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Boundary stabilization of a hybrid Euler–Bernoulli beam

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Abstract. We consider a problem of boundary stabilization of small flexural vibrations of a flexible structure modeled by an Euler–Bernoulli beam which is held by a rigid hub at one end and totally free at the other. The hub dynamics leads to a hybrid system of equations. By incorporating a condition of small rate of change of the deflection with respect to x as well as t, over the length of the beam, for appropriate initial conditions, uniform exponential decay of energy is established when a viscous boundary damping is present at the hub end.

Keywords. Boundary stabilization; Euler–Bernoulli beam equation; hybrid system; small deflection; exponential energy decay.

1. Introduction and mathematical formulation

We study the boundary stabilization of an Euler–Bernoulli beam of length l with a rigid hub of mass m^{H} capable of lateral motion at one end and the other end totally free, as in the case of a solar cell array, shown in figure 1 ([4, 5]). The objective here, is to study uniform stability of the overall system under suitable stabilizing force $Q(t)$ at the rigid-hub end only. Such a system is equivalent to a flexible space structure hoisted from a rigid hub. For small vibrations of the system, let $y^{\mathrm{H}}(t)$ be the transverse displacement of the hub and $y^{\mathrm{B}}(x,t)$ that of the beam at the position x relative to the hub at time t, then the total transverse deflection $y(x,t) = y^{\mathrm{H}}(t) + y^{\mathrm{B}}(x,t)$ satisfies the Euler–Bernoulli beam equation

$$my_{tt}(x,t) + EIy_{xxxx}(x,t) = 0, \quad 0 \leq x \leq l, \quad t \geq 0, \tag{1}$$

under the assumptions $|y(x,t)| \ll l$ and $|y_x(x,t)| \ll 1$. The constants EI and m are the flexural rigidity and mass per unit length of the beam respectively, and subscripts in y denote partial derivatives with respect to the corresponding variables.

The equation of motion of the hub on which the stabilizing force $Q(t)$ is assumed to act, yields the differential equation ([4, 5])

$$m^{\mathrm{H}}y^{\mathrm{H}}_{tt}(t) + EIy^{\mathrm{B}}_{xxx}(0,t) + Q(t) = 0.$$

The exact controlability of a similar problem has been investigated recently in Gorain and Bose [6]. To study boundary stabilization, we assume that $Q(t)$ is proportional to $y_t(0,t)$ say, $Q(t) = by_t(0,t)$ i.e., a viscous boundary damping (stabilizer) is present at the hub end, the constant $b > 0$ being the viscous damping parameter. Also $y(0,t) = y^{\mathrm{H}}(t)$ and

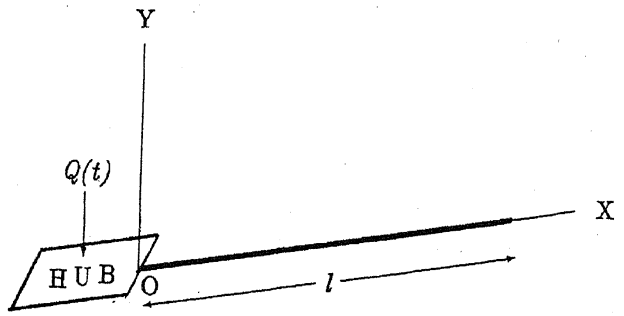

Figure 1. Schematic of the rigid hub and the beam.

$y_x(x,t) = y_x^{\mathrm{B}}(x,t)$, hence the above yields the hybrid boundary condition

$$y_{xxx}(0,t) + \alpha y_{tt}(0,t) + \lambda b y_t(0,t) = 0, \quad t \geq 0, \tag{2}$$

where $\alpha = m^{\mathrm{H}}/EI$ and $\lambda = 1/EI$. Assuming at $x = 0$, the beam is built-in position with the hub, we have

$$y_x(0,t) = 0, \quad t \geq 0. \tag{3}$$

At the free end of the beam

$$y_{xx}(l,t) = 0 \quad \text{and} \quad y_{xxx}(l,t) = 0, \quad t \geq 0, \tag{4}$$

and initially the beam is set to vibrations with

$$y(x,0) = y^0(x) \quad \text{and} \quad y_t(x,0) = y^1(x), \quad 0 \leq x \leq l. \tag{5}$$

The boundary stabilization for Euler–Bernoulli beam equation has been studied by Chen and Zhou [1], Chen *et al* [2], Littman and Markus [8], Morgül [9] and Rao [10]. All their investigations have shown the controlability and stabilization of Euler–Bernoulli beam equation, clamped at one end and feedback damping or control force (viscous damping) on the other end. Littman and Markus [8], and Chen and Zhou [1] in particular, have shown by calculating the eigenvalues of certain hybrid system that uniform stabilization is not possible because of the inclusion of infinitely large wave number k, during the passage of a wave along the length of the beam. Rao [10] concludes the same by semigroup theory.

The difficulty in proving uniform stability, appears to stem from not imposing any restriction that the beam remains approximately straight during vibration ([3, 11]). Motivated by this consideration, the rate of change in both x and t from the equilibrium position of the displacement $y(x,t)$ remains small, that is to say, $|y_{xt}(x,t)|$ remains small. The implication is that the time rate of variation of small slope remains small and also the gradient of the velocity along the length of the beam remains small. Therefore considering the totality along the length of the beam, we impose the restriction that $\int_0^l y_{xt}^2 \mathrm{d}x$ remains small. If we compare this quantity with a similar one, $\int_0^l y_{xx}^2 \mathrm{d}x$ which is actually $2/EI$ times the potential energy of bending of the beam and is thus finite, then accordingly the restriction on vibrations satisfying (1), is assumed to be governed by

$$\int_0^l y_{xt}^2 \mathrm{d}x \leq \frac{EI}{ml^2} \int_0^l y_{xx}^2 \mathrm{d}x \quad t > t_0, \tag{6}$$

for appropriate $y^0(x)$ and $y^1(x)$. Here EI/ml^2 is a dimensionality constant. For our purpose we have assumed (6) to hold for time $t > t_0$, where t_0 is finite but may be as large as we please.

In practice, it is important to translate the condition (6) in terms of initial data $\{y^0, y^1\}$. As far as our knowledge goes, this remains an open problem.

2. Energy of the system

Associated with each solution of (1)–(5), the total energy at time t is defined by the functional

$$E(t) = \frac{1}{2}\int_0^l (my_t^2 + EIy_{xx}^2)\mathrm{d}x + \frac{1}{2}m^{\mathrm{H}}y_t^2(0,t). \tag{7}$$

Now differentiating (7) with respect to t and replacing my_{tt} by $-EIy_{xxxx}$, we obtain

$$\dot{E}(t) = EI\int_0^l \frac{\partial}{\partial x}(y_{tx}y_{xx} - y_t y_{xxx})\mathrm{d}x + m^{\mathrm{H}}y_t(0,t)y_{tt}(0,t),$$

where dot represents the time derivative. Applying the boundary conditions (2)–(4), we get

$$\begin{aligned}\dot{E}(t) &= -EI\{\alpha y_{tt}(0,t) + \lambda b y_t(0,t)\}y_t(0,t) + m^{\mathrm{H}}y_t(0,t)y_{tt}(0,t)\\ &= -by_t^2(0,t) \leq 0,\end{aligned} \tag{8}$$

for all $t \geq 0$, since $\alpha = m^{\mathrm{H}}/EI$, $\lambda = 1/EI$. This implies

$$E(t) \leq E(0) \quad \text{for all } t \geq 0. \tag{9}$$

Hence the energy $E(t)$ is non-increasing with time and the system (1)–(5) is energy dissipating due to boundary damping at the hub end.

As the energy decays, our main interest is to obtain explicitly the uniform exponential energy decay estimate for the solution of (1)–(5), that is to establish the result of the form

$$E(t) \leq M\mathrm{e}^{-\mu t}E(0), \quad t \geq 0 \tag{10}$$

for some reals $\mu > 0$ and $M \geq 1$.

3. Uniform stability result

Theorem 1. *Let $y(x,t)$ be a solution of the system* (1)–(5) *corresponding to the initial conditions $\{y^0, y^1\}$ for which* (6) *holds and $E(0) < \infty$. Then $E(t)$ satisfies the relation* (10) *for some reals $\mu > 0$ and $M \geq 1$.*

Proof. Proceeding as in Komornik [7], when $0 \leq t \leq t_0$, where t_0 (may be large enough) is a finite number such that (6) holds, we have

$$\mathrm{e}^{1-t/t_0} \geq 1.$$

Evidently, we can write from (9) that

$$E(t) \leq E(0) \leq \mathrm{e}^{1-t/t_0}E(0) = M_1\mathrm{e}^{-\mu_1 t}E(0) \quad \text{for } 0 \leq t \leq t_0, \tag{11}$$

where $M_1 = e$ and $\mu_1 = 1/t_0$.

For the case $t > t_0$, the proof is as in the following: Let $\epsilon > 0$ be a fixed small constant. We define the scalar-valued function V as

$$V(t) = E(t) + \epsilon\rho(t) \tag{12}$$

for all $t \geq t_0$, where

$$\rho(t) = 2m \int_0^l x y_t y_x \mathrm{d}x. \tag{13}$$

Since $y_x(0,t) = 0$, by Wirtinger's inequality [12], we have

$$\int_0^l y_x^2 \mathrm{d}x \leq \frac{4l^2}{\pi^2} \int_0^l y_{xx}^2 \mathrm{d}x, \tag{14}$$

and also it can be easily established that

$$y_t^2(l,t) \leq 2\left(y_t^2(0,t) + l \int_0^l y_{xt}^2 \mathrm{d}x \right). \tag{15}$$

Now from (13) we can estimate $\rho(t)$ as

$$\begin{aligned} |\rho(t)| &\leq \frac{4l^2}{\pi} \sqrt{\frac{m}{EI}} \int_0^l |\sqrt{m} y_t| \left| \frac{\pi}{2l} \sqrt{EI} y_x \right| \mathrm{d}x \\ &\leq \frac{2l^2}{\pi} \sqrt{\frac{m}{EI}} \int_0^l \left(m y_t^2 + \frac{\pi^2}{4l^2} EI y_x^2 \right) \mathrm{d}x \leq \mu_0 E(t) \end{aligned} \tag{16}$$

by (14) and the energy equation (7), where

$$\mu_0 = \frac{4l^2}{\pi} \sqrt{\frac{m}{EI}}. \tag{17}$$

Thus from (12), we find

$$(1 - \epsilon\mu_0) E(t) \leq V(t) \leq (1 + \epsilon\mu_0) E(t). \tag{18}$$

Now differentiating (13) with respect to t, integrating by parts and applying the system of equations (1)–(4), it becomes

$$\begin{aligned} \dot{\rho}(t) &= \int_0^l x \frac{\partial}{\partial x} (m y_t^2 + EI y_{xx}^2) \mathrm{d}x + 2EI \int_0^l y_x y_{xxx} \mathrm{d}x \\ &= m^{\mathrm{H}} y_t^2(0,t) + ml y_t^2(l,t) - 2EI \int_0^l y_{xx}^2 \mathrm{d}x - 2E(t). \end{aligned} \tag{19}$$

Inserting the inequality (15) into (19), we obtain

$$\dot{\rho}(t) \leq (m^{\mathrm{H}} + 2ml) y_t^2(0,t) + 2\left(ml^2 \int_0^l y_{xt}^2 \mathrm{d}x - EI \int_0^l y_{xx}^2 \mathrm{d}x \right) - 2E(t),$$

and by the use of (6), we ultimately have

$$\dot{\rho}(t) \leq (m^{\mathrm{H}} + 2ml) y_t^2(0,t) - 2E(t). \tag{20}$$

Again differentiating (12) with respect to t, and inserting (8) and (20), we obtain the differential inequality

$$\dot{V}(t) \leq -2\epsilon E(t) - (b - \epsilon(m^{\mathrm{H}} + 2ml)) y_t^2(0,t). \tag{21}$$

If we choose $\epsilon \leq \epsilon_0$, where

$$\epsilon_0 = \min\{ b/(m^{\mathrm{H}} + 2ml),\ 1/2\mu_0 \}, \tag{22}$$

then from (21), it follows that for all $t > t_0$,

$$\dot{V}(t) + 2\epsilon E(t) \leq 0, \tag{23}$$

and at the same time we have from (18)

$$\mu_0 \epsilon E(t) \leq V(t) \leq (1 + \epsilon\mu_0)E(t). \tag{24}$$

With the help of (24), (23) yields

$$\dot{V}(t) + \mu_2 V(t) \leq 0, \tag{25}$$

where

$$\mu_2 = \frac{2\epsilon}{1 + \epsilon\mu_0} > 0. \tag{26}$$

Now multiplying (25) by $e^{\mu_2 t}$ and integrating over t_0 to t, we obtain

$$V(t) \leq e^{-\mu_2(t-t_0)} V(t_0). \tag{27}$$

Then finally, inserting (24), it follows from (27) that for $t > t_0$,

$$E(t) \leq \frac{1 + \epsilon\mu_0}{\mu_0 \epsilon} e^{-\mu_2(t-t_0)} E(t_0) \leq M_2 e^{-\mu_2 t} E(0), \tag{28}$$

in virtue of (9), where

$$M_2 = \frac{1 + \epsilon\mu_0}{\mu_0 \epsilon} e^{\mu_2 t_0}.$$

From (11) and (28), we conclude the result (10) for some reals $M = \max\{M_1, M_2\}$ and $\mu = \min\{\mu_1, \mu_2\}$.

Remark. It follows from (26) that exponential energy decay rate μ after passage of the time t_0 will be maximum for largest admissible value of ϵ, i.e., for $\epsilon = \epsilon_0$. Choosing ϵ_0 equal to $b/(m^{\mathrm{H}} + 2ml)$ or $1/2\mu_0$ according to (22), the maximum decay rate μ will be equal to either $2b(m^{\mathrm{H}} + 2ml + b\mu_0)^{-1}$ or $2/3\mu_0$, and since as in (17), μ_0 is proportional to l^2, the maximum energy decay rate μ decreases quadratically with increasing l after the elapse of the time t_0. Hence it appears, that the decay of the solution of the system will be slower for a longer beam, which is very significant to our problem as one end of the beam is totally free.

4. Conclusions

Here we have established uniform boundary stabilization of small flexural vibrations of a flexible Euler–Bernoulli beam attached to a movable rigid hub at one end and free at the other, and obtained a uniform exponential energy decay rate for the solution of this hybrid system by taking into account a natural restriction for small vibrations [11] of the beam. The motivation of considering this type of hybrid system arises from many practical systems which consists of two parts: coupled elastic part and rigid part, constituting the hybrid system such as solar cell array, space craft with flexible attachments, robot with flexible links and parts of many mechanical system. For these systems the situation generally occurs when it is very difficult or undesirable to apply the boundary control at the free end of the elastic part where as, it is easier to apply it on the rigid part to obtain a good performance of the overall system. For initial conditions $y^0(x)$ and $y^1(x)$, when the energy and the motion decay with time following (8) and the beam approaches its straight position, we have assumed (6) to hold at the stages of vibration after elapse of some time t_0, however large. Our discussion here, has significantly covered the cases of uniform stability of such type of small vibration problem from mathematical point of view.

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