'| > |v|-\delta > |v|\sqrt{\frac{a}{a'}}$. Since $g^\lambda \in L^p(\mathfrak{v})$ for all p we get by Hölder's inequality a constant $C > 0$ such that

$$\begin{aligned}C &\ge \int_{\mathfrak{v}} e^{\frac{1}{4a}|v-v'|^2}|f^\lambda(v-v')||g^\lambda(v')|dv'\\ &\ge e^{\frac{1}{4a'}|v|^2}\int_{\mathfrak{v}} |f^\lambda(v-v')|\ |g^\lambda(v')|dv'\end{aligned}$$

for all v with $|v| > \frac{\delta\sqrt{a'}}{\sqrt{a'}-\sqrt{a}}$. From the continuity of the function $(f*g)^\lambda$ it follows that

$$\begin{aligned}|(f*g)^\lambda(v)| &= |f^\lambda \times_\lambda g^\lambda(v)|\\ &\le \int_{\mathfrak{v}} |f^\lambda(v-v')||g^\lambda(v')|dv'\\ &\le Ce^{-\frac{1}{4a'}|v|^2}\end{aligned}$$

for all $v \in \mathfrak{v}$. Hypothesis on $\hat{f}(\lambda,\mu)$ implies that $\hat{f}(\lambda,\mu)e^{bH(\mu,d(\lambda))}$ is a bounded operator. Since

$$H(\mu,d(\lambda))U(d(\lambda))\Phi_\alpha = |\mu|^2 + \sum_{j=1}^{n}(2\alpha_j+1)d_j(\lambda)$$

for all $\alpha \in \mathbb{N}$, it follows that for some constant $C > 0$

$$\|\hat{f}(\lambda,\mu)U(d(\lambda))\Phi_\alpha\|_2 \le Ce^{-b|\mu|^2}\prod_{j=1}^{n} e^{-b(2\alpha_j+1)d_j(\lambda)}$$

for all $\alpha \in \mathbb{N}$, which shows that $\|\hat{f}(\lambda,\mu)\|^2_{\mathrm{HS}} \le Ce^{-2b|\mu|^2}$. Therefore

$$\begin{aligned}\|\widehat{f*g}(\lambda,\mu)\|_{\mathrm{HS}} &\le \|\hat{g}(\lambda,\mu)\|_{\mathrm{op}}\|\hat{f}(\lambda,\mu)\|_{\mathrm{HS}} \\ &\le Ce^{-b|\mu|^2}.\end{aligned}$$

From the proof of the previous theorem we conclude that $f * g = 0$ as $a < a' < b$. Let $g(v,t) = (2\pi)^{-\frac{k}{2}}h(v)e^{-\frac{|t|^2}{2}}$ where h is a compactly supported smooth function and $\int_{\mathfrak{v}} h(v)\, dv = 1$. Let $g_\epsilon(v,t) = \epsilon^{-(2n+k+r)}g(\frac{v}{\epsilon},\frac{t}{\epsilon})$ for $\epsilon > 0$. Then $\{g_\epsilon\}_{\epsilon>0}$ form an approximate identity. Since $f * g_\epsilon(v,t) = 0$ for all $\epsilon > 0$ whenever $a < b$ it follows that $f = 0$ for $a < b$.

□

4.2. **Step two stratified groups with MW- condition.** If G is stratified from the work of Jerison and Sánchez-Calle [12] it is known that $p_a(v,t) \le Ce^{-\frac{A}{a}|(v,t)|^2}$ for some $A > 0$, where $|\cdot|$ denotes a homogeneous norm on G. If G satisfies MW- condition then

$$\hat{p}_a(\lambda) = e^{-aH(d(\lambda))}$$

and

$$p_a^\lambda(x,y) = (4\pi)^{-n}\prod_{j=1}^{n}\left(\frac{d_j(\lambda)}{\sinh ad_j(\lambda)}\right)e^{-\frac{1}{4}d_j(\lambda)\coth ad_j(\lambda)(x_j^2+y_j^2)}.$$

Now we are ready to prove the following versions of Hardy's theorem and Cowling-Price theorem.

Theorem 4.3. *Let G be a step two stratified group with MW- condition and f be a function on G such that $f(p_a)^{-1} \in L^p(G)$, $1 \le p \le \infty$ and $\hat{f}(\lambda)(\hat{p}_b(\lambda))^{-1} \in \mathcal{S}_q$, $1 \le q \le \infty$ for every $\lambda \in \Lambda$. Then $f = 0$ whenever $a < b$.*

Proof. Since any two homogeneous norms are equivalent, using the estimate of p_a in terms of the homogeneous norm obtained from natural dilation, it can be shown that f^λ can be extended as a holomorphic function of λ in a strip of $\mathbb{C}^k$ and also using the explicit expression of p_a^λ we can conclude that $f^\lambda(x,y)e^{\frac{1}{4a}(|x|^2+|y|^2)} \in L^p(\xi_\lambda \bigoplus \eta_\lambda)$. As before from the hypothesis on $\hat{f}(\lambda)$ we get

$$\|\hat{f}(\lambda)U(d(\lambda))\Phi_\alpha\|_2 \le C\left(\prod_{j=1}^{n} e^{-b(2\alpha_j+1)d_j(\lambda)}\right)$$

for all $\alpha \in \mathbb{N}^n$.

Let us compute $\pi_\lambda(\xi',\xi'',0)^*U(d(\lambda))\Phi_0$.

$$\begin{aligned}&\pi_\lambda(\xi',\xi'',0)^*U(d(\lambda))\Phi_0(x)\\ =\;& \sum_\alpha \langle \pi_\lambda(\xi',\xi'',0)^*U(d(\lambda))\Phi_0, U(d(\lambda))\Phi_\alpha\rangle U(d(\lambda))\Phi_\alpha(x)\\ =\;& \sum_\alpha \langle U(d(\lambda))^*\pi_\lambda(\xi',\xi'',0)^*U(d(\lambda))\Phi_0, \Phi_\alpha\rangle U(d(\lambda))\Phi_\alpha(x)\\ =\;& \sum_\alpha \langle W(-\sqrt{d(\lambda)}(\xi'+i\xi''))\Phi_0, \Phi_\alpha\rangle U(d(\lambda))\Phi_\alpha(x)\\ =\;& \sum_\alpha \bar{\Phi}_{\alpha,0}\left(\sqrt{d(\lambda)}(\xi'+i\xi'')\right)U(d(\lambda))\Phi_\alpha(x).\end{aligned}$$

Using the estimate on $\|\hat{f}(\lambda)U(d(\lambda))\Phi_\alpha\|_2$ we will get the estimate

$$|\langle \hat{f}((\lambda),(\xi',\xi''))\phi_\lambda,\phi_\lambda\rangle| \le C \prod_{j=1}^{n} e^{-\frac{1}{4}d_j(\lambda)\left(1-e^{-4b'd_j(\lambda)}\right)(\xi_j'^2+\xi_j''^2)}$$

for $a<b'<b$ as in the case of H_d^n. Proceeding as in the proof of theorem 2.2 we conclude that $f^\lambda=0$ whenever $ad_j(\lambda)<\frac{1}{4}\left(1-e^{-4b'd_j(\lambda)}\right)$ for all $1\le j\le n$. We claim that there exits $\delta>0$ such that above inequality is true for all $\lambda\in\Lambda$ with $0<|\lambda|<\delta$ consequently we can conclude that $f=0$ a.e. As $a<b'<b$ and $d_j(\lambda)\to 0$ we can choose $\delta>0$ such that $ad_j(\lambda)<\frac{1}{4}\left(1-e^{-4b'd_j(\lambda)}\right)$ for all j and for all λ with $0<|\lambda|<\delta$. □

In the above proof we have used the estimate $\|\hat{f}(\lambda)U(d(\lambda))\Phi_\alpha\|_2^2 \le C \prod_{j=1}^{n} e^{-2b(2\alpha_j+1)d_j(\lambda)}$ which is also true if we assume that $\hat{f}(\lambda)^*\hat{f}(\lambda)\le C\ e^{-2bH(d(\lambda))}$. Therefore we have the following Hardy's theorem.

Theorem 4.4. *Let G be a stratified step two group satisfying MW -condition and f be a function on G satisfying $|f(v,t)|\le Cp_a(v,t)$, $\hat{f}(\lambda)^*f(\lambda)\le C\hat{p}_{2b}(\lambda)$ for some $a,b>0$. Then $f=0$ whenever $a<b$.*

5. General step two nilpotent Lie groups

In this section we prove an analogue of Beurling's theorem for all step two groups. Though the heat kernel may not exist for general step two nilpotent Lie group we find an alternative condition on f and prove Hardy's and Cowling-Price theorems. We also prove equality case of a version of Hardy's theorem left open in Astengo et al [1].

5.1. **Step two groups without MW condition.** For $\xi=(\xi',\xi'')\in\xi_\lambda\bigoplus\eta_\lambda$ we define

$$\hat{f}((\lambda,\mu),\xi)=\pi_{\lambda,\mu}(\xi',\xi'',0,0)\hat{f}(\lambda,\mu)\pi_{\lambda,\mu}(\xi',\xi'',0,0)^*$$

and call this the Fourier-Weyl transform of f on G. Then a simple calculation shows that

$$\hat{f}((\lambda,\mu),(\xi',\xi''))=\int_{\eta_\lambda}\int_{\xi_\lambda} e^{i\sum_{j=1}^{n}d_j(\lambda)(x_j\xi_j''-y_j\xi_j')} f^{\lambda,\mu}(x,y)\pi_{\lambda,\mu}(x,y,0,0)\ dx\ dy.$$

With this preparation we have the following version of Beurling's theorem for all step two nilpotent Lie group G without MW condition :

Theorem 5.1. *Let $f\in L^1\bigcap L^2(G)$ such that for all $\lambda\in\Lambda$, and $\mu\in r_\lambda^*$*

$$\int_G\int_{\eta_\lambda}\int_{\xi_\lambda}|f(x,y,z,t)|\ |\langle(\hat{f}(\lambda,\mu),(\xi',\xi''))\phi_\lambda,\phi_\lambda\rangle|\times$$
$$e^{|\sum_{j=1}^{n}d_j(\lambda)(x_j\xi_j''-y_j\xi_j')|}\ d\xi'\ d\xi''\ dx\ dy\ dz\ dt<\infty,$$

where $\phi_\lambda(x)=U(d(\lambda))\Phi_0(x)$. Then $f=0$ a.e.

Proof. Applying Beurling's thoerem on $\mathbb{R}^{2n}$ to the function $g_{\lambda,\mu}(x,y)=f^{\lambda,\mu}(x,y)\langle\pi_{\lambda,\mu}(x,y,0,0)\phi_\lambda,\phi_\lambda\rangle$ as in the case of H_d^n we will get $f^{\lambda,\mu}=0$ for all $\mu\in r_\lambda^*$ and $\lambda\in\Lambda$ and hence $f=0$ a.e. □

Heat kernel may not exit for general step two groups. For $\lambda\in\Lambda$ if $\lambda=r\omega$ for $r\in(0,\infty)$ and $\omega\in S^{k-1}$ write $v=(x(\omega),y(\omega),z(\omega))\in\mathfrak{v}$. For suitable function g on $\mathbb{R}^k$ its Radon transform $R_\omega g$ is a function on $\mathbb{R}$ and is defined by

$$R_\omega g(s)=\int_{\{t:<t,\omega>=s\}}g(t)\ dt,$$

where dt denotes Lebesgue measure on the hyperplane $\{t :< t, \omega >= r\}$. If we use the fact that

$$R_\omega g(r) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-isr} \int_{\mathbb{R}^k} e^{is<\omega,t>} g(t)\ dt\ ds$$

then from the first condition of theorem 4.1 we obtain

$$|R_\omega f((x(\omega), y(\omega), z(\omega)), s)| \leq Ce^{-\frac{1}{4a}|z(\omega)|^2} q_{a,d(\omega)}\left(x(\omega) + iy(\omega), s\right).$$

If we replace the first condition of the theorem 4.1 by above condition then for each $\omega \in S^{k-1}$

$$f^{r\omega}(x(\omega), y(\omega), z(\omega)) = \int_{-\infty}^{\infty} e^{irs} R_\omega f((x(\omega), y(\omega), z(\omega)), s)\ ds$$

extends to a holomorphic function of $r \in \mathbb{C}$ in the strip $|\Im(r)| < \frac{A}{a}$ for some $A > 0$. Also we have the estimate

$$|f^{r\omega}(x(\omega), y(\omega), z(\omega))| \leq Ce^{-\frac{1}{4a}(|x(\omega)^2| + |y(\omega)|^2 + |Z(\omega)|^2)}$$

for all $r \in (0, \infty)$ and $\omega \in S^{k-1}$. Now following the proof of theorem 4.1 we have following theorems for all step two groups.

Theorem 5.2. *Let G be a step two group without MW condition. Let f be a function on G such that $|R_\omega f(x(\omega), y(\omega), z(\omega), r)| \leq Ce^{-\frac{1}{4a}|z(\omega)|^2} q_{a,d(\omega)}\left(x(\omega) + iy(\omega), r\right)$ and $\hat{f}(\lambda, \mu)^* \hat{f}(\lambda, \mu) \leq Ce^{-2bH(\mu,d(\lambda))}$ for every $\mu \in r_\lambda^*$, and $\lambda \in \Lambda$. Then $f = 0$ whenever $a < b$.*

Also we have the following version of Cowling-Price theorem:

Theorem 5.3. *Let G be a step two group without MW condition. Let f be a function on G such that the function g defined by $g(x(\omega), y(\omega), z(\omega), r) = R_\omega f(x(\omega), y(\omega), z(\omega), r) e^{\frac{1}{4a}|z(\omega)|^2} \left(q_{a,d(\omega)}\right)^{-1} (x(\omega) + iy(\omega), r)$ is in $L^p(\mathfrak{v} \times \mathbb{R})$, $1 \leq p \leq \infty$ and $\hat{f}(\lambda, \mu) e^{bH(\mu,d(\lambda))} \in \mathcal{S}_q$, $1 \leq q \leq \infty$ for every $\mu \in r_\lambda^*$, and $\lambda \in \Lambda$. Then $f = 0$ whenever $a < b$.*

Proof. Observe that using the estimate on $R_\omega f(x(\omega), y(\omega), z(\omega), s)$ it can be shown $f^{r\omega}(x(\omega), y(\omega), z(\omega))$ can be extended as a holomorphic function of $r \in \mathbb{C}$ in the strip $|\Im(r)| < \frac{A}{a}$ for all $\omega \in S^{k-1}$ as in the case of H_d^n. Also it is easy to see that for each $r \in (0, \infty)$ and $\omega \in S^{k-1}$ $e^{\frac{1}{4a}|\cdot|^2} f^{r\omega}(\cdot) \in L^p(\mathfrak{v})$ as in the case of H_d^n. Now the rest of the proof will be same as theoem 4.2.

□

For all step two nilpotent Lie groups not satisfying MW-condition the following version of Hardy's theorem has been proved in [1] .(See also Ray [15] and Kaniuth-Kumar [13].)

Theorem 5.4. *Let $f \in L^1(G)$ satisfy the following two conditions :*

(*i*) $\int_{\mathfrak{v}} |f(x, y, z, t)|(1 + |(x, y, z)|)^{-k} dx\ dy\ dz \leq Ce^{-\alpha|t|^2},\ t \in \mathfrak{z}$

(*ii*) $\int_{r_\lambda^*} \|\hat{f}(\lambda, \mu)\|_{op} (1 + |\mu|)^{-k}\ d\mu \leq Cr(\lambda)^l e^{-\beta|\lambda|^2},\ \lambda \in \Lambda$

where $r(\lambda) = \sum_{j=1}^{n} (d_j(\lambda)^2 + d_j(\lambda)^{-2})$ for some integers $l, k \geq 0$ and $\alpha, \beta > 0$. Then $f = 0$ whenever $\alpha\beta > \frac{1}{4}$.

The equality case has been left open in Astengo et al [1] for which we have the following result.

Theorem 5.5. *Let $f \in L^1(G)$ satisfy the following conditions :*

(*i*) $\int_{\mathfrak{b}} |f(x, y, z, t)|(1 + |(x, y, z)|^2)^{-k} dx\ dy\ dz \leq Ce^{-\alpha|t|^2},\ t \in \mathfrak{z}$

(ii) $\int_{r_\lambda^*} \|\hat{f}(\lambda,\mu)\|_{op}(1+|\mu|)^{-k}\, d\mu \leq Cr(\lambda)^l e^{-\beta|\lambda|^2}, \ \lambda \in \Lambda$

where $r(\lambda) = \sum_{j=1}^{n}(d_j(\lambda)^2 + d_j(\lambda)^{-2})$ *for some integers* $l, k \geq 0$ *and* $\alpha, \beta > 0$. *then* $f = 0$ *for* $\alpha\beta > \frac{1}{4}$ *and* $f(x,y,z,t) = f(x,y,z,0)e^{-\alpha|t|^2}$ *for* $\alpha\beta = \frac{1}{4}$.

Proof. Let g be a Schwartz function on $\mathfrak{b}$ and $h(x,y,z,t) = f(x,y,z,t)(1+|(x,y,z)|^2)^{-k}$. Consider the function

$$\begin{aligned} F(x,y,z,t) &= h(\cdot,\cdot,\cdot,t) * \bar{g}(x,y,z) \\ &= \int_{r_\lambda}\int_{\eta_\lambda}\int_{\xi_\lambda} h(u,v,w,t)\bar{g}(x-u,y-v,z-w)\, du\, dv\, dw. \end{aligned}$$

Taking the Fourier transform in the t variable

$$\begin{aligned} F^\lambda(x,y,z) &= \int_{r_\lambda}\int_{\eta_\lambda}\int_{\xi_\lambda} \frac{f^\lambda(u,v,w)}{(1+|(u,v,w)|^2)^k}\bar{g}(x-u,y-v,z-w)\, du\, dv\, dw \\ &= (2\pi)^{-(n+r)}\mathrm{Pf}(\lambda)\int_{r_\lambda^*} \mathrm{tr}\left(\hat{f}(\lambda,\mu)W_{\lambda,\mu}\left(h'_{(x,y,z)}\right)^*\right) d\mu \end{aligned}$$

where $h'_{(x,y,z)}(u,v,w) = g(x-u,y-v,z-w)(1+|(u,v,w)|^2)^{-k}$. Note that $h'_{(x,y,z)}$ is from Schwartz class and hence

$$\begin{aligned} |F^\lambda(x,y,z)| &\leq (2\pi)^{-(n+r)}|\mathrm{Pf}(\lambda)|\int_{r_\lambda^*} \left|\mathrm{tr}\left(\hat{f}(\lambda,\mu)W_{\lambda,\mu}(h'_{(x,y,z)})^*\right)\right| d\mu \\ &\leq C\ |\mathrm{Pf}(\lambda)|\int_{r_\lambda^*} \frac{\|\hat{f}(\lambda,\mu)\|_{\mathrm{op}}}{(1+|\mu|)^k}(1+|\mu|)^k\ \mathrm{tr}\left(|W_{\lambda,\mu}(h'_{(x,y,z)})|\right)\ d\mu \\ &\leq C(x,y,z)\ r(\lambda)^{k'} e^{-\beta|\lambda|^2} \end{aligned}$$

for some positive integer k' using the second hypothesis of the theorem and estimate on $(1+|\mu|)^k\mathrm{tr}\left(\left|W_{\lambda,\mu}h'_{(x,y,z)}\right|\right)$ as mentioned in subsection 3.1. Let $D = \{\omega \in S^{k-1} : d_j(\omega) \neq 0 \text{ for all } j\}$. Note that we have $d_j(\lambda) > 0$ for all j whenever $\lambda \in \Lambda$. Also $d_j(r\omega) = rd_j(\omega)$. If ν denotes the surface measure on S^{k-1} then $\nu(S^{k-1}\setminus D) = 0$ as Λ is a set of full measure on $\mathbb{R}^k$. For each $\omega \in D$ consider the Radon transform $R_\omega F(x,y,z,s)$of the function $F(x,y,z,t)$ in the last variable t. So we can conclude that

$|\widehat{R_\omega F}(x,y,z,s)| \leq C(x,y,z)C'(\omega)(s^{2m}+s^{-2m})e^{-\beta s^2}$ for sufficiently large positive integer m . Also we have $|R_\omega F(x,y,z,s)| \leq Ce^{-\alpha s^2}$ as $|F(x,y,z,t)| \leq Ce^{-\alpha|t|^2}$ using the first hypothesis of the theorem. Now applying Hardy's theorem to the function $R_\omega F(x,y,z,\cdot)$ we conclude that for all $\omega \in D$, $F^{s\omega}(x,y,z) = C(x,y,z)C'(\omega)e^{-\beta s^2}$ whenever $\alpha\beta = \frac{1}{4}$ and $F^{s\omega}(x,y,z) = 0$ for $\alpha\beta > \frac{1}{4}$. Since F^λ is a continuous function of λ we conclude that $F^\lambda(x,y,z) = C(x,y,z)e^{-\beta|\lambda|^2}$ for $\alpha\beta = \frac{1}{4}$ and $F^\lambda(x,y,z) = 0$ whenever $\alpha\beta > \frac{1}{4}$ for all $\lambda \in \mathfrak{z}^*$. Finally we get $F(x,y,z,t) = h(\cdot,\cdot,\cdot,t) * \bar{g}(x,y,z) = C(x,y,z)e^{-\alpha|t|^2}$ for $\alpha\beta = \frac{1}{4}$ and $F(x,y,z,t) = h(\cdot,\cdot,\cdot,t) * \bar{g}(x,y,z) = 0$ for $\alpha\beta > \frac{1}{4}$. Choosing g from approximate identity $\{g_m\}_m$ where each g_m is coming from Schwartz class on $\mathfrak{v}$ we conclude that $h(\cdot,\cdot,\cdot,t) * \bar{g}_m(x,y,z) = 0$ for $\alpha\beta > \frac{1}{4}$ and $h(\cdot,\cdot,\cdot,t) * \bar{g}_m(x,y,z) = C_m(x,y,z)e^{-\alpha|t|^2}$ for $\alpha\beta = \frac{1}{4}$. But $h(\cdot,\cdot,\cdot,t) * \bar{g}_m(x,y,z)$ converges to $h(x,y,z,t)$ as $m \to 0$. Hence $C_m(x,y,z) \to C(x,y,z)$ as $m \to 0$. Therefore $f(x,y,z,t) = C(x,y,z)(1+|(x,y,z)|^2)^k e^{-\alpha|t|^2} = f(x,y,z,0)e^{-\alpha|t|^2}$ for $\alpha\beta = \frac{1}{4}$ and $f(x,y,z,t) = 0$ whenever $\alpha\beta > \frac{1}{4}$. □

5.2. **Step two groups with MW- condition.** For $(\xi', \xi'') \in \xi_\lambda \bigoplus \eta_\lambda$ we define

$$\hat{f}(\lambda, (\xi', \xi'')) = \pi_\lambda(\xi', \xi'', 0)\hat{f}(\lambda)\pi_\lambda(\xi', \xi'', 0)^*$$

and call the Fourier-Weyl transform of f on G.

In the course of proof of the theorem 5.1 we have proved following Beurling's theorem for all step two groups with MW- condition.

Theorem 5.6. *Let G be a step two group satisfying MW- condition, let $f \in L^1 \bigcap L^2(G)$ be such that for every $\lambda \in \Lambda$*

$$\int_G \int_{\mathbb{R}^{2n}} |f(x,y,t)||(\hat{f}(\lambda,(\xi',\xi''))\phi_\lambda, \phi_\lambda)|e^{|\sum_{j=1}^n d_j(\lambda)(x_j\xi_j'' - y_j\xi')|} d\xi' \, d\xi'' dx \, dy \, dt < \infty$$

where $\phi_\lambda = U(d(\lambda))\Phi_0$. Then $f = 0$ a.e.

Proof. Consider the function $g_\lambda(x,y) = f^\lambda(x,y)\langle \pi_\lambda(x,y,0)\phi_\lambda, \phi_\lambda\rangle$ and proceed as in the case of H_d^n. □

For the case of step two groups with MW- condition theorem 5.2 and theorem 5.3 will be as follows:

Theorem 5.7. *Let G be a step two group with MW- condition. Let f be a function on G such that $|R_\omega f(x(\omega), y(\omega), r)| \le C q_{a,d(\omega)}(x(\omega) + iy(\omega), r)$ and $\hat{f}(\lambda)^*\hat{f}(\lambda) \le Ce^{-2bH(d(\lambda))}$ for every $\lambda \in \Lambda$. Then $f = 0$ whenever $a < b$.*

Proof. As before it can be shown that $f^{r\omega}(x(\omega), y(\omega))$ can be extended as a holomorphic function of r in the strip $|\Im(r)| < \frac{A}{a}$ for some $A > 0$, for each $\omega \in S^{k-1}$ and $|R_\omega f(x(\omega), y(\omega), r)| \le C \; q_{a,d(\omega)}(x(\omega) + iy(\omega), r)$. The rest will follow from the proof of theorem 4.4. □

Theorem 5.8. *Let G be a step two group with MW- condition. Let f be a function on G such that the function g defined by $g(x(\omega), y(\omega), r) = R_\omega f(x(\omega), y(\omega), r) \left(q_{a,d(\omega)}\right)^{-1}(x(\omega) + iy(\omega), r)$ is in $L^p(\mathfrak{v} \times \mathbb{R})$, $1 \le p \le \infty$ and $\hat{f}(\lambda)e^{bH(d(\lambda))} \in \mathcal{S}_q$, $1 \le q \le \infty$ for every $\lambda \in \Lambda$. Then $f = 0$ whenever $a < b$.*

Proof. As above it can be shown that $f^{r\omega}(x(\omega), y(\omega))$ can be extended as a holomorphic function of r in the strip $|\Im(r)| < \frac{A}{a}$ for some $A > 0$ and $e^{\frac{1}{4a}|\cdot|^2} f^{r\omega}(\cdot) \in L^p(\mathfrak{v})$ for all $r \in (0, \infty)$, which is true for all $\omega \in S^{k-1}$. Then the rest of the proof will follow from the proof of theorem 4.3. □

We state below the theorem 5.5 for the case of step two groups with MW-condition.

Theorem 5.9. *Let G be as in theorem 5.6 and $f \in L^1(G)$ satisfy the following conditions :*

(i) $\int_{\mathfrak{b}} |f(x,y,t)|(1 + |(x,y)|^2)^{-k} dx \, dy \le Ce^{-\alpha|t|^2}, \; t \in \mathfrak{z}$

(ii) $\|\hat{f}(\lambda)\|_{op} \le Cr(\lambda)^l e^{-\beta|\lambda|^2}, \; \lambda \in \Lambda$

where $r(\lambda) = \sum_{j=1}^n (d_j(\lambda)^2 + d_j(\lambda)^{-2})$ for some integers $l, k \ge 0$ and $\alpha, \beta > 0$. then $f = 0$ for $\alpha\beta > \frac{1}{4}$ and $f(x,y,t) = f(x,y,0)e^{-\alpha|t|^2}$ for $\alpha\beta = \frac{1}{4}$.

Proof. Let g be a Schwartz function on $\xi_\lambda \bigoplus \eta_\lambda$ and $h(x,y,t) = f(x,y,t)(1 + |(x,y)|^2)^{-k}$. Consider the function

$$\begin{aligned} F(x,y,t) &= h(\cdot,\cdot,t) * \bar{g}(x,y) \\ &= \int_{\eta_\lambda}\int_{\xi_\lambda} h(u,v,t)\bar{g}(x-u, y-v, t) dx \, dy. \end{aligned}$$

Taking the Fourier transform in the t variable

$$\begin{aligned} F^\lambda(x,y) &= \int_{\eta_\lambda}\int_{\xi_\lambda} \frac{f^\lambda(u,v)}{(1+|(u,v)|^2)^k}\bar{g}(x-u,y-v)du\ dv \\ &= (2\pi)^{-n}\ \mathrm{Pf}\ (\lambda)\mathrm{tr}\left(\hat{f}(\lambda)W'_\lambda(h'_{(x,y)})^*\right), \end{aligned}$$

where $h'_{x,y}(u,v) = g(x-u,y-v)(1+|(u,v)|^2)^{-k}$. Now arguing as in theorem 5.5 we have the desired result. □

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