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Degenerations of the moduli spaces of vector bundles on curves I

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Abstract. Let Y be a smooth projective curve degenerating to a reducible curve X with two components meeting transversally at one point. We show that the moduli space of vector bundles of rank two and odd determinant on Y degenerates to a moduli space on X which has nice properties, in particular, it has normal crossings. We also show that a nice degeneration exists when we fix the determinant. We give some conjectures concerning the degeneration of moduli space of vector bundles on Y with fixed determinant and arbitrary rank.

Keywords. Nodal curve; torsion free sheaf; moduli.

1. Introduction

Let Y be a smooth projective curve specializing to a projective curve X with nodes as the only singularities. One knows that the theory of moduli spaces of vector bundles on a smooth projective curve extends in a nice manner to the case of X, using torsion free sheaves; further these moduli spaces have good specialization properties, namely the moduli space $M(Y)$ on Y (fixing certain invariants) specializes to a corresponding space $M(X)$ on X (see [11, 8, 7, 13]). Now $M(X)$ is closely related to a corresponding moduli space $M(\tilde{X})$ on the normalization $\tilde{X}$ of X, though working out this relationship is a non-trivial exercise. These facts serve as a tool for an indepth study of the moduli space $M(Y)$, based on induction on the genus of Y, for the genus of any irreducible component of $\tilde{X}$ is strictly less than that of Y. For example, the work of Narasimhan and Ramadas concerning the Verlinde formulae is motivated by these considerations (see [7]).

The work of Gieseker proving a conjecture of Newstead and Ramanan on the vanishing of certain Chern classes of the moduli space $M(Y)$ of rank two and odd degree, is also motivated by similar considerations (see [4]). However, Gieseker does *not* use torsion free sheaves. He takes X to be *irreducible* with only *one node* and gives a specialization $M'(X)$ of $M(Y)$, where $M'(X)$ is a scheme intrinsically attached to X with its singularities as normal crossings. Further the total space which gives this specialization is regular, if the total space which gives the specialization of Y to X is regular. Gieseker obtains $M'(X)$ by using only *vector bundles* on some curves semi-stably equivalent to X.

The aim of this work is to show that if we take X to be *reducible with only one node p*, the moduli spaces using torsion free sheaves give all the good properties given by Gieseker. To be more precise, a suitable moduli space $M(X)$ of rank two torsion free sheaves on X has only two smooth projective irreducible components M_1, M_2 which intersect transversally on a (smooth) irreducible variety N. One can describe M_1, M_2 by a certain moduli problem of vector bundles of rank two on the normalization $\tilde{X}$ of X, which is the disjoint union of the irreducible components X_1, X_2 of X. The variety N turns out to be a product of certain parabolic moduli spaces on X_1, X_2 respectively. Further, the moduli space $M(Y)$ of vector bundles on Y of rank two and odd degree specializes to $M(X)$ such that the total space which gives this specialization is regular if the total space which gives the specialization of Y to X is regular (see § 4). The crucial point which leads to the above nice properties is that if $M(X)$ is chosen as the moduli space of semi-stable, rank two, torsion free sheaves for a *generic polarization* on X, then if $V \in M(X)$, V *cannot* be of the form $(m_{X,p} \oplus m_{X,p})$ locally at the node p of X ($m_{X,p}$ = maximal ideal of the stalk $O_{X,p}$ of the structure sheaf of X at p), i.e. either (a) V is locally free at p, or (b) V is of the form $m_{X,p} \oplus \mathcal{O}_{X,p}$ locally at p. By deformation arguments as in [3], it is not difficult to see that the functor defining $M(X)$ at V of the form as in (b), is *formally smooth* over the local ring $\mathcal{O}_{X,p}$, which leads to the required normal crossing.

That torsion free sheaves of the form $m_{X,p} \oplus m_{X,p}$ locally at p, may not appear in a well-chosen $M(X)$, could be seen intuitively as follows. The moduli problem corresponding to $M(X)$ can be translated as one on vector bundles on the normalization $\tilde{X}$ of X. A vector bundle V of rank two on $\tilde{X}$ gives rise to a pair $V = (V_1, V_2)$, where V_i is a rank two vector bundle on X_i, $i = 1, 2$. Then the moduli problem on $\tilde{X}$ is the moduli problem of vector bundles $V = (V_1, V_2)$ together with a homomorphism A of the fibre of V_1 into that of V_2 at p, i.e. the moduli problem of *triples* (V_1, V_2, A) (see § 2). An element of $\mathbb{G}_m \times \mathbb{G}_m$ gives a canonical automorphism of V (scalar multiplication on each component). Thus a triple (V_1, V_2, A) is isomorphic to $(V_1, V_2, \lambda A)$, λ a non-zero scalar. Thus the moduli problem of triples (V_1, V_2, A), $A \neq 0$, is equivalent to the moduli problem of triples (V_1, V_2, A') where A' is an element of the *projective space* associated to the vector space of homomorphisms of this fibre of V_1 into that of V_2 at p. Now this defines a functor which is *proper* over the moduli functor of vector bundles on $\tilde{X}$. Hence one could expect a *proper scheme* to solve this moduli problem by using GIT techniques. Now the triples of the form $(V_1, V_2, 0)$, correspond precisely to torsion free sheaves, which are of the form $m_{X,p} \oplus m_{X,p}$, locally at p, and the foregoing argument shows that we could expect to avoid them.

The notion of a triple is an improvement on the considerations in [11] trying to relate the moduli problem of torsion free sheaves on X (with a fixed Hilbert polynomial) to one on the normalization $\tilde{X}$ of X. The crucial point is that this can be translated as a moduli problem of vector bundles on X with a *fixed Hilbert polynomial* (this does not figure in [11] and is inspired by the GPB's of Usha Bhosle [2]).

It seems likely that the degeneration $M(X)$ of $M(Y)$ that we give, could also be used to prove the above mentioned conjecture of Newstead and Ramanan on the vanishing of certain Chern classes of $M(Y)$ (moduli space of rank two and odd degree) as well as the Verlinde formulae (for the rank two case). To do this, one has to generalize our moduli space $M(X)$ by taking parabolic structures at a finite number of smooth points ‹n X (as is done in the work of Narasimhan and Ramadas, cited above) to have correct induction machinery.

Consider now the general case of a smooth projective curve Y specializing to X with nly nodes. Let $\mathscr{L}$ be a line bundle on Y of degree d specializing to a torsion free sheaf , of rank one and degree d on X. Let $M(Y, n, d)$ denote the moduli of semi-stable vector undles V of rank n and degree d and $M(Y, n, \mathscr{L})$ denote the subvariety of $M(Y, n, d)$ uch that $\det V = \overset{n}{\wedge} V = \mathscr{L}$. Then the question how $M(Y, n, \mathscr{L})$ specializes when (specializes to X, has not been properly investigated. In particular, one could ask whether $M(Y,n,\mathscr{L})$ specializes to an intrinsically defined subscheme of a moduli space $M(X,n,d)$ to which $M(Y,n,d)$ specializes, which we could then denote by $M(X,n,L)$. The difficulty in this problem is that, in general, there does not seem to be a well-defined morphism det: $M(X,n,d)\to M(X,1,d)$.

An interesting corollary of our method is that in our case (X reducible with one node) we get a well-defined morphism det: $M(X,2,d)\to M(X,1,d)$, which shows that $M(Y,2,\mathscr{L})$ specializes to a nice subscheme $M(X,2,L)$ of $M(X,2,d)$ (see § 7).

Take now X to be irreducible with only one node. Then isomorphism classes of torsion free sheaves of rank one with a fixed degree d, is a compactification $M(X,1,d)$ of the generalized Jacobian $J^d(X)$ of isomorphism classes of line bundles on X of degree d. Let $M(X,n,d)$ be the moduli space of semi-stable torsion-free sheaves of rank n and degree d and $M(X,n,d)^0$ the open subscheme consisting of vector bundles. Then we have a morphism

$$\det\colon M(X,n,d)^0\to J^d(X),$$
$$V\mapsto \overset{n}{\Lambda}V.$$

This does not seem to extend to a morphism of $M(X,n,d)$ into $M(X,1,d)$. However, we give a precise conjecture which gives the specialization of $M(Y,n,\mathscr{L})$ as an intrinsically defined subscheme $M(X,n,L)$ of $M(X,n,d)$. In case L is a line bundle, $M(X,n,L)$ is just the closure of the fibre $(\det)^{-1}(L)$ in $M(X,n,d)$.

After we submitted the manuscript to the Journal, Balaji pointed out the paper of Huashi Xia [14]. Some of the results in our paper are also contained in [14] but the methods are different.

2. Triples associated to a torsion free sheaf on a reducible nodal curve

Let X be the nodal curve which is a union of two smooth curves X_1 of genus g_1 and X_2 of genus g_2 meeting at one point. Let p be the node of X.

Locally at the node p of X we have the following: If $m_{X,p}$ is the maximal ideal of the local ring $\mathcal{O}_{X,p}$ then there exists $x_1,x_2\in m_{X,p}$ such that $X_1=\{x_2=0\}$ and $X_2=\{x_1=0\}$.

PROPOSITION 2.1

Let $\mathscr{F}$ be a coherent $\mathcal{O}_X$-module. Then $\mathscr{F}$ has depth 1 (i.e. depth 1 at every $x\in X$) if and only if $\mathscr{F}$ is pure of dimension 1 i.e. for all $\mathcal{O}_X$-submodules $\mathscr{G}\subsetneq\mathscr{F}$, $\mathscr{G}\neq 0$, dimension of Supp $(\mathscr{G})=1$.

Proof. Since depth is a local property and proposition is clear at the smooth points of X, we need to prove the proposition at the node p of X. Hence we assume $\mathscr{F}$ is a finitely generated $\mathcal{O}_{X,p}$-module. Suppose $\mathscr{F}$ is not pure. Then there exists a subsheaf $T\subsetneq\mathscr{F}$, $(T\neq 0)$ such that T is supported at p. This implies $m_{X,p}^r T=0$. Take r minimal; so that there exists a $t\in T$, $t\neq 0$ such that $a\cdot t\neq 0$ for some $a\in m_{X,p}^{r-1}$. Set $b=a\cdot t\in T$. Then $x\cdot b=0$ for every $x\in m_{X,p}$. This implies that the depth of $\mathscr{F}$ is zero, which contradicts the assumption that the depth of $\mathscr{F}$ is one. This proves the only if part of the proposition. To prove the if part of the proposition, assume that $\mathscr{F}$ is not of depth one. i.e. for all

$a \in m_{X,p}$, $a \neq 0$, $m_a: \mathscr{F} \to \mathscr{F}$ (multiplication by a) is not injective. Then consider

$$\mathrm{Ann}(a) = \{f \in \mathscr{F} \mid a \cdot f = 0\},$$

where a is such that its restriction to every X_i is not zero (i.e. a is a part of an $m_{X,p}$ sequence). Then the dimension of the support of $\mathrm{Ann}(a)$ is zero and $\mathrm{Ann}(a) \subsetneq \mathscr{F}$, and $\mathrm{Ann}(a) \neq 0$. This shows that $\mathscr{F}$ is not pure.

DEFINITION 2.1

A depth 1 sheaf $\mathscr{F}$ on X is called a torsion free sheaf. We say that $\mathscr{F}$ is of rank (r_1, r_2) on X if $\mathscr{F}$ restricted to X_i is of rank r_i, $i = 1, 2$. We say $\mathscr{F}$ is of rank r if $r_1 = r_2 = r$.

Note. One of the r_i (not both) could be zero for a torison free sheaf.

Notations. Let $\mathscr{F}$ be a torsion free sheaf on X. Let $\bar{\bar{\mathscr{F}}}_1$ be the restriction of $\mathscr{F}$ to X_1 and $\bar{\bar{\mathscr{F}}}_2$ be the restriction of $\mathscr{F}$ to X_2. Note that locally at p we have $\bar{\bar{\mathscr{F}}} = \mathscr{F}/x_2.\mathscr{F}$ and $\bar{\bar{\mathscr{F}}}_2 = \mathscr{F}/x_1.\mathscr{F}$. Let $\mathscr{F}_1 = \bar{\bar{\mathscr{F}}}_1/(\text{torsion})$ and $\mathscr{F}_2' = \bar{\bar{\mathscr{F}}}_2/(\text{torsion})$.

PROPOSITION 2.2

Let $\mathscr{F}$ be a torsion free sheaf of rank (r, s) on X. The canonical homomorphisms

$$\mathscr{F} \to \bar{\bar{\mathscr{F}}}_1 \oplus \bar{\bar{\mathscr{F}}}_2$$

and

$$\mathscr{F} \to \mathscr{F}_1 \oplus \mathscr{F}'_2$$

are injective. Besides if we set

$$0 \to \mathscr{F} \to \bar{\bar{\mathscr{F}}}_1 \oplus \bar{\bar{\mathscr{F}}}_2 \to \bar{T} \to 0,$$

$$0 \to \mathscr{F} \to \mathscr{F}_1 \oplus \mathscr{F}'_2 \to T \to 0,$$

the supports of $\bar{T}$ and T are at p and in fact

$$m_{X,p}\, T = m_{X,p}\, \bar{T} = 0.$$

Proof. The injectivity assertions are immediate, for the canonical homomorphisms

$$\mathscr{F} \to \bar{\bar{\mathscr{F}}}_1 \oplus \bar{\bar{\mathscr{F}}}_2$$

and

$$\mathscr{F} \to \mathscr{F}_1 \oplus \mathscr{F}'_2$$

are isomorphisms outside p. Hence the kernels of these homomorphisms are supported at p and $\mathscr{F}$ being pure, the kernels reduce to zero.

To prove the remaining assertions, note that they are really local at the mode p of X and hence we assume that $\mathscr{F}$ is a finitely generated $\mathcal{O}_{X,p}$-module. Note that we have the following exact sequence of $\mathcal{O}_{X,p}$-modules

$$0 \to x_2\mathscr{F} \to \mathscr{F} \to \bar{\bar{\mathscr{F}}}_1 \to 0,$$

$$0 \to x_1\mathscr{F} \to \mathscr{F} \to \bar{\bar{\mathscr{F}}}_2 \to 0.$$

Consider the canonical homomorphism $x_2\mathscr{F} \to \mathscr{F}'_2$ given by composite

$$x_2\mathscr{F} \subsetneq \mathscr{F} \to \bar{\bar{\mathscr{F}}}_2 \to \mathscr{F}'_2.$$

This is an isomorphism outside p and since $x_2\mathscr{F}$ (being a subsheaf of $\mathscr{F}$) is pure, it follows that $x_2\mathscr{F}\to\mathscr{F}'_2$ is injective. Besides, we see that $x_2\mathscr{F}$ maps onto $x_2\mathscr{F}'_2$ and hence $x_2\mathscr{F}$ identifies with $x_2\mathscr{F}'_2$ (this is immediate, for $\mathscr{F}$ maps onto $\bar{\bar{\mathscr{F}}}_2$ and hence maps onto $\mathscr{F}'_2$). Similarly $x_1\mathscr{F}\to\mathscr{F}_1$ is injective and $x_1\mathscr{F}$ identifies with $x_1\mathscr{F}_1$. By the same argument we see that the canonical homomorphisms

$$x_1\mathscr{F}\to\bar{\bar{\mathscr{F}}}_1, x_2\mathscr{F}\to\bar{\bar{\mathscr{F}}}_2$$

are injective and hence we get the following natural isomorphisms

$$x_1\mathscr{F}\simeq x_1\bar{\bar{\mathscr{F}}}_1, x_2\mathscr{F}\simeq x_2\bar{\bar{\mathscr{F}}}_2.$$

(In particular it follows that x_2 annihilates the torsion of $\bar{\bar{\mathscr{F}}}_2$ and since x_1 annihilates $\bar{\bar{\mathscr{F}}}_2$, it follows that $m_{X,p}$ annihilates the torsion of $\bar{\bar{\mathscr{F}}}_2$.)

Now to show that, say

$$m_{X,p}T=0,$$

we have only to show that

$$x_i(\mathscr{F}_1\oplus\mathscr{F}'_2)\subset\mathscr{F}\ (i=1,2).$$

Take say x_2, then $x_2\bar{\bar{\mathscr{F}}}_1=0$ by definition and hence $x_2\mathscr{F}_1=0$. Besides we have seen $x_2\mathscr{F}'_2\simeq x_2\mathscr{F}$ thus

$$x_2(\mathscr{F}_1\oplus\mathscr{F}'_2)\subset\mathscr{F}.$$

Similarly

$$x_1(\mathscr{F}_1\oplus\mathscr{F}'_2)\subset\mathscr{F}.$$

(The injectivity of say

$$\mathscr{F}\to\bar{\bar{\mathscr{F}}}_1\oplus\bar{\bar{\mathscr{F}}}_2$$

also follows in an explicit way, wince the kernel of this map is $x_2\mathscr{F}\cap x_1\mathscr{F}$, which is annihilated by $m_{X,p}$, and since $\mathscr{F}$ is pure, this kernel is zero.) This proves the proposition.

Notations. Let $\mathscr{F}(p), \mathscr{F}_1(p), \mathscr{F}'_2(p)$ etc. denote the fibres at p i.e., $\mathscr{F}(p)=\mathscr{F}/m_{X,p}\mathscr{F}$ etc.

Remark 2.1. Set

$$\mathbf{N}=\operatorname{Ker}(\mathscr{F}_1(p)\oplus\mathscr{F}'_2(p)\to T).$$

Then the canonical projections

$$\mathbf{N}\to\mathscr{F}_1(p) \text{ and } \mathbf{N}\to\mathscr{F}'_2(p)$$

are surjective.

Proof. Since $\mathscr{F}\to\mathscr{F}_1$ and $\mathscr{F}\to\mathscr{F}'_2$ are surjective, we see that the canonical homomorphisms $\mathscr{F}(p)\to\mathscr{F}_1(p)$ and $\mathscr{F}(p)\to\mathscr{F}'_2(p)$ are surjective.

Note that the sequence

$$\mathscr{F}(p)\to\mathscr{F}_1(p)\oplus\mathscr{F}'_2(p)\to T\to 0$$

is exact and hence $\mathbf{N}$ = image of the canonical homomorphism $(\mathscr{F}(p)\to\mathscr{F}_1(p)\oplus\mathscr{F}'_2(p))$.

Therefore the surjections

$$\mathscr{F}(p)\to\mathscr{F}_1(p) \text{ and } \mathscr{F}(p)\to\mathscr{F}'_2(p)$$

pass through $\mathbf{N}$ and hence

$$\mathbf{N}\to\mathscr{F}_1(p) \text{ and } \mathbf{N}\to\mathscr{F}'_2(p)$$

are surjective.

Remark 2.2. From the above remark it follows that $\mathscr{F}$ can be identified with the subsheaf of $\mathscr{F}_1\oplus\mathscr{F}'_2$ consisting of all $f=(f_1,f_2)$ such that the evaluation of f at p is in $\mathbf{N}$. More generally if $\mathscr{G}_i$ is a vector bundle on X_i for $i=1,2$ and $\mathbf{M}$ is a linear subspace of $\mathscr{G}_1(p)\oplus\mathscr{G}_2(p)$ then the subsheaf

$$\mathscr{G}=\{g=(g_1,g_2)\,|\,g_i\in\mathscr{G}_i, i=1,2 \text{ and } g(p)=(g_1(p),g_2(p))\in\mathbf{M}\}$$

of $\mathscr{G}_1\oplus\mathscr{G}_2$ is a torsion free $\mathscr{O}_X$-submodule.

Remark 2.3. Let $K=\mathrm{Ker}\{\text{canonical homomorphism } \mathbf{N}\to\mathscr{F}_1(p)\}$. Then we see $K\subset\mathscr{F}'_2(p)$. Let $i:\mathscr{F}'_2\to\mathscr{F}_2$ be the Hecke modification with $\mathrm{Ker}(i_p)=K$ (see below for the definition of a Hecke modification.) Set $\mathbf{N}^1=\text{Image }(\mathbf{N})$ under the canonical homomorphism

$$\mathscr{F}_1(p)\oplus\mathscr{F}'_2(p)\xrightarrow{\Theta}\mathscr{F}_1(p)\oplus\mathscr{F}_2(p).$$

Then we see that

a) the canonical homomorphism of $\mathbf{N}^1$ onto $\mathscr{F}_1(p)$ is surjective and since $\dim(\mathbf{N}^1)=\dim(\mathscr{F}_1(p))$, the map $\mathbf{N}^1\to\mathscr{F}_1(p)$ is in fact an isomorphism.

b) $\mathbf{N}=\Theta^{-1}(\mathbf{N}^1)$.

From the following commutative diagram of $\mathscr{O}_X$ modules

$$\begin{array}{ccc}\mathscr{F} & \subsetneq & \mathscr{F}_1\oplus\mathscr{F}'_2\\ \| & & \downarrow\\ \mathscr{F} & \subsetneq & \mathscr{F}_1\oplus\mathscr{F}_2\end{array}$$

we see that

$$\mathscr{F}=\{f=(f_1,f_2)\in\mathscr{F}_1\oplus\mathscr{F}_2\,|\text{ evaluation of } f \text{ at } p \text{ is in } \mathbf{N}^1\}.$$

Now giving $\mathbf{N}^1$ is equivalent to defining a homomorphism

$$A:\mathscr{F}_1(p)\to\mathscr{F}_2(p)$$

and hence we can also define $\mathscr{F}$ as

$$\mathscr{F}=\{f=(f_1,f_2)\in\mathscr{F}_1\oplus\mathscr{F}_2\,|\,A(f_1(p))=f_2(p)\}.$$

Remark 2.4. Let V be a vector bundle on a smooth curve Y and K be a subspace of V_p (p be a point of Y). Then there are two canonical constructions called Hecke modifications defined as follows:

I) $W\xrightarrow{\phi}V$, $\mathrm{Im}(W_p)=K$, where W is a vector bundle and ϕ is a homomorphism of vector bundles, which is an isomorphism outside p.

(II) $V \xrightarrow{\phi} W$, $\mathrm{Ker}(\phi_p) = K$, where W is a vector bundle and ϕ is a homomorphism of vector bundles, which is an isomorphism outside p.

W and ϕ in (I) are defined as follows:

Let $T = V_p/K$ and $j: V \to T$ be the canonical $\mathscr{O}_X$-module homomorphism.

Then $W = \mathrm{Ker}(j)$ and ϕ be the natural homomorphism.

To define W and ϕ of (II) we proceed as follows:

Let $V_p \times V_p^* \to k(= k(p))$ be the canonical dual pairing and $K^\perp$ denote the orthogonal of K under the dual pairing. Let V^* be the dual of V. Define W^* and ϕ^* as in (I) such that

$$W^* \xrightarrow{\phi^*} V^* \quad \mathrm{Im}(W_p^*) = K^\perp.$$

Let ϕ be the dual of ϕ^* and W be the dual of W^*. Then $W \xrightarrow{\phi} V$ is a homomorphism of vector bundles and ϕ_p is the dual of ϕ_p^*. Also

$$\mathrm{Ker}(\phi) = (\mathrm{Im}(\phi^*))^\perp = (K^\perp)^\perp = K.$$

Thus we get the W and ϕ of (II) with the required properties.

Note. a) If $K = 0$ in the above remark, then in both the constructions we have $W = V$.
b) If $K = V_p$ in the above remark, then in case (I) we get $W = V \otimes \mathscr{O}_Y(-p)$ and in case (II) we get $W = V \otimes \mathscr{O}_Y(p)$.

DEFINITION 2.2

Let $\mathscr{F}_1$ (resp. $\mathscr{F}_2$) be a locally free sheaf on X_1 (resp. on X_2). Let $A: \mathscr{F}_1(p) \to \mathscr{F}_2(p)$ be a linear map. Then we call $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ a triple, where $\vec{A}$ denotes that A is a homomorphism from $\mathscr{F}_1(p)$ to $\mathscr{F}_2(p)$.

Remark 2.5. As we have seen above every torsion free sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ gives rise to a triple.

Note that we could have similarly defined a canonical Hecke modification $\mathscr{F}_1 \to \mathscr{F}'_1$ and a homomorphism $B: \mathscr{F}'_2(p) \to \mathscr{F}'_1(p)$ (here $\mathscr{F}_1, \mathscr{F}'_2$ are as in the notations after the proof of the Proposition 2.1) so that

$$\mathscr{F} = \{ f = (f_1, f_2) \in \mathscr{F}'_1 \oplus \mathscr{F}'_2 \mid B(f_2(p)) = f_1(p) \}.$$

We write this triple as $(\mathscr{F}'_1, \mathscr{F}'_2, \overleftarrow{B})$. Thus to $\mathscr{F}$ we can associate canonically two triples.

We denote the triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ or $(\mathscr{F}'_1, \mathscr{F}'_2, \overleftarrow{B})$ by $\mathscr{F}$ and often by abuse of notation $\mathscr{F}$ also denotes the $\mathscr{O}_X$-module defined by the triples.

Note that given a triple $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, the other triple associated to the $\mathscr{O}_X$-module $\mathscr{F}$ can be obtained directly as follows: Consider the diagram

$$\begin{array}{c} \mathscr{F}_1(p) \xrightarrow{i_p} \mathscr{F}'_1(p) \\ \downarrow A \\ \mathscr{F}_1(p) \xleftarrow{j_p} \mathscr{F}'_2(p) \end{array}$$

where $\mathscr{F}_1 \xrightarrow{i} \mathscr{F}'_1$ (resp. $\mathscr{F}_2 \xleftarrow{j} \mathscr{F}'_2$) is the canonical Hecke modification such that $\mathrm{Ker}(i_p) = \mathrm{Ker}(A)$ (resp. $\mathrm{Im}(j_p) = \mathrm{Im}(A)$). Then there is a canonically defined homomorphism

$$B: \mathscr{F}'_2(p) \to \mathscr{F}'_1(p).$$

To see this, for $x \in \mathscr{F}'_2(p)$ let $f \in \mathscr{F}_1(p)$ be such that $A(f) = j_p(x)$. Now if we set $B(x) = i_p(f)$ then it is clearly well defined.

Remark 2.6. The above construction of the triple associated to a torsion free $\mathcal{O}_X$-module $\mathscr{F}$ gives the local structure of $\mathscr{F}$ at p, namely locally at p

$$\mathscr{F}_p \simeq \mathcal{O}^a_{X,p} \oplus \mathcal{O}^b_{X_1,p} \oplus \mathcal{O}^c_{X_2,p}$$

(see [11], Huitiême Partie; Proposition 3).) In particular, if $\mathscr{F}$ is a torsion free sheaf of rank r then

$$\mathscr{F}_p \simeq \mathcal{O}^a_{X,p} \oplus m^b_{X,p}, \quad a + b = r.$$

Proof. Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be the triple associated to $\mathscr{F}$. We can choose a basis of the free $\mathcal{O}_{X_1,p}$-module $\mathscr{F}_{1,p}$ (resp. a basis of the free $\mathcal{O}_{X_2,p}$-module $\mathscr{F}_{2,p}$) $e_1, \ldots, e_{r_1}$ (resp. $f_1, \ldots, f_{r_2}$) such that $A(e_i) = f_i$, $(1 \leqslant i \leqslant s)$ and $A(e_i) = 0$, $(s \leqslant i \leqslant r_1)$. Then by setting $a = s, b = r_1 - s$ and $c = r_2 - s$ we get the desired result.

Remark 2.7. Let $\mathscr{F}$ be a torsion free $\mathcal{O}_X$-module. Let $\bar{\mathscr{F}}_1, \bar{\mathscr{F}}_2, \mathscr{F}_1, \mathscr{F}'_2$ be as in the notation before Proposition 2.2 and $K \subset \mathscr{F}'_2(p)$ be as in Remark 2.3. Let $T_{1,p}$ = torsion of $\bar{\mathscr{F}}_1$. Then $T_{1,p}$ maps isomorphically onto the subspace K under the canonical map $T_{1,p} \to \mathscr{F}'_2(p)$.

Proof. To prove this note that (see Remark 2.6)

$$\mathscr{F}_p \simeq \mathcal{O}^a_{X,p} \oplus \mathcal{O}^b_{X_1,p} \oplus \mathcal{O}^c_{X_2,p} \text{ and } \bar{\mathscr{F}}_{1,p} \simeq \mathcal{O}^{a+b}_{X_1,p} \oplus (\mathcal{O}_{X,p}/m_{X,p})^c.$$

Now if $\mathbf{N}$ is as in Remark 2.1, we clearly see that $\mathbf{N}$ maps onto $\mathscr{F}_1(p)$ with kernel K which is isomorphic to $T_{1,p}$ as required.

Remark 2.8. Given a directed arrow, say $\to$, the association to a torsion free $\mathcal{O}_X$-module $\mathscr{F}$ the canonical triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, is functorial (note that triples form a category in an obvious way).

If $\mathscr{F} \xrightarrow{\phi} \mathscr{G}$ is given, and $\mathscr{F}$ and $\mathscr{G}$ are represented by triples $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ and $(\mathscr{G}_1, \mathscr{G}_2, \vec{B})$ respectively, it then give rise to in a canonical way an $\mathcal{O}_{X_1}$-module (resp. $\mathcal{O}_{X_2}$-module) map $\phi_1 : \mathscr{F}_1 \to \mathscr{G}_1$ (resp. $\phi_2 : \mathscr{F}_2 \to \mathscr{G}_2$), and a commutative diagram

$$\begin{array}{ccc} \mathscr{F}_1(p) & \xrightarrow{\phi_1 \otimes k(p)} & \mathscr{G}_1(p) \\ \downarrow A & & \downarrow B \\ \mathscr{F}_2(p) & \xrightarrow{\phi_2 \otimes k(p)} & \mathscr{G}_2(p) \end{array}.$$

Proof. This follows because in the construction of the triple from a torsion free $\mathcal{O}_X$-module $\mathscr{F}$, the constructions $\mathscr{F}_i, i = 1, 2$ and A are all functorial.

Lemma 2.3. *Let $\vec{C}$ be the category of triples $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$. Then the functor*

$$\mathscr{F} \mapsto (\mathscr{F}_1, \mathscr{F}_2, \vec{A})$$

from the category of torsion free $\mathcal{O}_X$-modules to $\vec{C}$ is an equivalence.

Proof. The inverse of

$$\mathscr{F} \mapsto (\mathscr{F}_1, \mathscr{F}_2, \vec{A})$$

is given by

$$(\mathscr{F}_1, \mathscr{F}_2, \vec{A}) \mapsto \mathscr{F} = \{f = (f_1, f_2) \in \mathscr{F}_1 \oplus \mathscr{F}_2 \mid A(f_1(p)) = f_2(p)\}.$$

Now the lemma follows from the Remark 2.8 above.

Remark 2.9. If $\overleftarrow{C}$ is the category of triples $(\mathscr{F}_1, \mathscr{F}_2, \overleftarrow{A})$ then again the functor

$$\mathscr{F} \mapsto (\mathscr{F}_1, \mathscr{F}_2, \overleftarrow{A})$$

is an equivalence of categories.

2.1 *Subtriple*

DEFINITION 2.3

Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be a triple as before. A triple $(\mathscr{G}_1, \mathscr{G}_2, \vec{B})$ is said to be a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ if $\mathscr{G}_1 \overset{i_1}{\subsetneq} \mathscr{F}_1$ (resp. $\mathscr{G}_2 \overset{i_2}{\subsetneq} \mathscr{F}_2$) is an inclusion of $\mathscr{O}_{X_1}$-modules (resp. $\mathscr{O}_{X_2}$-modules) and the following diagram

$$\begin{array}{ccc} \mathscr{G}_1(p) & \xrightarrow{i_1 \otimes k(p)} & \mathscr{F}_1(p) \\ \downarrow B & & \downarrow A \\ \mathscr{G}_2(p) & \xrightarrow{i_2 \otimes k(p)} & \mathscr{F}_2(p) \end{array}$$

is commutative.

Note. If $\mathscr{G} \subsetneq \mathscr{F}$ is an $\mathscr{O}_X$- submodule of $\mathscr{F}$, then there is a $\mathscr{G}'$, $\mathscr{G} \subsetneq \mathscr{G}' \subsetneq \mathscr{F}$ such that $\mathscr{F}/\mathscr{G}'$ is torsion free and $\mathscr{G}'/\mathscr{G}$ is torsion. One has only to take the inverse image in $\mathscr{F}$ of the torsion subsheaf of $\mathscr{F}/\mathscr{G}$. Note if $\mathscr{G} \subsetneq \mathscr{F}$ is such that $\mathscr{F}/\mathscr{G}$ is torsion free, then if $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ (resp. $(\mathscr{G}_1, \mathscr{G}_2, \vec{B})$) is the triple associated to $\mathscr{F}$ (resp. $\mathscr{G}$), then for $\mathscr{G}_i \subsetneq \mathscr{F}_i$, it need not be true that $\mathscr{F}_i/\mathscr{G}_i$ is torsion free. For example, let $\mathscr{F}$ be a vector bundle on X and let $\mathscr{F}'(= x_1 . \mathscr{F})$ be the subsheaf of $\mathscr{F}$ vanishing on X_2. Then $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ where $\mathscr{F}_i = \mathscr{F}|_{X_i}$ and $\mathscr{F}' = (\mathscr{G}, 0, \vec{0})$, where $\mathscr{G} = \mathscr{F}'$ considered as a sheaf (vector bundle) on X_1. Now $\mathscr{F}/\mathscr{F}'$ is torsion free; however for $\mathscr{G} \subsetneq \mathscr{F}_1$, $\mathscr{F}_1/\mathscr{G}$ is not torsion free.

Remark 2.10. If $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is a triple and $\mathscr{F}$ is the corresponding depth one $\mathscr{O}_X$-module, then is a 1–1 correspondence between subtriples of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ and $\mathscr{O}_X$-submodules of $\mathscr{F}$.

Proof. This follows from Lemma 2.3.

2.2 *Euler characterstic of a triple*

DEFINITION 2.4

Given a triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ we define its Euler characteristic by

$$\chi_X((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2) - \mathrm{rk}(\mathscr{F}_2),$$

where $\chi_{X_i}(\mathscr{F}_i)$ is the usual Euler characteristics of the sheaf of $\mathscr{O}_{X_i}$- module $\mathscr{F}_i (i=1,2)$ and $\mathrm{rk}(\mathscr{F}_2)$ is the rank of the $\mathscr{O}_{X_2}$- module $\mathscr{F}_2$. Similarly, given a triple $(\mathscr{F}_1, \mathscr{F}_2, \overleftarrow{A})$, we define its Euler characteristic by

$$\chi_X((\mathscr{F}_1, \mathscr{F}_2, \overleftarrow{A})) = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2) - \mathrm{rk}(\mathscr{F}_1).$$

Remark 2.11. If $\mathscr{F}$ is the $\mathscr{O}_X$-module associated to a triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ (resp. $(\mathscr{F}'_1, \mathscr{F}'_2, \overleftarrow{B})$) then

$$\chi_X((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi_X(\mathscr{F}) = \chi_X((\mathscr{F}'_1, \mathscr{F}'_2, \overleftarrow{B})).$$

Proof. Given a triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ we get an exact sequence

$$0 \to \mathscr{F} \to \mathscr{F}_1 \oplus \mathscr{F}_2 \to T_A \to 0$$

where $\mathscr{F}$ is the $\mathscr{O}_X$ -module associated to the triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ (see Lemma 2.3) and T_A is a torsion sheaf supported at the node p of X. In fact T_A is a vector space over the residue field $k(p)$ at p and its dimension is equal to the rank of $\mathscr{F}_2$. Thus we see that

$$\chi_X((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi_X(\mathscr{F}).$$

Similarly,

$$\chi_X(\mathscr{F}'_1, \mathscr{F}'_2, \overleftarrow{B})) = \chi_X(\mathscr{F}).$$

Remark 2.12. If $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ and $(\mathscr{G}_1, \mathscr{G}_2, \vec{B})$ are two triples such that $\chi_{X_1}(\mathscr{F}_1) = \chi_{X_1}(\mathscr{G}_1), \chi_{X_2}(\mathscr{F}_2) = \chi_{X_2}(\mathscr{G}_2)$ and rank $(\mathscr{F}_2) =$ rank $(\mathscr{G}_2)$, then

$$\chi_X((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi_X((\mathscr{G}_1, \mathscr{G}_2, \vec{B})).$$

A similar result holds for triples of type $(\mathscr{F}_1, \mathscr{F}_2, \overleftarrow{A})$.

2.3 *Stability of triples*

Fix a polarization (a_1, a_2) on X with $a_i > 0$ rational and $a_1 + a_2 = 1$, i.e. we take an ample line bundle L on X such that if $L_i = L|_{X_i}$, then

$$\frac{\deg(L_1)}{\deg(L_2)} = \frac{a_1}{a_2}.$$

DEFINITION 2.5

Let$(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be a triple. Then we define

$$\mu((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \mu_{(X,a_1,a_2)}((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) := \frac{\chi_X((\mathscr{F}_1, \mathscr{F}_2, \vec{A}))}{a_1 \mathrm{rk}(\mathscr{F}_1) + a_2 \mathrm{rk}(\mathscr{F}_2)}.$$

DEFINITION 2.6

Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be a triple. We say that the triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is stable (resp. semi-stable) if for every proper subtriple $(\mathscr{G}_1, \mathscr{G}_2, \vec{B})$

$$\mu((\mathscr{G}_1, \mathscr{G}_2, \vec{B})) < \mu((\mathscr{F}_1, \mathscr{F}_2, \vec{A}))$$

(resp. $\leqslant$).

Note. In the definition of semi-stability (resp. stability) $\mathscr{G}_i$ are only subsheaves of $\mathscr{F}_i$ and we cannot suppose that (unlike the case of a smooth projective curve) it suffices to take $\mathscr{G}_i$ as subbundles of $\mathscr{F}_i$ (see the note before Remark (2.10)).

Let g be the genus of X and $g_i(i=1,2)$ be the genus of $X_i(i=1,2)$. Note that $g = g_1 + g_2$. Let d be an integer and let $S((a_1, a_2), (2,2), \chi = d + 2(1-g))$ denote the set of all isomorphism classes of semi-stable $\mathscr{O}_X$-modules which are torsion free of type (2, 2) and Euler characteristic $\chi = d + 2(1-g)$ (here type (2, 2) means the $\mathscr{O}_X$ -module when restricted to each component has rank 2) (see [11], ch. (VII.II)).

Remark 2.13. Let $\mathscr{F}$ be an $\mathscr{O}_X$ -module such that

$$[\mathscr{F}] \in S((a_1, a_2), (2,2), \chi = d + 2(1-g)).$$

Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be the triple which corresponds to $\mathscr{F}$ (see Lemma 2.3). Then $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is semi-stable. Conversely, if $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is a semi-stable triple such that $\mathrm{rk}\mathscr{F}_1 = 2 = \mathrm{rk}\mathscr{F}_2$ and Euler characteristic $d + 2(1-g)$, then if $\mathscr{F}$ is the corresponding torsion free sheaf, we have $[\mathscr{F}] \in S((a_1, a_2), (2,2), \chi = d + 2(1-g))$. Moreover, $\mathscr{F}$ is stable iff the corresponding triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is stable.

Proof. This follows from Remark 2.10, Remark 2.11 and Lemma 2.3.

3. Euler characteristic of the components of a semi-stable triple

Notations are as in the previous section. Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be a triple. Recall that by definition

$$\chi_X((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2) - \mathrm{rk}(\mathscr{F}_2).$$

Now if

$$[\mathscr{F}] \in S((a_1, a_2), (2,2), \chi = d + 2(1-g))$$

and $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ the triple which corresponds to $\mathscr{F}$, then

$$\chi := \chi_X((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi_X(\mathscr{F}) = d + 2(1-g))$$

is fixed. We show below that if

$$[\mathscr{F}] \in S((a_1, a_2), (2,2), d + 2(1-g))$$

and $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is the triple which corresponds to $\mathscr{F}$, then the following inequalities hold; $a_1 \cdot \chi \leqslant \chi_{X_1}(\mathscr{F}_1) \leqslant a_1 \cdot \chi + 2$ and $a_2 \cdot \chi \leqslant \chi_{X_2}(\mathscr{F}_2) \leqslant a_2 \cdot \chi + 2$. This has some interesting consequences.

Let

$$[\mathscr{F}] \in S((a_1, a_2), (2,2), d + 2(1-g))$$

and let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be the triple which corresponds to $\mathscr{F}$.

Case 1. $\vec{A}$ is invertible.

Since the canonical map $(\mathscr{F}_1 \otimes \mathscr{O}_{X_1}(-p))(p) \to \mathscr{F}_1(p)$ is zero, $(\mathscr{F}_1 \otimes \mathscr{O}_{X_1}(-p), 0, \vec{0})$ is a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$. Hence by the definition of semi-stability of the triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ (see Remark 2.13), we get

$$\frac{\chi_{X_1}(\mathscr{F}_1) - 2}{2a_1} \leqslant \frac{\chi}{2}.$$

Thus

$$\chi_{X_1}(\mathscr{F}_1) \leqslant a_1\chi + 2. \tag{1}$$

Similarly, since $(0, \mathscr{F}_2, \vec{0})$ is a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, we get

$$\frac{\chi_{X_2}(\mathscr{F}_2) - 2}{2a_2} \leqslant \frac{\chi}{2}.$$

Thus

$$\chi_{X_2}(\mathscr{F}_2) \leqslant a_2\chi + 2. \tag{2}$$

On the other hand

$$\chi = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2) - 2.$$

Using this in (2) we get

$$\chi - \chi_{X_1}(\mathscr{F}_1) \leqslant a_2\chi.$$

Since $a_1 + a_2 = 1$, the last equation together with (1) gives

$$a_1\chi \leqslant \chi_{X_1}(\mathscr{F}_1) \leqslant a_1\chi + 2.$$

Similarly we see that

$$a_2\chi \leqslant \chi_{X_2}(\mathscr{F}_2) \leqslant a_2\chi + 2.$$

Case 2. $\mathrm{rk}(\vec{A}) = 1$.

Let $\mathscr{F}'_1 \to \mathscr{F}_1$ be the Hecke modification such that $\mathrm{Im}(\mathscr{F}'_1(p)) = \mathrm{Ker}\,(A)$ (see Remark 2.4). Then clearly $(\mathscr{F}'_1, 0, \vec{0})$ is a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$. Again by the semi-stability of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, we get

$$\frac{\chi_{X_1}(\mathscr{F}'_1)}{2a_1} \leqslant \frac{\chi}{2}.$$

But $\chi_{X_1}(\mathscr{F}'_1) = \chi_{X_1}(\mathscr{F}_1) - 1$, hence

$$\chi_{X_1}(\mathscr{F}_1) \leqslant a_1\chi + 1. \tag{3}$$

Again, since $(0, \mathscr{F}_2, \vec{0})$ is a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, we get

$$\frac{\chi_{X_2}(\mathscr{F}_2) - 2}{2a_2} \leqslant \frac{\chi}{2}.$$

Thus

$$\chi_{X_2}(\mathscr{F}_2) \leqslant a_2\chi + 2. \tag{4}$$

Since

$$\chi = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2) - 2,$$

by (4) we get

$$\chi - \chi_{X_1}(\mathscr{F}_1) \leqslant a_2\chi.$$

Also, since $a_1 + a_2 = 1$, the last equation together with (3) gives

$$a_1\chi \leqslant \chi_{X_1}(\mathscr{F}_1) \leqslant a_1\chi + 1.$$

Similarly

$$a_2\chi + 1 \leqslant \chi_{X_2}(\mathscr{F}_2) \leqslant a_2\chi + 2.$$

Case 3. $\mathrm{rk}(\vec{A}) = 0$.

In this case note that $A = 0$. Then clearly $(\mathscr{F}_1, 0, \vec{0})$ is a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{0})$. Again by the semi-stability of $(\mathscr{F}_1, \mathscr{F}_2, \vec{0})$ we get

$$\frac{\chi_{X_1}(\mathscr{F}_1)}{2a_1} \leqslant \frac{\chi}{2}.$$

Hence

$$\chi_{X_1}(\mathscr{F}_1) \leqslant a_1\chi. \tag{5}$$

Again, since $(0, \mathscr{F}_2, \vec{0})$ is a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{0})$, we get

$$\frac{\chi_{X_2}(\mathscr{F}_2) - 2}{2a_2} \leqslant \frac{\chi}{2}.$$

Thus

$$\chi_{X_2}(\mathscr{F}_2) \leqslant a_2\chi + 2. \tag{6}$$

Now, as above, from (5) and (6) we get

$$\chi_{X_1}(\mathscr{F}_1) = a_1\chi.$$

and

$$\chi_{X_2}(\mathscr{F}_2) = a_2\chi + 2.$$

Thus we have proved the following:

Theorem 3.1. (a) *Let $\chi \neq 0$ and let (a_1, a_2) be a polarization (i.e. $a_i > 0$ rational and $a_1 + a_2 = 1$). Assume that $a_1\chi$ is not an integer. Let (d_1, d_2) be the unique tuple satisfying the inequalities*

$$a_1 \cdot \chi < d_1 < a_1 \cdot \chi + 1, \quad a_2 \cdot \chi + 1 < d_2 < a_2 \cdot \chi + 2$$

and $d_1 + d_2 = \chi + 2$. If $[\mathscr{F}] \in S((a_1, a_2), (2, 2), \chi = d + 2(1 - g))$ and $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is the triple which corresponds to $\mathscr{F}$, then $\mathrm{rank}(A) \geqslant 1$ and we must have either $\chi(\mathscr{F}_1) = d_1, \chi(\mathscr{F}_2) = d_2$ or $\chi(\mathscr{F}_1) = d_1 + 1, \chi(\mathscr{F}_2) = d_2 - 1$.

Moreover, if rank $(A) = 1$, then we must have $\chi(\mathscr{F}_1) = d_1$ and $\chi(\mathscr{F}_2) = d_2$. (b) *Let $\chi = 0$ and let (a_1, a_2) be any polarization (i.e. $a_i > 0$ rational and $a_1 + a_2 = 1$). If*

$$[\mathscr{F}] \in S((a_1, a_2), (2, 2), \chi = 0)$$

and $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is the triple which corresponds to $\mathscr{F}$, then

(i) $\chi(\mathscr{F}_1) = 0$ *and* $\chi(\mathscr{F}_2) = 2$, *or*

(ii) $\chi(\mathscr{F}_1) = 1$ *and* $\chi(\mathscr{F}_2) = 1$, *or*

(iii) $\chi(\mathscr{F}_1) = 2$ *and* $\chi(\mathscr{F}_2) = 0$.

Moreover, if $\mathrm{rk}(A) = 1$ *(resp.* $\mathrm{rk}(A) = 0$*), then either* $\chi(\mathscr{F}_1) = 0$ *and* $\chi(\mathscr{F}_2) = 2$ *or* $\chi(\mathscr{F}_1) = 1$ *and* $\chi(\mathscr{F}_2) = 1$ *(resp.* $\chi(\mathscr{F}_1) = 2$ *and* $\chi(\mathscr{F}_2) = 0$*).*

COROLLARY 3.1

With the notations of Theorem 3.1 (*a*), *if*

$$[\mathscr{F}] \in S((a_1, a_2), (2; 2), \chi = d + 2(1-g))$$

then either $\mathscr{F}$ *is locally free or a torsion free* $\mathcal{O}_X$ *-module with* $\mathscr{F}_p \simeq \mathcal{O}_{X,p} \oplus m_{X,p}$.

Remark 3.1. Let $\mathscr{G}_s (s \in S)$ be a family of (a_1, a_2) semi-stable torsion free sheaves on X of rank two and Euler characteristic $\chi \neq 0$ (with $a_1\chi$ is not an integer), parametrized by an irreducible variety S. Then by Theorem 3.1 we can choose a direction of the arrow such that $\chi_{X_1}(\mathscr{G}_{s,1})$ (resp. $\chi_{X_2}(\mathscr{G}_{s,2})$) is independent of $s \in S$, where $\mathscr{G}_{s,1}$ (resp. $\mathscr{G}_{s,2}$) is the bundle on X_1 (resp. X_2) in the triple associated to $\mathscr{G}_s$.

4. Properties of the moduli space of rank two torsion free sheaves on X and its deformations

4.1 *Infinitesimal deformations of rank two torsion free sheaves of type* $\mathcal{O}_{X,p} \oplus m_{X,p}$ *on* X

Let $\mathscr{F}_0$ be a torsion free $\mathcal{O}_X$-module. Assume that $\mathscr{F}_0 \otimes \mathcal{O}_{X,p} \simeq \mathcal{O}_{X,p} \oplus m_{X,p}$ and $\mathscr{F}_0 \otimes \mathcal{O}_{X,x} \simeq \mathcal{O}_{X,x} \oplus \mathcal{O}_{X,x}$ for all $x \in X - p$, where $m_{X,p}$ is the maximal ideal at the node p.

Let $\underline{C}$ be the category of Artin local k algebras. Let $\mathbf{F}$: $\underline{C} \to (\underline{\text{Sets}})$ be the functor defined by

$$\mathbf{F}(A) = \left\{ \begin{matrix} \text{Isomorphism class of coherent } \mathcal{O}_{\mathrm{Spec}(A) \times X}\text{-modules } \mathscr{F} \\ \text{flat over } \mathrm{Spec}(A) \text{ and } \mathscr{F}|_{\mathrm{Spec}(A/m_A) \times X} \simeq \mathscr{F}_0 \end{matrix} \right\}.$$

Note that $\mathbf{F}$ has a versal deformation space.

Let $\mathbf{FM}: \underline{C} \to (\underline{\text{Sets}})$ be the functor defined by

$$\mathbf{FM}(A) = \left\{ \begin{matrix} \text{Isomorphism class of } \mathcal{O}_{X,p} \otimes A\text{-modules } M \\ \text{flat over } A \text{ and } M \otimes A/m_A \simeq \mathcal{O}_{X,p} \oplus m_{X,p} \end{matrix} \right\}.$$

There is a natural morphism of functors

$$\Phi: \mathbf{F} \to \mathbf{FM}$$

defined by

$$\mathscr{F} \mapsto \mathscr{F} \otimes (\mathcal{O}_{X,p} \otimes A),$$

$$\mathbf{F}(A) \to \mathbf{FM}(A).$$

PROPOSITION 4.1

The morphism $\Phi: \mathbf{F} \to \mathbf{FM}$ *is formally smooth.*

Proof. From Grothendieck's criterion [10] it is enough to verify the following. If

$$A' \to A \to 0$$

is a small extension in $\underline{C}$, then the natural map

$$\mathbf{F}(A') \to \mathbf{F}(A) \times_{\mathbf{FM}(A)} \mathbf{FM}(A')$$

is surjective.

Let $A' \to A \to 0$ be a small extension in $\underline{C}$, and $\mathscr{F} \in \mathbf{F}(A)$ and $N \in \mathbf{FM}(A')$ be such that

$$\mathscr{F} \otimes (\mathcal{O}_{X,p} \otimes A) \simeq N \otimes_{A'} A.$$

We show that there is an $\mathscr{F}' \in \mathbf{F}(A')$ such that $\mathscr{F}' \otimes A \simeq \mathscr{F}$ and $\mathscr{F}' \otimes (\mathcal{O}_{X,p} \otimes A') \simeq N$. Let $U_i (i = 1, 2)$ be affine open subschemes of X (U_i need not be connected) with

(i) $X = U_1 \cup U_2$

(ii) $p \in U_1$

(iii) $\mathscr{F}|_{(U_1 - p) \times \mathrm{Spec}(A)}$ and $\mathscr{F}|_{(U_2) \times \mathrm{Spec}(A)}$ are free.

(Note that the above assumption implies $p \notin U_2$). Then $\mathscr{F}$ is given by the patching data

$$(a_{ij}) \in GL_2(\mathcal{O}_{(U_1 \cap U_2) \times \mathrm{Spec}(A)}) = GL_2(\mathcal{O}_{(U_1 \cap U_2)} \otimes A).$$

Since $A' \to A$ is a small extension there exists

$$(a'_{ij}) \in GL_2(\mathcal{O}_{(U_1 \cap U_2) \times \mathrm{Spec}(A')}) = GL_2(\mathcal{O}_{(U_1 \cap U_2)} \otimes A')$$

such that under the natural map

$$GL_2(\mathcal{O}_{(U_1 \cap U_2)} \otimes A') \to GL_2(\mathcal{O}_{(U_1 \cap U_2)} \otimes A),$$

(a'_{ij}) maps to (a_{ij}). Let (a'_{ij}) be one such element and let $\mathscr{G}$ be a sheaf of $\mathcal{O}_{(U_1) \times \mathrm{Spec}(A')}$-modules such that $\mathscr{G}$ is free on $(U_1 - p) \times \mathrm{Spec}(A)$ and $\mathscr{G} \otimes (\mathcal{O}_{X,p} \otimes A') \simeq N$. Let $\mathscr{F}'$ be the sheaf of $\mathcal{O}_{X \times \mathrm{Spec}(A')}$-modules obtained by patching $\mathscr{G}$ and $\mathcal{O}^2_{U_2 \times \mathrm{Spec}(A')}$ along $(U_1 \cap U_2) \times \mathrm{Spec}(A')$ using (a'_{ij}). Now it is clear that $\mathscr{F}' \in \mathbf{F}(A')$ and maps to $(\mathscr{F}, N) \in \mathbf{F}(A) \times_{\mathbf{FM}(A)} \mathbf{FM}(A')$ under the natural map

$$\mathbf{F}(A') \to \mathbf{F}(A) \times_{\mathbf{FM}(A)} \mathbf{FM}(A').$$

Since $(\mathscr{F}, N) \in \mathbf{F}(A) \times_{\mathbf{FM}(A)} \mathbf{FM}(A')$ was arbitrary, we get the desired surjection. Thus $\Phi: \mathbf{F} \to \mathbf{FM}$ is formally smooth.

Next consider the following functors from $\underline{C}$ to ($\underline{\text{Sets}}$).

$$\mathbf{EFM}(A) = \left\{ \begin{array}{l} \text{Sub } (\mathcal{O}_{X,p} \otimes A)\text{-modules } \mathscr{M} \text{ of } (\mathcal{O}_{X,p} \otimes A)^2 \\ \text{such that } (\mathcal{O}_{X,p} \otimes A)^2/\mathscr{M} \text{ is flat over } A \\ \text{and is a deformation of } \mathcal{O}_{X,p}/m_{X,p} \end{array} \right\},$$

$$\mathbf{M}(A) = \left\{ \begin{array}{l} \text{Isomorphism classes of } (\mathcal{O}_{X,p} \otimes A)\text{-modules which are} \\ \text{flat over } A \text{ and are deformations of } \mathcal{O}_{X,p}/m_{X,p} \end{array} \right\}.$$

Then we get the following morphisms between functors:

$$\Psi: \mathbf{EFM} \to \mathbf{FM}$$

defined by $\mathbf{EFM}(A) \ni \mathscr{M} \mapsto [\mathscr{M}] \in \mathbf{FM}(A)$, where $[\mathscr{M}]$ denotes the isomorphism class of $\mathscr{M}$, and

$$\Lambda: \mathbf{EFM} \to \mathbf{M}$$

defined by $\mathbf{EFM}(A) \ni \mathscr{M} \mapsto [(\mathcal{O}_{X,p} \otimes A)^2/\mathscr{M}] \in \mathbf{M}(A)$ where $[(\mathcal{O}_{X,p} \otimes A)^2/\mathscr{M}]$ denotes the isomorphism class of $(\mathcal{O}_{X,p} \otimes A)^2/\mathscr{M}$.

We claim that Ψ and Λ are formally smooth (in the sense of Schlessinger [10]). That Λ is formally smooth is immediate (see [1]). To show that Ψ is formally smooth, we need the following.

Lemma 4.2. *Let $A \to B$ be a local homomorphism of noetherian local rings. Let N and L be B modules of finite type with L flat over A. Then a B homomorphism $f: N \to L$ is injective with an A flat cokernel if and only if $f \otimes_A k: N \otimes_A k \to L \otimes_A k$ is injective, where k is the residue field of A.*

Proof. See [9], Appendix.

Lemma 4.3. *Let $A \to B$ be a local homomorphism of noetherian local rings. Let N be a B module of finite type which is flat over A and satisfies* $\mathrm{Ex}(1/B)(\bar{N}, \bar{B}) = 0$, *where $\bar{B} = B \otimes_A k$ and $\bar{N} = N \otimes_A k$ and k is the residue field of A. Then* $\mathrm{Hom}_B(N, B)$ *is A-flat and* $\mathrm{Hom}_B(N, B) \otimes_A k = \mathrm{Hom}_{\bar{B}}(\bar{N}, \bar{B})$.

Proof. See [9], corollary in the Appendix.

PROPOSITION 4.4.

The morphism $\Psi: \mathbf{EFM} \to \mathbf{FM}$ *is formally smooth.*

Proof. From Grothendieck's criterion [10] it is enough to verify the following. If $A' \to A \to 0$ is small extension in $\underline{C}$, then the natural map

$$\mathbf{EFM}(A') \to \mathbf{EFM}(A) \times_{\mathbf{FM}(A)} \mathbf{FM}(A')$$

is surjective.

Let $A' \to A \to 0$ be a small extension in $\underline{C}$, and $\mathscr{M} \in \mathbf{EFM}(A)$ and $[\mathscr{M}'] \in \mathbf{FM}(A')$ be such that

$$[\mathscr{M}' \otimes_{A'} \otimes A] = [\mathscr{M}].$$

Now applying Lemma 4.3 to the local homomorphism $A' \to (\mathscr{O}_{X,p} \otimes A')$ and to the $(\mathscr{O}_{X,p} \otimes A')$-module $\mathscr{M}'$ (note that the hypothesis of the lemma $\mathrm{Ext}^1_{\mathscr{O}_{X,p}}(m_{X,p}, \mathscr{O}_{X,p}) = 0$, is true in this case (see [3])), we conclude that $(\mathscr{M}')^* = \mathrm{Hom}_{\mathscr{O}_{X,p} \otimes A'}(\mathscr{M}', \mathscr{O}_{X,p} \otimes A')$ is A' flat and hence $(\mathscr{M}')^* \otimes_{A'} A = (\mathscr{M})^*$. *A priori*, we can lift the inclusion $\mathscr{M} \subsetneq (\mathscr{O}_{X,p} \otimes A)^2$ to $\phi: \mathscr{M}' \to (\mathscr{O}_{X,p} \otimes A')^2$. Using Lemma 4.2, we conclude that ϕ is injective and $(\mathscr{O}_{X,p} \otimes A')^2/\mathrm{Im}(\phi)$ is flat over A'. Hence $\mathscr{M}' \in \mathbf{EFM}(A')$ as required. This proves the proposition.

Remark 4.1. Let (a_1, a_2) be as in Theorem 3.1 and let $\mathscr{O}_X(1)$ be a line bundle of type (a_1, a_2) on X. Let **Quot** be the quot scheme of rank two quotients of $\mathscr{O}_X(1)^N$ with a fixed Hibert polynomial. Now using the above results we conclude that if $v \in \mathbf{Quot}$ corresponds to a torsion free coherent sheaf of $\mathscr{O}_X$ module $\mathscr{F}_0$ such that $\mathscr{F}_0 \otimes \mathscr{O}_{X,p} \simeq \mathscr{O}_{X,p} \oplus m_{X,p}$ and $\mathscr{F}_0 \otimes \mathscr{O}_{X,x} \simeq \mathscr{O}_{X,x} \oplus \mathscr{O}_{X,x}$ for all $x \in X - p$, then there is an integer l such that the local ring $\hat{\mathscr{O}}_{\mathbf{Quot},v}[[u_1, \ldots, u_l]]$ is isomorphic to $\hat{\mathscr{O}}_{X,p}[[t_1, \ldots, t_s]]$ for some s, where $\hat{\mathscr{O}}_{X,p}$ is the completion of the local ring $\mathscr{O}_{X,p}$ with respect to its maximal ideal.

Theorem 4.1. *Let $\chi \neq 0$ be an integer. Let (a_1, a_2) be a polarization. Assume that $a_1.\chi$ is not an integer. Then the moduli space $M(2,(a_1,a_2),\chi)$ of (a_1,a_2)-semi-stable rank two torsion free sheaves on X with Euler characteristics χ is a reduced, connected projective scheme and with exactly two irreducible components. Moreover, when χ is odd the moduli space is a union of two smooth varieties intersecting transversally.*

Proof. Existence of the moduli space in question as a projective scheme follows from the very general result of ([11], septième partie, III, Théorèm 15). By Theorem 3.1(a) we see that a (a_1, a_2)-semistable rank two torsion free sheaf is either a locally free sheaf or a torsion free sheaf of type $\mathcal{O}_{X,p} \oplus m_{X,p}$. Remaining assertions of the theorem now follows from Remark 4.1 and the general results of GIT quotients (see [6]). This proves the theorem.

For a different proof of the above theorem see Theorem 6.1.

4.2 *Infinitesimal deformation of sheaves of type $\mathcal{O}_{X,p} \oplus m_{X,p}$ on X along a smooth deformation of X*

Let R be a complete discrete valuation ring with residue field $\mathbb{C}$. Let χ be a regular two dimensional scheme proper and flat over $S := \mathrm{Spec}(R)$ and the closed fibre of the characteristic morphism $\chi \to S$ is isomorphic to X ($=$ union of two smooth curves meeting transversally along a point p). Let $\mathcal{F}_0$ be a coherent sheaf $\mathcal{O}_X$ -modules such that $\mathcal{F}_0 \otimes \mathcal{O}_{X,p} \simeq \mathcal{O}_{X,p} \oplus m_{X,p}$ and $\mathcal{F}_0 \otimes \mathcal{O}_{X,x} \simeq \mathcal{O}_{X,x} \oplus \mathcal{O}_{X,x}$ for all $x \in X - p$.

We have the following functors:

1) Let $\underline{F}: \underline{C} \to \underline{(\mathrm{Sets})}$ be the functor defined by

$$\underline{F}(A) = \left\{ (f, [\mathcal{F}]) \;\middle|\; \begin{array}{l} f: R \to A \text{ a local homomorphism and } [\mathcal{F}] \text{ is an} \\ \text{isomorphism class of coherent } \mathcal{O}_{\mathrm{Spec}(A) \times_f \chi}\text{-modules} \\ \mathcal{F} \text{ flat over } \mathrm{Spec}(A) \text{ and } \mathcal{F}|_{\mathrm{Spec}(A/m_A) \times \chi} \simeq \mathcal{F}_0 \end{array} \right\}.$$

2) Let $x_0 \in \chi$ be the closed point which corresponds to p on X. Note that $R \subsetneq \mathcal{O}_{x_0}, \chi$. Let $\underline{\mathbf{FM}}: \underline{C} \to \underline{(\mathrm{Sets})}$ be the functor defined by

$$\underline{\mathbf{FM}}(A) = \left\{ (f, [M]) \;\middle|\; \begin{array}{l} f: R \to A \text{ a local homomorphism and } [M] \text{ is an} \\ \text{isomorphism class } (\mathcal{O}\chi_{x_0} \otimes_R A) \text{ modules} \\ M \text{ flat over } A \text{ and } M \otimes_R R/m_R \simeq \mathcal{O}_{X,p} \oplus m_{p,X} \end{array} \right\}.$$

3) Let $\underline{\mathbf{EFM}}: \underline{C} \to \underline{(\mathrm{Sets})}$ be the functor defined by

$$\underline{\mathbf{EFM}}(A) = \left\{ (f, M) \;\middle|\; \begin{array}{l} f: R \to A \text{ a local homomorphism and } M \text{ is a} \\ \mathcal{O}\chi_{x_0} \otimes_R A) \text{ submodule of } \mathcal{O}\chi_{x_0} \otimes_R A)^2 \\ \text{such that } \mathcal{O}\chi_{x_0} \otimes_R A)^2/M \text{ is flat over } A \\ \text{and is a deformation of } \mathcal{O}_{X,p}/m_{X,p} \end{array} \right\}.$$

4) Let $\underline{\mathbf{M}}: \underline{C} \to \underline{(\mathrm{Sets})}$ be the functor defined by

$$\underline{\mathbf{M}}(A) = \left\{ (f, N) \;\middle|\; \begin{array}{l} f: R \to A \text{ a local homomorphism and } N \text{ is a} \\ (\mathcal{O}_{x_0}, \chi \otimes_R A) \text{ module flat over } A \\ \text{and is a deformation of } \mathcal{O}_{X,p}/m_{X,p} \end{array} \right\}.$$

Now arguments similar to the one used in the case of 'infinitesimal deformations of rank two torsion free sheaves of type $\mathcal{O}_X \oplus m_{X,p}$ on X' shows that the natural morphisms of functors

$$\begin{aligned} \underline{\mathbf{F}} &\to \underline{\mathbf{FM}} \\ \underline{\mathbf{EFM}} &\to \underline{\mathbf{FM}} \\ \underline{\mathbf{EFM}} &\to \underline{\mathbf{M}} \end{aligned}$$

are all formally smooth.

Remark 4.2. Let (a_1, a_2), S and χ be as above. Let $\mathcal{O}_\chi(1)$ be a relative ample line bundle such that $\mathcal{O}_\chi(1)|_X$ is a polarization of type (a_1, a_2). Let $\mathbf{Quot}^P/S$ be the quot scheme of quotients of $\mathcal{O}_\chi(1)^N$ which are of rank two and with a fixed Hilbert polynomial P along the fibres. Now using the above results, we conclude that if $v \in \mathbf{Quot}^P/S$ corresponds to a torsion free coherent $\mathcal{O}_X$-module $\mathcal{F}_0$ such that $\mathcal{F}_0 \otimes \mathcal{O}_{X,p} \simeq \mathcal{O}_{X,p} \oplus m_{X,p}$ and $\mathcal{F}_0 \otimes \mathcal{O}_{X,x} \simeq \mathcal{O}_{X,x} \oplus \mathcal{O}_{X,x}$ for all $x \in X - p$, then there is an integer l such that the local ring $\hat{\mathcal{O}}_{\mathrm{Quot}^P/S,v}[[u_1, \ldots, u_l]]$ is isomorphic to $\hat{\mathcal{O}}_{\chi,x_0}[[t_1, \ldots, t_s]]$ for some s, where $\mathcal{O}_{\chi,\underline{x_0}}$ is the completion of the local ring $\mathcal{O}_{\chi,\underline{x_0}}$ with respect to the maximal ideal $\underline{x_0}$ of χ.

Remark 4.3. From Remark 4.2, it follows that if $v \in \mathbf{Quot}^P/S$ corresponds to a torsion free coherent $\mathcal{O}_X$-module $\mathcal{F}_0$ such that $(\mathcal{F}_0 \otimes \mathcal{O}_{X,p}) \simeq \mathcal{O}_{X,p} \oplus m_{X,p}$ and $(\mathcal{F}_0 \otimes \mathcal{O}_{X,x}) \simeq \mathcal{O}_{X,x} \oplus \mathcal{O}_{X,x}$ for all $x \in X - p$, then v is a smooth point of $\mathbf{Quot}^P/S$.

Theorem 4.2. *Notations are as above. Then there exists a family*

$$\mathcal{M}(2, \mathcal{O}_{\mathcal{X}}, \chi \neq 0) \to S$$

of moduli spaces of rank two torsion free sheaves along the fibers of $\chi \to S$ with Euler characteristic χ and semi-stable with respect to $\mathcal{O}(1)$. Moreover,

$$\mathcal{M}(2, \mathcal{O}_{\mathcal{X}}, \chi \neq 0) \to S$$

is an integral scheme which is proper and flat over S. If χ is odd $\mathcal{M}(2, \mathcal{O}_{\mathcal{X}}, \chi \neq 0)$ is a regular scheme, if $\chi \neq 0$ is even (in particular degree of the bundles under consideration is zero, i.e., $\chi = 2 - 2g$) the scheme $\mathcal{M}(2, \mathcal{O}_{\mathcal{X}}, \chi \neq 0)$ gives a degeneration of the general fibre.

Proof. Using the results of geometric invariant theory (GIT) over an arbitrary base (see [13]) one can construct a family

$$\mathcal{M}(2, \mathcal{O}_{\mathcal{X}}, \chi \neq 0) \to S$$

of moduli spaces of rank two torsion free sheaves along the fibers $\mathcal{X} \to S$ with Euler characteristic χ and semi-stable with respect to $\mathcal{O}(1)$, as a GIT quotient of the appropriate Quot scheme $\mathbf{Quot}^P/S$. Moreover,

$$\mathcal{M}(2, \mathcal{O}_{\mathcal{X}}, \chi \neq 0) \to S$$

is an integral scheme which is proper and flat over S. Then by the results proved above, we see that $\mathcal{M}(2, \mathcal{O}_{\mathcal{X}}, \chi \neq 0)$ is a regular scheme if χ is odd and if $\chi \neq 0$ is even (in particular degrees of the underlying bundle is zero, i.e., $\chi = 2 - 2g$) the scheme $\mathcal{M}(2, \mathcal{O}_{\mathcal{X}}, \chi \neq 0)$ gives a degeneration of the general fibre, which 'essentially' has all the nice properties of the general fiber.

5. Direct construction of the moduli space of triples

From now on, for simplicity, we assume that the genus of $X_i (i = 1, 2)$ is $\geqslant 2$ even though the results still hold if the genus of $X_i (i = 1, 2)$ is $\geqslant 0$. Let $\chi \neq 0$ be an integer. Let $0 < a_1 < 1$ be a rational number such that $a_1\chi$ is not an integer. Set $a_2 = 1 - a_1$.

It follows, from the existence of moduli space of rank two torsion free sheaves on X (see Theorem 4.1) and the results of § 2 of this paper, that the moduli space of triples exists. In this section we indicate a direct construct of the moduli space of (a_1, a_2)-semistable triples $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ where $\mathscr{F}_1$ (resp. $\mathscr{F}_2$) is a rank two vector bundle on X_1 (resp. X_2) with $a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1$ (resp. $a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$) and

$$\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi.$$

Using this we constuct (see § 6) moduli space of torsion free sheaves on X. This shows that the moduli space of torsion free sheaves of rank two on X can be constructed directly from the space of vector bundles with some extra data on the normalization. We believe that this could be useful for applications (for example, as in [4, 7]).

First some preliminaries.

Let

$$[\mathscr{F}] \in S((a_1, a_2), (2, 2), \chi = d + 2(1 - g))$$

and let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be the triple which corresponds to $\mathscr{F}$. By Theorem 3.1 we have the following inequalities;

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 2,$$
$$a_2\chi < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$$

and $\chi = \chi_X(\mathscr{F}) = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2) - 2$. Moreover, if $\mathscr{F}$ is not locally free, then we have the following sharper inequalities;

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1,$$
$$a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$$

with $\chi = \chi_X(\mathscr{F}) = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2) - 2$ and $\mathrm{rk}(A) = 1$.

Write $a_1 = a/b$ with a, b relatively prime positive integers. Let $\mathscr{L}_1$ be a line bundle on X_1 of degree a and let $\mathscr{L}_2$ be a line bundle on X_2 of degree $b - a$. Let $\mathscr{O}_X(1)$ be an ample line bundle on X such that $\mathscr{O}_X(1)\,|_{X_1} = \mathscr{L}_1$ and $\mathscr{O}_X(1)\,|_{X_2} = \mathscr{L}_2$. Since $a/(b - a) = a_1/a_2$, observe that $\mathscr{O}_X(1)$ defines a polarization of type (a_1, a_2) on X. If $\mathscr{F}$ is a torsion free sheaf of $\mathscr{O}_X$-modules of rank 2 and if $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is the triple corresponding to it, then the triple $(\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes n}, \mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes n}, \overrightarrow{A \otimes \lambda^{\otimes n}})$ corresponds to the $\mathscr{O}_X$-module $\mathscr{F} \otimes \mathscr{O}_X(n)$, where λ: $\mathscr{L}_1(p) \to \mathscr{L}_2(p)$ is the isomorphism defining $\mathscr{O}_X(1)$. Since the bi-degree of the line bundle $\mathscr{O}_X(1)$ is $(a, b - a)$ we have

$$\chi_{X_1}(\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m}) = \chi_{X_1}(\mathscr{F}_1) + 2am,$$
$$\chi_{X_2}(\mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes m}) = \chi_{X_2}(\mathscr{F}_2) + 2(b - a)m,$$
$$\chi(\mathscr{F} \otimes \mathscr{O}_X(m)) = \chi(\mathscr{F}) + 2bm.$$

Thus, if

$$[\mathscr{F}] \in S((a_1, a_2), (2, 2), \chi \neq 0)$$

and $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ the triple which corresponds to $\mathscr{F}$, then

$$[\mathscr{F} \otimes \mathcal{O}_X(m)] \in S((a_1, a_2), (2,2), \chi + 2mb \neq 0)$$

and the corresponding triple $(\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m}, \mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes n}, \overrightarrow{A \otimes \lambda^{\otimes n}})$ satisfies the following inequalities;

$$a_1\chi(\mathscr{F} \otimes \mathcal{O}_X(m)) < \chi_{X_1}(\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m}) < a_1\chi(\mathscr{F} \otimes \mathcal{O}_X(m)) + 2,$$
$$a_2\chi(\mathscr{F} \otimes \mathcal{O}_X(m)) < \chi_{X_2}(\mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes m}) < a_2\chi(\mathscr{F} \otimes \mathcal{O}_X(m)) + 2$$

and $\chi_X(\mathscr{F} \otimes \mathcal{O}_X(m)) = \chi_{X_1}(\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m}) + \chi_{X_2}(\mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes m}) - 2$. Moreover if $\mathscr{F}$ is not locally free then $\mathscr{F} \otimes \mathcal{O}_X(m)$ is not locally free and we have the following sharper inequalities;

$$a_1\chi(\mathscr{F} \otimes \mathcal{O}_X(m)) < \chi_{X_1}(\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m}) < a_1\chi(\mathscr{F} \otimes \mathcal{O}_X(m)) + 1,$$
$$a_2\chi(\mathscr{F} \otimes \mathcal{O}_X(m)) + 1 < \chi_{X_2}(\mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes m}) < a_2\chi(\mathscr{F} \otimes \mathcal{O}_X(m)) + 2$$

with $\chi_X(\mathscr{F} \otimes \mathcal{O}_X(m)) = \chi_{X_1}(\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m}) + \chi_{X_2}(\mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes m}) - 2$ and $\mathrm{rk}(A \otimes \lambda^{\otimes m}) = 1$.

Remark 5.1. Let $\chi \neq 0$ and $0 < a_1 < 1$ be such that $a_1\chi$ is not an integer. Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be a $(a_1, a_2 = 1 - a_1)$ semistable triple of rank $(2, 2)$ with $\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi$ and

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1,$$
$$a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2.$$

Then $\mathscr{F}_1$ is a semistable $\mathcal{O}_{X_1}$-module and $\mathscr{F}_2$ is a semi-stable $\mathcal{O}_{X_2}$-module.

Proof. Let $L_2 \subset \mathscr{F}_2$ be a line subbundle. Then $(0, L_2, \vec{0})$ is a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$. Now by the semi-stability of the triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, we conclude that

$$\chi(L_2) - 1 \leqslant a_2\frac{\chi}{2}.$$

Then from the assumption on $\chi(\mathscr{F}_2)$, we get

$$\chi(L_2) \leqslant a_2\frac{\chi}{2} + 1 < \frac{\chi_{X_2}(\mathscr{F}_2) - 1}{2} + 1.$$

Thus

$$\chi(L_2) < \frac{\chi_{X_2}(\mathscr{F}_2)}{2} + \frac{1}{2}.$$

Since $\chi(L_2)$ is an integer, we get

$$\chi(L_2) \leqslant \frac{\chi_{X_2}(\mathscr{F}_2)}{2}.$$

This proves that $\mathscr{F}_2$ is a semi-stable $\mathcal{O}_{X_2}$-module.

Let $L_1 \subset \mathscr{F}_1$ be a line subbundle. Then $L_1 \otimes \mathcal{O}_{X_1}(-p) \subset \mathscr{F}_1 \otimes \mathcal{O}_{X_1}(-p)$ hence $(L_1 \otimes \mathcal{O}_{X_1}(-p), 0, \vec{0})$ is a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$. Now by the semistability of the triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, we conclude that

$$\chi(L_1) - 1 \leqslant a_1\frac{\chi}{2}.$$

Thus

$$\chi_{X_1}(L_1) \leqslant a_1 \frac{\chi}{2} + 1 < \frac{\chi_{X_1}(\mathscr{F}_1)}{2} + 1. \tag{1}$$

Note that, if $\chi_{X_1}(\mathscr{F}_1)$ is even, then (1) gives

$$\chi_{X_1}(L_1) \leqslant \frac{\chi_{X_1}(\mathscr{F}_1)}{2}$$

and hence $\mathscr{F}_1$ is semistable. Thus from now on we assume $\chi_{X_1}(\mathscr{F}_1)$ is odd. Then from (1) we get

$$\chi_{X_1}(L_1) \leqslant \frac{\chi_{X_1}(\mathscr{F}_1)}{2} + \frac{1}{2}. \tag{2}$$

Claim.

$$\chi_{X_1}(L_1) \leqslant \frac{\chi_{X_1}(\mathscr{F}_1)}{2}$$

for every line subbundle L_1 of $\mathscr{F}_1$

To prove the claim we assume the contrary and get a contradiction. Assume that there is a line subbundle L_1 of $\mathscr{F}_1$ such that

$$\chi_{X_1}(L_1) > \frac{\chi_{X_1}(\mathscr{F}_1)}{2}.$$

Then for any such subbundle L_1 of $\mathscr{F}_1$ we have (by (2) above)

$$\chi_{X_1}(L_1) = \frac{\chi_{X_1}(\mathscr{F}_1)}{2} + \frac{1}{2}. \tag{3}$$

Let L_1 be such a line bundle. Note that $(L_1, \mathscr{F}_2, \overrightarrow{A|_{L_1}})$ is a subtriple of the semistable triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, hence we get

$$\frac{\chi(L_1) + \chi(\mathscr{F}_2) - 2}{a_1 + 2a_2} \leqslant \frac{\chi}{2}.$$

Since $a_1 + a_2 = 1$, the last inequality gives

$$\chi(L_1) + \chi(\mathscr{F}_2) - 2 \leqslant \frac{\chi}{2} + a_2 \frac{\chi}{2}.$$

Using (3) in this equation we get

$$\frac{\chi_{X_1}(\mathscr{F}_1)}{2} + \frac{1}{2} + \chi(\mathscr{F}_2) - 2 \leqslant \frac{\chi}{2} + a_2 \frac{\chi}{2}.$$

Hence

$$\chi_{X_1}(\mathscr{F}_1) + 2\chi(\mathscr{F}_2) - 3 \leqslant \chi + a_2\chi.$$

Since $\chi = \chi_X(\mathscr{F}) = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2) - 2$, we get

$$\chi + \chi(\mathscr{F}_2) - 1 \leqslant \chi + a_2\chi.$$

Hence

$$\chi(\mathscr{F}_2) \leqslant a_2\chi + 1,$$

which is a contradiction to the assumption that

$$a_2\chi + 1 < \chi(\mathscr{F}_2).$$

This proves the claim, and completes the proof of the remark.

Theorem 5.1. *Let $\chi \neq 0$ and $0 < a_1 < 1$ be such that $a_1\chi$ is not an integer. Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be a $(a_1, a_2 = 1 - a_1)$ semistable triple of rank $(2,2)$ with $\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi$. Then $\mathscr{F}_1$ is a semi-stable $\mathscr{O}_{X_1}$-module and $\mathscr{F}_2$ is a semi-stable $\mathscr{O}_{X_2}$-module.*

Proof. If the semi-stable triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is such that rank$(\vec{A}) = 1$, then by Theorem 3.1 the inequality in Remark 5.1 is the only possibility, hence $\mathscr{F}_1$ is a semi-stable $\mathscr{O}_{X_1}$-module and $\mathscr{F}_2$ is a semi-stable $\mathscr{O}_{X_2}$-module. For the other cases one can use the arrow in the other direction and the fact that the nature of the problem is symmetric in X_1 and X_2. This proves the theorem.

Let χ and (a_1, a_2) as above. Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be a rank $(2,2)$ triple such that $\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi$ and

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1,$$

$$a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2.$$

Let

$$S_1 = \left\{ \mathscr{G}_1 \left| \begin{array}{l} \mathscr{G}_1 \subset \mathscr{F}_1 \text{ is a } \mathscr{O}_{X_1}\text{-submodule of one of the following types:} \\ \text{i) } \mathscr{G}_1 = 0, \\ \text{ii) } \mathscr{G}_1 \text{ is a subbundle,} \\ \text{iii) } \mathscr{F}_1 \otimes \mathscr{O}_{X_1}(-p) \subset \mathscr{G}_1 \subset \mathscr{F}_1. \end{array} \right. \right\}$$

and

$$S_2 = \left\{ \mathscr{G}_2 \left| \begin{array}{l} \mathscr{G}_2 \subset \mathscr{F}_2 \text{ is a } \mathscr{O}_{X_2}\text{-submodule of one of the following types:} \\ \text{i) } \mathscr{G}_2 = 0, \\ \text{ii) } \mathscr{G}_2 \text{ is a subbundle,} \\ \text{iii) } \mathscr{F}_2 \otimes \mathscr{O}_{X_2}(-p) \subset \mathscr{G}_2 \subset \mathscr{F}_2. \end{array} \right. \right\}.$$

Finally, let

$$S = \left\{ (\mathscr{G}_1, \mathscr{G}_2, \vec{B}) \left| \begin{array}{l} (\mathscr{G}_1, \mathscr{G}_2, \vec{B}) \text{ is a subtriple of } (\mathscr{F}_1, \mathscr{F}_2, \vec{A}) \\ \text{with } \mathscr{G}_1 \in S_1 \text{ and } \mathscr{G}_2 \in S_2 \end{array} \right. \right\}.$$

Lemma 5.2. The triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is (a_1, a_2)-stable (resp. semi-stable) if and only if for all the subtriples $(\mathscr{G}_1, \mathscr{G}_2, \vec{B}) \in S$

$$\chi((\mathscr{G}_1, \mathscr{G}_2, \vec{B})) < (a_1 \mathrm{rk}(\mathscr{G}_1) + a_2 \mathrm{rk}(\mathscr{G}_2)) \frac{\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A}))}{2}$$

(resp. $\leqslant$).

Proof. Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be (a_1, a_2) stable (resp. semi-stable). Then

$$\chi((\mathscr{G}_1, \mathscr{G}_2, \vec{B})) < (a_1 \mathrm{rk}(\mathscr{G}_1) + a_2 \mathrm{rk}(\mathscr{G}_2)) \frac{\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A}))}{2}$$

(resp. $\leqslant$)

for all subtriples $(\mathscr{G}_1, \mathscr{G}_2, \vec{B})$ and hence in particular for the subtriples in S. This proves the only if part of the lemma.

To prove the if part of the lemma, assume that the above strict inequalities (resp. inequalities) hold for all subtriples $(\mathscr{G}_1, \mathscr{G}_2, \vec{B}) \in S$ of the triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$. Let $(\mathscr{G}_1, \mathscr{G}_2, \vec{B}) \notin S$ be a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$. Then we see that one of the following holds:

(i) $(\mathscr{G}_1, \mathscr{G}_2, \vec{B}) = (0, \mathscr{G}_2, \vec{0})$ and $\mathscr{G}_2$ is neither a subbundle of $\mathscr{F}_2$ nor $\mathscr{F}_2 \otimes \mathcal{O}_{X_2}(-p) \subset \mathscr{G}_2$.
(ii) rank $(\mathscr{G}_1) = 1$ and $\mathscr{G}_1$ is a subbundle of $\mathscr{F}_1$ but $\mathscr{G}_2$ is neither a subbundle of $\mathscr{F}_2$ nor $\mathscr{F}_2 \otimes \mathcal{O}_{X_2}(-p) \subset \mathscr{G}_2$.
(iii) rank $(\mathscr{G}_1) = 1$ and $\mathscr{G}_1$ is a not subbundle of $\mathscr{F}_1$.
(iv) rank $(\mathscr{G}_1) = 2$ and $\mathscr{F}_1 \otimes \mathcal{O}_{X_1}(-p)$ is not contained in $\mathscr{G}_1$.

In the cases (i) and (ii), if we take $\mathscr{L}_1 = \mathscr{G}_1$ and $\mathscr{L}_2$ to be the smallest $\mathcal{O}_{X_2}$-subbundle of $\mathscr{F}_2$ containing $\mathscr{G}_2$ and $C = B$, then $(\mathscr{L}_1, \mathscr{L}_2, \vec{C}) \in S$. Since rank $(\mathscr{G}_1) =$ rank $(\mathscr{L}_1)$, rank $(\mathscr{G}_2) =$ rank $(\mathscr{L}_2)$, clearly we get

$$\chi((\mathscr{G}_1, \mathscr{G}_2, \vec{B})) < \chi((\mathscr{L}_1, \mathscr{L}_2, \vec{C})) < (a_1 \mathrm{rk}(\mathscr{G}_1) + a_2 \mathrm{rk}(\mathscr{G}_2)) \frac{\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A}))}{2}$$

(resp. $\leqslant$).

In the case (iii), take $\mathscr{L}_1$ to be the smallest $\mathcal{O}_{X_1}$-subbundle of $\mathscr{F}_1$ generated by $\mathscr{G}_1$. Then rank $(\mathscr{L}_1) = 1$ and $\chi(\mathscr{G}_1) \leqslant \chi(\mathscr{L}_1) - 1$. Now if rank $(\mathscr{G}_2) = 2$, then set $\mathscr{L}_2 = \mathscr{F}_2$ and $C = A|_{\mathscr{L}_1(p)}$. Clearly $(\mathscr{L}_1, \mathscr{L}_2, \vec{C}) \in S$ and

$$\chi((\mathscr{G}_1, \mathscr{G}_2, \vec{B})) < \chi((\mathscr{L}_1, \mathscr{L}_2, \vec{C})) < (a_1 \mathrm{rk}(\mathscr{G}_1) + a_2 \mathrm{rk}(\mathscr{G}_2)) \frac{\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A}))}{2}$$

(resp. $\leqslant$)

as required. On the other hand if rank $(\mathscr{G}_2) = 1$, let $\mathscr{L}_2$ be the smallest $\mathcal{O}_{X_2}$-subbundle of $\mathscr{F}_2$ generated by $\mathscr{G}_2$. Then rank $(\mathscr{L}_2) = 1$, $\chi(\mathscr{G}_2) \leqslant \chi(\mathscr{L}_2)$ and $(0, \mathscr{L}_2, \vec{0}) \in S$ and hence

$$\chi(\mathscr{G}_2) - 1 \leqslant \chi(\mathscr{L}_2) - 1 < a_2 \frac{\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A}))}{2} < \frac{\chi(\mathscr{F}_2))}{2} - \frac{1}{2}$$

(resp. $\leqslant$).

Thus

$$\chi(\mathscr{G}_1) + \chi(\mathscr{G}_2) - 1 \leqslant \chi(\mathscr{L}_1) + \chi(\mathscr{L}_2) - 2 < \chi(\mathscr{L}_1) - 1 + \frac{\chi((\mathscr{F}_2))}{2} - \frac{1}{2}. \quad (1)$$

But, if $C = A|_{\mathscr{L}_1(p)}$, then $(\mathscr{L}_1, \mathscr{F}_2, \vec{C}) \in S$ and hence

$$\chi(\mathscr{L}_1, \mathscr{F}_2, \vec{C}) = \chi(\mathscr{L}_1) + \chi(\mathscr{F}_2) - 2 < \frac{\chi}{2} + a_2 \frac{\chi}{2}$$

(resp. $\leqslant$).

Thus

$$\chi(\mathscr{L}_1) - 1 \leqslant \frac{\chi}{2} + a_2 \frac{\chi}{2} - (\chi(\mathscr{F}_2) - 1).$$

Using this in the inequality (1) we get

$$\chi(\mathscr{G}_1) + \chi(\mathscr{G}_2) - 1 \leqslant \frac{(1 + a_2)\chi + 1 - \chi(\mathscr{F}_2)}{2}.$$

Now, since by our assumption $a_2\chi - \chi(\mathscr{F}_2) < -1$, we get

$$\chi(\mathscr{G}_1) + \chi(\mathscr{G}_2) - 1 < \frac{\chi}{2}.$$

This proves the required inequality in this case.

This proof in case (iv) is divided into following three subcases:

(a) $\mathscr{G}_2 = 0$. Consider the canonical surjection

$$\mathscr{F}_1 \to \mathscr{F}_1(p)$$

and let $\mathscr{L}_1$ be the inverse image of ker (A) under this surjection. Then $(\mathscr{L}_1, 0, \vec{0}) \in S$ and $\mathscr{G}_1 \subset \mathscr{L}_1$ with $\chi(\mathscr{G}_1) \leqslant \chi(\mathscr{L}_1)$. Hence again we get the desired result.

(b) Rank $(\mathscr{G}_2) = 1$. Let $\mathscr{L}_2$ be the $\mathscr{O}_{X_2}$-subbundle of $\mathscr{F}_2$ generated by $\mathscr{G}_2$. Let V denote the image of $\mathscr{L}_2$ in $\mathscr{F}_2(p)$ under the natural map $\mathscr{F}_2 \to \mathscr{F}_2(p)$ and V' be the inverse image of V under the map A. Let $\mathscr{L}_1$ be the inverse image of V' under the natural surjection

$$\mathscr{F}_1 \to \mathscr{F}_1(p)$$

and set $C = A|_{V'}$. Now, clearly $(\mathscr{L}_1, \mathscr{L}_2, \vec{C}) \in S$, and $\mathscr{G}_i \subsetneq \mathscr{L}_i (i = 1, 2)$. Also, since rank $(\mathscr{L}_i) =$ rank $(\mathscr{G}_i)$ $(i = 1, 2)$, we see that $\chi(\mathscr{G}_i) \leqslant \chi(\mathscr{L}_i)$ $(i = 1, 2)$. Thus we get the desired result.

(c) Rank $(\mathscr{G}_2) = 2$. In this case, since $\mathrm{rk}(\mathscr{G}_i) = \mathrm{rk}(\mathscr{F}_i) = 2$ $(i = 1, 2)$ and $\mathscr{G}_i \subsetneq \mathscr{F}_i$ $(i = 1, 2)$, the required inequality is obvious.

This completes the proof of the lemma.

Theroem 5.3. *Let $\chi \neq 0$ be an integer and $0 < a_1 < 1$ be a rational number. Assume that $a_1\chi$ is not an integer. Then there is an irreducible projective variety $M(2, a_1, \vec{\chi})$ which is a course moduli space of $(a_1, a_2 = 1 - a_1)$-semistable triples $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ where $\mathscr{F}_1$ (resp. $\mathscr{F}_2$) is a rank two vector bundle on X_1 (resp. X_2), A: $\mathscr{F}_1(p)$ $\mathscr{F}_2(p)$* is a linear map and

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1, a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2,$$
$$\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A}) = \chi.$$

Moreover, when χ is odd $M(2, a_1, \vec{\chi})$ is smooth.

Proof. Let T be the set of isomorphism classes of (a_1, a_2)-semi-stable triples $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$, where $\mathscr{F}_1$ (resp. $\mathscr{F}_2$) is a rank two vector bundle on X_1 (resp. X_2), $A: \mathscr{F}_1(p) \to \mathscr{F}_2(p)$ is a linear map and

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1, a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2,$$
$$\chi((\mathscr{F}_1, \mathscr{F}_2, \vec{A})) = \chi.$$

Let

$$T_1 = \{[\mathscr{F}_1] | [(\mathscr{F}_1, \mathscr{F}_2, \vec{A})] \in T \quad \text{for some} \quad \mathscr{F}_2 \text{ and } A: \mathscr{F}_1(p) \to \mathscr{F}_2(p)\}$$

and

$$T_2 = \{[\mathscr{F}_2{}^1] | [(\mathscr{F}_1, \mathscr{F}_2, \vec{A})] \in T \quad \text{for some} \quad \mathscr{F}_1 \text{ and } A: \mathscr{F}_1(p) \to \mathscr{F}_2(p)\},$$

where $[\cdot]$ denote the isomorphism class of the object. Then by Theorem 5.1, T_1 (resp. T_2) consists of isomorphism classes of semistable vector bundles of rank two on X_1 (resp. X_2) of fixed Euler characteristics and hence T_1 (resp. T_2) is bounded (see [8], ch. (5.3)).

Hence there is an integer m_0 such that for all $m \geqslant m_0$ and for all $[\mathscr{F}_1] \in T_1$ (resp. $[\mathscr{F}_2] \in T_2$), $\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m}$ (resp. $\mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes m}$) is generated by global sections and the first cohomology group $H^1(\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m}) = 0$ (resp. $H^1(\mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes m}) = 0$). Similarly, we see that given any real number θ_1 (resp. θ_2), there is an integer $m(\theta_1)$ (resp. $m(\theta_2)$) such that for all $m \geqslant m(\theta_1)$ (resp. $m \geqslant m(\theta_2)$) and for any subbundle $\mathscr{G}_1$ of $\mathscr{F}_1$ (resp. $\mathscr{G}_2$ of $\mathscr{F}_2$) with $[\mathscr{F}_1] \in T_1$ (resp. $[\mathscr{F}_2] \in T_2$), degree of $\mathscr{G}_1$ (resp. $\mathscr{G}_2$) $\geqslant m(\theta_1)$ (resp. $\geqslant m(\theta_2)$), $\mathscr{G}_1 \otimes \mathscr{L}_1^{\otimes m}$ (resp. $\mathscr{G}_2 \otimes \mathscr{L}_2^{\otimes m}$) is generated by global sections and the first cohomology group $H^1(\mathscr{G}_1 \otimes \mathscr{L}_1^{\otimes m}) = 0$ (resp. $H^1(\mathscr{G}_2 \otimes \mathscr{L}_2^{\otimes m}) = 0$).

Set $\theta_1 = -2(g-8)$ (resp. $\theta_2 = -2(g-8)$), where g is the genus of X. Note that, since we have assumed that the genus of X_i, $i = 1, 2$ is atleast 2, $g \geqslant 4$. Let m be a fixed integer which is $\geqslant \max\{m(\theta_1), m(\theta_2)\}$, where $m(\theta_1)$ (resp. $m(\theta_2)$) is as above for the given choice of θ_1 (resp. θ_2). Let k_1 (resp. k_2) be the common dimension of all the vector spaces $H^0(\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m})$ (resp. $H^0(\mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes m})$) with $[\mathscr{F}_1] \in T_1$ (resp. $[\mathscr{F}_2] \in T_2$) Let E_1 (resp. E_2) be a vector space of dimension k_1 (resp. k_2). Let $\mathbf{Q}_1$ (resp. $\mathbf{Q}_2$) be the 'Quot' scheme of coherent sheaves on X_1 (resp. X_2), which are quotients of $E_1 \otimes \mathscr{O}_{X_1}$ (resp. $E_2 \otimes \mathscr{O}_{X_2}$) and whose Hilbert polynomial is that of $\mathscr{F}_1 \otimes \mathscr{L}_1^{\otimes m}$ (resp. $\mathscr{F}_2 \otimes \mathscr{L}_2^{\otimes m}$) for $[\mathscr{F}_1] \in T_1$ (resp. $[\mathscr{F}_2] \in T_2$). Denote by R_1 (resp. R_2) the open subset of $\mathbf{Q}_1$ (resp. $\mathbf{Q}_2$) consisting of those points $q_1 \in \mathbf{Q}_1$ (resp. $q_2 \in \mathbf{Q}_2$) such that if

$$E_1 \otimes \mathscr{O}_{X_1} \to \mathscr{F}_{q_1} \to 0 \quad (\text{resp. } E_2 \otimes \mathscr{O}_{X_2} \to \mathscr{F}_{q_2} \to 0)$$

is the corresponding quotient, then $H^1(\mathscr{F}_{q_1}) = 0$ (resp. $H^1(\mathscr{F}_{q_2}) = 0$), the natural map induces an isomorphism $H^0(\mathscr{F}_{q_1}) \simeq E_1$ (resp. $H^0(\mathscr{F}_{q_2}) \simeq E_2$) and $\mathscr{F}_{q_1}$ (resp. $\mathscr{F}_{q_2}$) is locally free. It is known that (see [12]) R_1 (resp. R_2) is a non-singular variety of dimension $k_1^2 + 4(g_1 - 1)$ (resp. $k_2^2 + 4(g_2 - 1)$).

Let

$$E_1 \otimes \mathscr{O}_{\mathbf{Q}_1 \times X_1} \to \mathbf{F}_1 \to 0 \quad (\text{resp. } E_2 \otimes \mathscr{O}_{\mathbf{Q}_2 \times X_2} \to \mathbf{F}_2 \to 0)$$

be the universal quotient sheaf on $\mathbf{Q}_1 \times X_1$ (resp. $\mathbf{Q}_2 \times X_2$). Set $F_1 = p_1^*(\mathbf{F}_1|_{R_1 \times p})$ (resp. $F_2 = p_2^*(\mathbf{F}_2|_{R_2 \times p})$) on $R_1 \times R_2$. Note that $F_i (i = 1, 2)$, are locally free of rank two. Thus $\mathrm{Hom}(F_1, F_2)$ is a rank four vector bundle on $R_1 \times R_2$. Let n be a large positive integer (which we will choose later). Let $r_1, \ldots, r_{na}$ (resp. $r_{na+1}, \ldots, r_{nb}$) be distinct points on X_1 (resp. X_2), where a and b are integers chosen at the beginning of this section. Define a morphism

$$\tau_n : \mathrm{Hom}(F_1, F_2) \to H_2(E_1 \oplus E_2)^{nb+1},$$

$$\phi_{(q_1, q_2)} \mapsto (\mathbf{F}_1|_{q_1 \times r_1}, \ldots, \mathbf{F}_1|_{q_1 \times r_{na}}, \mathbf{F}_2|_{q_2 \times r_{na+1}}, \ldots, \mathbf{F}_1|_{q_2 \times r_{nb}}, \Delta(\phi)),$$

where $H_2(E_1 \oplus E_2)$ is the Grassmanian of two-dimensional quotients of $E_1 \oplus E_2$ and $\Delta(\phi)$ is the two-dimensional quotient given by the composite of the surjections

$$E_1 \oplus E_2 \to \mathbf{F}_1|_{q_1 \times p} \oplus \mathbf{F}_2|_{q_2 \times p} \to \frac{\mathbf{F}_1|_{q_1 \times p} \oplus \mathbf{F}_2|_{q_2 \times p}}{\mathrm{Graph}(\phi)}.$$

Let

$$G = (GL(E_1) \times GL(E_2)) \cap SL(E_1 \oplus E_2).$$

Note that G acts on $\mathrm{Hom}(F_1, F_2)$ and on $H_2(E_1 \oplus E_2)^{nb+1}$ and τ_n is equivariant for this action.

Lemma 5.4. *A point*

$$(x_1, \ldots, x_{nb+1}) \in H_2(E_1 \oplus E_2)^{nb+1}$$

is semi-stable (resp. stable) for the action of G with respect to the linearization $(\varepsilon_1,\ldots,\varepsilon_1,\varepsilon)$ *if and only if for any subvector space of the form* $F_1\oplus F_2$ *with* $F_1\subset E_1, F_2\subset E_2$ *and* $F_1\oplus F_2\neq E_1\oplus E_2$, *we have*

$$-(\dim F_1+\dim F_2)(2nb\varepsilon_1+2\varepsilon)+(\dim E_1+\dim E_2)\left(\varepsilon_1\sum_{i=1}^{nb}\dim V_i+\varepsilon\dim V_{nb+1}\right)$$

$\geqslant 0$ (*resp.* >0), *where* $V_i=\mathrm{Im}(F_1\oplus F_2\to x_i), i=1,\ldots,nb+1$.

Proof. Let D be the maximal torus

$$\left\{(t_1,\ldots,t_{n_1},l_1,\ldots,l_{n_2})\,\middle|\,\left(\prod_{i=1}^{n_1}t_i\prod_{i=1}^{n_2}l_i\right)=1\right\}$$

of $SL_{n_1+n_2}(\simeq SL(E_1\oplus E_2))$, where $n_1=\dim E_1$ and $n_2=\dim E_2$. Note that $D\subset(GL_{n_1}\times GL_{n_2})\cap SL_{n_1+n_2}(\simeq G)$. Thus D is also a maximal torus of G. Note that any one-parameter subgroup of D is of the form

$$\left\{(t^{a_1},\ldots,t^{a_{n_1}},t^{b_1}\ldots,t^{b_{n_2}})\,|\,t\in k^*,a_i,b_i\in\mathbb{Z}\quad\text{and}\quad\left(\sum_{i=1}^{n_1}a_i+\sum_{i=1}^{n_2}b_i\right)=0\right\}.$$

Now since any one-parameter subgroup of GL_{n1+n2} can be conjugated to a one-parameter subgroup of the diagonal subgroup of the form

$$\left\{(t^{a_1},\ldots,t^{a_{n1}},t^{b_1},\ldots,t^{b_{n2}})\,\middle|\,\begin{array}{l}t\in k^*,a_i,b_i\in\mathbb{Z},\left(\sum_{i=1}^{n1}a_i+\sum_{i=1}^{n2}b_i\right)=0\\ \text{and } a_1\geqslant\cdots\geqslant a_{n1}\geqslant b_1\cdots\geqslant b_{n2}\end{array}\right\},$$

it follows that any one-parameter subgroup of G can be conjugated to a one-parameter subgroup of the form

$$\left\{(t^{a_1},\ldots,t^{a_{n1}},t^{b_1},\ldots,t^{b_{n2}})\,\middle|\,\begin{array}{l}t\in k^*,a_i,b_i\in\mathbb{Z},\left(\sum_{i=1}^{n1}a_i+\sum_{i=1}^{n2}b_i\right)=0\\ \text{and } a_1\geqslant\cdots\geqslant a_{n_1},b_1\geqslant\cdots\geqslant b_{n2}\end{array}\right\}.$$

On the other hand, we clearly see that a point

$$(x_1,\ldots,x_{nb+1})\in H_2(E_1\oplus E_2)^{nb+1}$$

is semi-stable (resp. stable) for the action of G with respect to the linearization $(\varepsilon_1,\ldots,\varepsilon_1,\varepsilon)$ if and only if for every one-parameter c of G

$$\mu(x_1,\ldots,x_{nb+1};c)=\varepsilon_1\sum_{i=1}^{nb}\mu(x_i;c)+\varepsilon\mu(x_{nb+1};c)\geqslant 0\quad(\text{resp.}>0)$$

(see[11], Proposition (28), Premiére partie. III).

Let c be a one-parameter subgroup of the form

$$\left\{(t^{a_1},\ldots,t^{a_{n_1}},t^{b_1},\ldots,t^{b_{n_2}})\,\middle|\,\begin{array}{l}t\in k^*,a_i,b_i\in\mathbb{Z},\left(\sum_{i=1}^{n_1}a_i+\sum_{i=1}^{n_2}b_i\right)=0\\ \text{and } a_1\geqslant\cdots\geqslant a_{n_1},\ b_1\geqslant\cdots\geqslant b_{n_2}\end{array}\right\}.$$

Then

$$\mu(x_1,\ldots,x_{nb+1};c)\geqslant 0 \quad (\text{resp.} > 0)$$

if and only if for all natural numbers p, p_1, p_2 such that $1\leqslant p<n_1+n_2, 0\leqslant p_1\leqslant n_1$, $0\leqslant p_2\leqslant n_2$ and $p_1+p_2=p$,

$$\mu(x_1,\ldots,x_{nb+1};c_{p,p_1,p_2})\geqslant 0 \quad (\text{resp.} > 0),$$

where

$$c_{p,p_1,p_2}=\Bigg\{\Bigg(\underbrace{t^{n_1+n_2-p},\ldots,t^{n_1+n_2-p}}_{p_1},\underbrace{t^{-p},\ldots,t^{-p}}_{n_1-p_1},$$
$$\underbrace{t^{n_1+n_2-p},\ldots,t^{n_1+n_2-p}}_{p_2},\underbrace{t^{-p},\ldots,t^{-p}}_{n_2-p_2}\Bigg)\Bigg|t\in k^*\Bigg\}.$$

That is, every c of the above form is a positive linear combination of $\{c_{p,p_1,p_2}\}$ and μ is linear on the set of all c of the above form.

Note that if $E_1\oplus E_2\to V$ is a quotient of dimension two, then

$$\mu(V;c_{p,p_1,p_2})=-2(p_1+p_2)+(n_1+n_2)\dim(V'),$$

where $V'=\mathrm{Im}(F_1\oplus F_2\to V)$ with F_1 as the subspace of E_1 spanned by the eigenvectors of c_{p,p_1,p_2} with eigenvalue $t^{n_1+n_2-p}$ and F_2 as the subspace of E_2 spanned by the eigenvectors of c_{p,p_1,p_2} with eigenvalue $t^{n_1+n_2-p}$. Hence

$$\mu(x_1,\ldots,x_{nb+1};c_{p,p_1,p_2})=\varepsilon_1\sum_{i=1}^{nb}(\dim(V_i)(n_1+n_2)-2p)$$
$$+\varepsilon(\dim(V_{nb+1})(n_1+n_2)-2p),$$

where $V_i=\mathrm{Im}(F_1\oplus F_2\to x_i), i=1,\ldots,nb+1$. From this the Lemma follows immediately.

Continuing the proof of Theorem 5.3, note that each point of $\mathrm{Hom}(F_1,F_2)$ gives rise to a triple as follows; if $\phi_{(q_1,q_2)}\in\mathrm{Hom}(F_1,F_2)_{(q_1,q_2)}$, then it defines a homomorphism

$$A:\mathscr{F}|_{q_1\times X_1}(p)\to\mathscr{F}|_{q_2\times X_2}(p)$$

and thus gives rise to a triple $(\mathscr{F}|_{q_1\times X_1},\mathscr{F}|_{q_2\times X_2},\vec{A})$.

DEFINITION 5.1

A point $\phi_{(q_1,q_2)}\in\mathrm{Hom}(F_1,F_2)$ is said to be semistable (resp. stable) if the corresponding triple $(\mathscr{F}|_{q_1\times X_1},\mathscr{F}|_{q_2\times X_2},\vec{A})$ is semi-stable (resp. stable).

Lemma 5.5. *If n is sufficiently large and $H_2(E_1\oplus E_2)^{nb+1}$ is G linearized with respect to the polarization $(\varepsilon_1,\ldots,\varepsilon_1,\varepsilon)$, then*

(1) *A point $\phi_{(q_1,q_2)}\in\mathrm{Hom}(F_1,F_2)$ is semi-stable $\Rightarrow\tau_n(\phi_{(q_1,q_2)})$ is semi-stable for the G action.*
(2) *A point $\phi_{(q_1,q_2)}\in\mathrm{Hom}(F_1,F_2)$ is not semi-stable $\Rightarrow\tau_n(\phi_{(q_1,q_2)})$ is not semi-stable for the G action.*
(3) *$\tau_n(\phi_{(q_1,q_2)})$ is stable for the G action if and only if $\phi_{(q_1,q_2)}$ is stable.*

Proof. Let $\phi_{(q_1,q_2)}$ be stable. Then to show $\tau_n(\phi_{(q_1,q_2)})=(x_1,\ldots,x_{nb+1})$ is stable for the G action, by Lemma 5.4, we need to show that for any subvector space of the form $K_1\oplus K_2$ with

$K_1 \subset E_1, K_2 \subset E_2$ and $K_1 \oplus K_2 \neq E_1 \oplus E_2$, the real number

$$-(\dim K_1 + \dim K_2)(2nb\varepsilon_1 + 2\varepsilon) + (\dim E_1 + \dim E_2)(\varepsilon_1 \textstyle\sum_{i=1}^{nb} \dim V_i + \varepsilon \dim V_{nb+1}) > 0, \qquad (*)$$

where $V_i = \mathrm{Im}(K_1 \oplus K_2 \to x_i)\, i = 1, \ldots, nb+1$.

Let $K_1 \oplus K_2$ with $K_1 \subset E_1, K_2 \subset E_2$ and $K_1 \oplus K_2 \neq E_1 \oplus E_2$ be given. Assume that there exists a subtriple $(\mathscr{G}_1, \mathscr{G}_2, \vec{B})$ of $(\mathscr{F}|_{q_1 \times X_1} \otimes \mathscr{L}_1^{\otimes -m}, \mathscr{F}|_{q_2 \times X_2} \otimes \mathscr{L}_2^{\otimes -m}, \overrightarrow{A \otimes \lambda^{\otimes -m}})$ such that

(i) $\mathscr{G}_i \subset \mathscr{F}|_{q_i \times X_i} \otimes \mathscr{L}_i^{\otimes -m}$ is a subbundle for $i = 1, 2$.
(ii) $H^0(\mathscr{G}_1 \otimes \mathscr{L}_1^{\otimes m}) = K_1,\ H^0(\mathscr{G}_2 \otimes \mathscr{L}_2^{\otimes m}) = K_2$.
(iii) $H^1(\mathscr{G}_1 \otimes \mathscr{L}_1^{\otimes m}) = 0,\ H^1(\mathscr{G}_2 \otimes \mathscr{L}_2^{\otimes m}) = 0$.

By assumption $(\mathscr{F}|_{q_1 \times X_1}, \mathscr{F}|_{q_2 \times X_2}, \vec{A})$ is stable, hence we get

$$\dim K_1 + \dim K_2 - \mathrm{rk}\mathscr{G}_2 < (a_1 \mathrm{rk}\mathscr{G}_1 + a_2 \mathrm{rk}\mathscr{G}_2)\left(\frac{\dim E_1 + \dim E_2}{2} - 1\right). \qquad (**)$$

On the other hand, we see that $V_i = \mathrm{rk}\mathscr{G}_1\ (1 \leqslant i \leqslant na)$ and $V_i = \mathrm{rk}\mathscr{G}_2 (na+1 \leqslant i \leqslant nb)$. Hence the L.H.S. of $(*)$ is equal to

$$-(\dim K_1 + \dim K_2)(2nb\varepsilon_1 + 2\varepsilon) + (n_1 + n_2)(na\varepsilon_1 \mathrm{rk}\mathscr{G}_1 + \varepsilon_1(nb - na)\mathrm{rk}\mathscr{G}_2 + \varepsilon \dim V_{nb+1}).$$

Dividing this by $2nb$ and recalling that $a_1 = a/b$ and $a_2 = 1 - a_1 = (b-a)/b$, we get that inequality $(*)$ which is equivalent to the inequality

$$-(\dim K_1 + \dim K_2)\left(\varepsilon_1 + \frac{\varepsilon}{nb}\right) + \frac{n_1 + n_2}{2}\left(\varepsilon_1 a_1 \mathrm{rk}\mathscr{G}_1 + \varepsilon_1 a_2 \mathrm{rk}\mathscr{G}_2 + \frac{\varepsilon}{nb}\dim V_{nb+1}\right) > 0.$$

Now set $\varepsilon = 2nb/(n_1 + n_2)$. Then the above inequality becomes

$$-(\dim K_1 + \dim K_2)\varepsilon_1 - \frac{\dim K_1 + \dim K_2}{(n_1 + n_2)/2} + \frac{n_1 + n_2}{2}(\varepsilon_1 a_1 \mathrm{rk}\mathscr{G}_1 + \varepsilon_1 a_2 \mathrm{rk}\mathscr{G}_2 + \dim V_{nb+1}) > 0. \qquad (***)$$

Note that

$$\frac{\dim K_1 + \dim K_2}{(n_1 + n_2)/2} = \frac{d(\mathscr{G}_1) + (\mathrm{rk}\mathscr{G}_1)ma + (\mathrm{rk}\mathscr{G}_1)(1 - g_1) + d(\mathscr{G}_2) + (\mathrm{rk}\mathscr{G}_2)m(b-a) + (\mathrm{rk}\mathscr{G}_2)(1 - g_2)}{(d(\mathscr{G}_1)/2) + ma + (1 - g_1) + (d(\mathscr{G}_2)/2) + m(b-a) + (1 - g_2)}$$

$$= \frac{\dfrac{d(\mathscr{G}_1) + (\mathrm{rk}\mathscr{G}_1)(1 - g_1)}{mb} + \dfrac{d(\mathscr{G}_2) + (\mathrm{rk}\mathscr{G}_2)(1 - g_2)}{mb} + (\mathrm{rk}\mathscr{G}_1)a_1 + (\mathrm{rk}\mathscr{G}_2)a_2}{\dfrac{d(\mathscr{F}_1) + 2(1 - g_1) + d(\mathscr{F}_2) + 2(1 - g_2)}{2mb} + 1},$$

where $\mathscr{F}_i = \mathscr{F}|_{q_i \times X_i} \otimes \mathscr{L}_i^{\otimes -m}$ $(i=1,2)$, $d(\mathscr{F}_i) =$ degree $(\mathscr{F}_i)$ $(i=1,2)$ and $d(\mathscr{G}_i) =$ degree $(\mathscr{G}_i)$ $(i=1,2)$.

Now set

$$\frac{1}{\varepsilon_1} = \frac{d(\mathscr{F}_1) + 2(1-g_i) + d(\mathscr{F}_2) + 2(1-g_2)}{2mb} + 1.$$

Now dividing the inequality (∗∗∗) by ε_1 we get

$$\left(\frac{n_1+n_2}{2} - 1\right)(a_1 \mathrm{rk}\mathscr{G}_1 + a_2 \mathrm{rk}\mathscr{G}_2) + \dim V_{nb+1} - (\dim K_1 + \dim K_2)$$
$$+ \dim V_{nb+1}\left(\frac{d(\mathscr{F}_1) + 2(1-g_1) + d(\mathscr{F}_2) + 2(1-g_2)}{2mb}\right)$$
$$-\left(\frac{d(\mathscr{G}_1) + (\mathrm{rk}\mathscr{G}_1)(1-g_1)}{mb} + \frac{d(\mathscr{G}_2) + (\mathrm{rk}\mathscr{G}_2)(1-g_2)}{mb}\right) > 0.$$

Now our assumptions on the subtriple $(\mathscr{G}_1, \mathscr{G}_2, \vec{B})$ of

$$(\mathscr{F}|_{q_1 \times X_1} \otimes \mathscr{L}_1^{\otimes -m}, \mathscr{F}|_{q_2 \times X_2} \otimes \mathscr{L}_2^{\otimes -m}, \overrightarrow{A \otimes \lambda^{\otimes -m}}$$

imply that $\dim V_{nb+1} = \mathrm{rk}\mathscr{G}_2$, hence the above inequality is a consequence of the inequality (∗∗) and Theorem 5.1.

The remaining part of the proof of the lemma is similar to the proof of the Proposition 4.2 of [5].

Proof of Theorem 5.3 (*continued*). For sufficiently large n it can be proved that (proof is similar to the proof of Theorem (5.6) of [8] and also see [11])

$$\tau_n : \mathrm{Hom}(F_1, F_2) \to H_2(E_1 \oplus E_2)^{nb+1}$$

is injective and by Lemma 5.5, it follows that the restriction to semistable points induces a morphism

$$\tau_n^{ss} : \mathrm{Hom}(F_1, F_2)^{ss} \to (H_2(E_1 \oplus E_2)^{nb+1})^{ss},$$

which maps the set of stable points $\mathrm{Hom}(F_1, F_2)^s$ into the set of stable points $(H_2(E_1 \oplus E_2)^{nb+1})^s$. Thus $\mathrm{Hom}(F_1, F_2)//G = \mathrm{Hom}(F_1, F_2)^{ss}/G$ exists by [6]. Since the normalization of this quotient can be identified with an irreducible component of the moduli space of rank 2 torsion free sheaves on X, it is irreducible and projective by ([11], septième partie, III, Théorèm 15). This is the required variety $M(2, a_1, \vec{\chi})$. Moreover, when χ is odd, we see that "semistable" = "stable", hence the variety $M(2, a_1, \vec{\chi})$ is smooth.

Note. In the proof of the above theorem, we used the fact that the moduli space of torsion free sheaves is proper to conclude that the moduli space of triples $M(2, a_1, \vec{\chi})$ is proper. We believe that, using methods similar to the Langton's proof of properness of moduli of semi-stable vector bundles on a smooth curve, one can also directly prove the properness of the moduli space of triples.

6. Another proof of Theorem 4.1

Note that by Theorem 3.1, if $\mathscr{F}$ is a torsion free (a_1, a_2) semi-stable sheaf on X of rank two and Euler characteristic χ, then there exists a unique triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ (resp. $(\mathscr{F}'_1, \mathscr{F}'_2, \overleftarrow{B})$) such that $\mathscr{F}_i$ (resp. $\mathscr{F}'_i$) is locally free of rank 2 on X_i $(i = 1, 2)$, satisfying

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 2 \text{ and } a_2\chi < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$$
$$(\text{resp. } a_1\chi < \chi_{X_1}(\mathscr{F}'_1) < a_1\chi + 2 \text{ and } a_2\chi < \chi_{X_2}(\mathscr{F}'_2) < a_2\chi + 2)$$

and

$$\chi_X(\mathscr{F}) + 2 = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_1}(\mathscr{F}_2) (\text{resp. } \chi_X(\mathscr{F}) + 2 = \chi_{X_1}(\mathscr{F}'_1) + \chi_{X_1}(\mathscr{F}'_2)$$

with

$$\vec{A}: \mathscr{F}_1(p) \to \mathscr{F}_2(p) (\text{resp. } \overleftarrow{B}: \mathscr{F}'_2(p) \to F'_1(p)),$$

a nonzero linear map. Moreover,

(a) if $\mathscr{F}$ is locally free, then $\vec{A}$ is invertible and $\mathscr{F}_i = \mathscr{F}'_i (i = 1, 2)$ with $\overleftarrow{B} = (\vec{A})^{-1}$,
(b) if $\mathscr{F}$ is not locally free, then $\vec{A}$ (resp. $\overleftarrow{B}$) has rank one and

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1 \text{ and } a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$$
$$(\text{resp. } a_1\chi + 1 < \chi_{X_1}(\mathscr{F}'_1) < a_1\chi + 2 \text{ and } a_2\chi < \chi_{X_2}(\mathscr{F}'_2) < a_2\chi + 1).$$

The triples $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ and $(\mathscr{F}'_1, \mathscr{F}'_2, \overleftarrow{B})$ are related by the following diagram:

$$\begin{array}{ccc} \mathscr{F}_1(p) & \xrightarrow{i(p)} & \mathscr{F}'_1(p) \\ \downarrow \vec{A} & & \overleftarrow{B}\uparrow \\ \mathscr{F}_2(p) & \xleftarrow{j(p)} & \mathscr{F}'_2(p) \end{array},$$

where $j: \mathscr{F}'_2 \to \mathscr{F}_2$ (resp. $i: \mathscr{F}_1 \to \mathscr{F}'_1$) is the canonical Hecke modification such that $\mathrm{Im}(j(p)) = \mathrm{Im}(\vec{A})$ (resp. $\mathrm{Ker}(i(p)) = \mathrm{Ker}(\vec{A})$) and $\mathrm{Ker}(j(p)) = \mathrm{Ker}(\overleftarrow{B})$ (resp. $\mathrm{Im}(i(p)) = \mathrm{Im}(\overleftarrow{B})$ (see Remark 2.4).

Let $M_1 = M(2, a_1, \bar{\chi})$ be the moduli space of (a_1, a_2)-semi-stable triples $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ of rank two with

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1 \text{ and } a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$$

and $\chi_X(\mathscr{F}) + 2 = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2)$ (this space is constructed in Theorem 5.3). Let $M_2 = M(2, a_1, \bar{\chi})$ be the moduli of (a_1, a_2) semi-stable triples $(\mathscr{F}_1, \mathscr{F}_2, \overleftarrow{B})$ of rank two with

$$a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 2 \text{ and } a_2\chi < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 1$$

and $\chi_X(\mathscr{F}) + 2 = \chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2)$ (the construction of M_2 is exactly similar to that of M_1). Let

$$M'_1 = \{[(\mathscr{F}_1, \mathscr{F}_2, \vec{A})] \in M_1 | \mathrm{rk}(\vec{A}) = 1\}$$

and

$$M'_2 = \{[(\mathscr{F}'_1, \mathscr{F}'_2, \overleftarrow{B})] \in M_2 | \mathrm{rk}(\overleftarrow{B}) = 1\}.$$

From (b) above, we observe that there is a natural isomorphism $M'_1 \to M'_2$. Now by lemma 2.3 and Remark 2.9, it follows that $M(2, (a_1, a_2), \chi)$ is isomorphic to $M_1 \cup M_2$ with the natural identification of the closed subscheme M'_1 of M_1 with the closed subscheme M'_2 of M_2. Moreover, when χ is odd, both M_1 and M_2 are smooth (see Theorem 5.3) and M'_1 (resp. M'_2) is a smooth closed subvariety of M_1 (resp. M_2) (see Theorem 6.1 below). Thus, if χ is odd, $M(2, (a_1, a_2), \chi)$ is a normal crossing variety. This proves the theorem.

Theorem 6.1. *Notations are as above. Let P_1 be the moduli space of parabolic semi-stable ($=$ parabolic stable) bundles $(\mathscr{F}_1, 0 \subset F^2\mathscr{F}_1(p) \subset \mathscr{F}_1(p))$ of rank two on X_1 with parabolic weight $(a_1/2, a_2/2)$ (if $a_1 < a_2$, otherwise $(a_2/2, a_1/2)$) and degree of $\mathscr{F}_1$ equal to χ_1, where χ_1, is an integer satisfying $a_1\chi < \chi_1 < a_1\chi + 1$ with χ as in the theorem above. Similarly let P_2 be the moduli space of parabolic semi-stable ($=$ parabolic stable) bundles $(\mathscr{F}_2, 0 \subset F^2\mathscr{F}_2(p) \subset \mathscr{F}_2(p))$ of rank two on X_2 with parabolic weight $(a_1/2, a_2/2)$ (if $a_1 < a_2$, otherwise $(a_2/2, a_1/2)$) and degree of $\mathscr{F}_2$ equal to χ_2, where χ_2, is an integer satisfying $a_2\chi + 1 < \chi_2 < a_2\chi + 2$ with χ as above. Moreover, we see that P_1 and P_2 are smooth. By sending $[(\mathscr{F}_1, \mathscr{F}_2, \vec{A})] \in M'_1$ to $(\mathscr{F}_1, 0 \subset F^2\mathscr{F}_1(p) \subset \mathscr{F}_1(p)) \times (\mathscr{F}_2, 0 \subset F^2\mathscr{F}_2(p) \subset \mathscr{F}_2(p)) \in P_1 \times P_2$, with $F^2\mathscr{F}_1(p) = \mathrm{Ker}(\vec{A})$ and $F^2\mathscr{F}'_1(p) = \mathrm{Im}(\vec{A})$, we can identify M'_1 with $P_1 \times P_2$. Similarly we can identify M'_2 also with $P_1 \times P_2$.*

Proof. Let $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ be a semi-stable triple such that $[(\mathscr{F}_1, \mathscr{F}_2, \vec{A})] \in M'_1$. Then we prove that $(\mathscr{F}_1, 0 \subset F^2\mathscr{F}_1(p) \subset \mathscr{F}_1(p))$(resp. $(\mathscr{F}_2, 0 \subset F^2\mathscr{F}_2(p) \subset \mathscr{F}_2(p))$) is parabolic stable on X_1 (resp. X_2) with respect to the weights (a_1, a_2). By Theorem (5.1) we know that $\mathscr{F}_1$ (resp. $\mathscr{F}_2$) is semi-stable on X_1 (resp. X_2). Thus for any line subbundle $\mathscr{L}_1$ (resp. $\mathscr{L}_1$) of $\mathscr{F}_1$ (resp. $\mathscr{F}_2$) we have (assuming $a_1 < a_2$)

$$\chi(\mathscr{L}_1) + a_1/2 < \chi(\mathscr{F}_1)/2 + 1/4 \text{ (resp. } \chi(\mathscr{L}_2) + a_1/2 < \chi(\mathscr{F}_2)/2 + 1/4).$$

Thus, if either $\mathscr{L}_1(p) \neq F^2\mathscr{F}_1(p)$ or $\chi(\mathscr{L}_1) < \frac{1}{2}\chi(\mathscr{F}_1)$ (resp. if either $\mathscr{L}_2(p) \neq F^2\mathscr{F}_2(p)$ or $\chi(\mathscr{L}_2) < \frac{1}{2}\chi(\mathscr{F}_2)$) we see that par deg $(\mathscr{L}_1) <$ par $\mu(\mathscr{F}_1)$(resp. par deg$(\mathscr{L}_2) <$ par $\mu(\mathscr{F}_2)$). We next show that there is no subbundle $\mathscr{L}_1$ of $\mathscr{F}_1$ such that $\chi(\mathscr{L}_1) = \chi(\mathscr{F}_1)/2$ and $\mathscr{L}_1(p) = F^2\mathscr{F}_1(p)$ (resp. $\mathscr{L}_2$ of $\mathscr{F}_2$ such that $\chi(\mathscr{L}_2) = \chi(\mathscr{F}_2)/2$ and $\mathscr{L}_2(p) = F^2\mathscr{F}_2(p)$). If such a line-bundle $\mathscr{L}_1$ exists, then $(\mathscr{L}_1, 0, \vec{0})$ is a subtriple of $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ but semi-stability of the triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ implies $\chi(\mathscr{L}_1) \leqslant a_1\chi/2$. Thus $\chi(\mathscr{F}_1) \leqslant a_1\chi$, which contradicts the fact $a_1\chi < \chi(\mathscr{F}_1)$. (The proof that there is no line subbundle $\mathscr{L}_2$ of $\mathscr{F}_2$ such that $\chi(\mathscr{L}_2) = \chi(\mathscr{F}_2)/2$ and $\mathscr{L}_2(p) = F^2\mathscr{F}_2(p)$ is similar.)

This implies that we get a morphism from the moduli space of triples $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ with $\mathrm{rk}(\vec{A}) = 1$ into the product of parabolic moduli spaces under consideration. This moduli space of triples is proper, being a closed subset of the moduli space of all semi-stable triples. Further the above morphism is injective and both the spaces have the same dimension. Hence the morphism is bijective and the required implication follows. In fact, the required assertion can be checked directly by a case-by-case analysis.

7. 'det' of semi-stable torsion free sheaf of rank 2 on X

Notations are as in the previous sections. Let $\mathscr{F}$ be a rank two, (a_1, a_2)-semi-stable torsion free sheaf on X. Assume that $\chi = \chi(\mathscr{F}) \neq 0$ and $a_1\chi$ is not an integer. If $\mathscr{F}$ is a vector bundle of rank two, then its second exterior power $\overset{2}{\wedge}(\mathscr{F})$ is a line bundle denoted by $\det(\mathscr{F})$. Moreover, if $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ (resp. $(\mathscr{F}'_1, \mathscr{F}'_2, \overleftarrow{B})$) is the triple representing $\mathscr{F}$, then we see that $\det(\mathscr{F})$ is represented by the triple $(\overset{2}{\wedge}(\mathscr{F}_1), \overset{2}{\wedge}(\mathscr{F}_2), \overrightarrow{\det(A)})$ (resp. $(\overset{2}{\wedge}(\mathscr{F}'_1), \overset{2}{\wedge}(\mathscr{F}'_2), \overleftarrow{\det(B)})$). Note that in this case $\mathscr{F}'_1 \simeq \mathscr{F}_1$ and $\mathscr{F}'_2 \simeq \mathscr{F}_2$ and $B = A^{-1}$. If $\mathscr{F}$ is not a vector bundle, then we see by Theorem 3.1 that $\mathscr{F}_p \simeq \mathcal{O}_{X,p} \oplus m_{X,p}$ and $\mathscr{F}_x \simeq \mathcal{O}_{X,x} \oplus \mathcal{O}_{X,x}$ for all $x \in X - p$. Thus $\overset{2}{\wedge}(\mathscr{F})_p/(\text{torsion}) \simeq m_{X,p}$ and $\overset{2}{\wedge}(\mathscr{F})_x \simeq \mathcal{O}_{X,x}$ for all $x \in X - p$. Also, when $\mathscr{F}$ is not a vector bundle, we see that if $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$

(resp. $(\mathscr{F}'_1, \mathscr{F}'_2, \overleftarrow{B})$) is the triple representing $\mathscr{F}$, then $\overset{2}{\wedge}(\mathscr{F})/(\text{torsion})$ is represented by the triple $(\overset{2}{\wedge}\mathscr{F}_1, \overset{2}{\wedge}\mathscr{F}_2, \vec{0})$ (resp. $(\overset{2}{\wedge}\mathscr{F}'_1, \overset{2}{\wedge}\mathscr{F}'_2, \overleftarrow{0})$). Note that in this case we have the following exact sequences

$$0 \to \mathscr{F}_1 \to \mathscr{F}'_1 \to k(p) \to 0,$$

$$0 \to \mathscr{F}'_2 \to \mathscr{F}_2 \to k(p) \to 0,$$

where $k(p)$ denotes the residue field at p. Thus we see that $\overset{2}{\wedge}\mathscr{F}'_1 \simeq \overset{2}{\wedge}\mathscr{F}_1 \otimes \mathscr{O}_{X_1}(p)$ and $\overset{2}{\wedge}\mathscr{F}'_2 \simeq \overset{2}{\wedge}\mathscr{F}_2 \otimes \mathscr{O}_{X_2}(-p)$. Consider the triple $(\overset{2}{\wedge}\mathscr{F}_1, \overset{2}{\wedge}\mathscr{F}_2, \vec{\lambda})$ (resp. $(\overset{2}{\wedge}\mathscr{F}'_1, \overset{2}{\wedge}\mathscr{F}'_2, \overleftarrow{\mu})$), where λ (resp. μ) is an isomorphism of one dimensional vector spaces $\overset{2}{\wedge}\mathscr{F}_1(p) \to \overset{2}{\wedge}\mathscr{F}_2(p)$ (resp. $\overset{2}{\wedge}\mathscr{F}'_1(p) \to \overset{2}{\wedge}\mathscr{F}'_2(p)$). If L_1 (resp. L_2) is the line bundle on X associated to the triple $(\overset{2}{\wedge}\mathscr{F}_1, \overset{2}{\wedge}\mathscr{F}_2, \vec{\lambda})$, resp. $((\overset{2}{\wedge}\mathscr{F}'_1, \overset{2}{\wedge}\mathscr{F}'_2, \overleftarrow{\mu}))$, then

$$L_1 \in J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2)$$

$$(\text{resp. } L_2 \in J_{\chi_1+1-(1-g_1)}(X_1) \times J_{\chi_2-1-(1-g_2)}(X_2)),$$

where χ_1, χ_2 are integers satisfying $a_1\chi < \chi_1 < a_1\chi + 1$, $a_2\chi + 1 < \chi_2 < a_2\chi + 2$, with $\chi_1 + \chi_2 - 2 = \chi$ and $J_{\chi_i-(1-g_i)}(X_i)$ is the Jacobian of line bundles on X_i of Euler characteristics $\chi_i - (1-g_i)$ for $i = 1, 2$. Also, note that under the natural isomorphism

$$J_{\chi_1+1-(1-g_1)}(X_1) \times J_{\chi_2-1-(1-g_2)}(X_2) \to J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2)$$

$$(\mathscr{L}_1, \mathscr{L}_2) \mapsto (\mathscr{L}_1 \otimes \mathscr{O}_{X_1}(-p), \mathscr{L}_2 \otimes \mathscr{O}_{X_2}(p)),$$

L_2 maps to L_1. Moreover,

$$\chi(\overset{2}{\wedge}\mathscr{F}/(\text{torsion})) = \chi(L_1) = \chi(L_2)$$

(see Remark 2.12).

Let $M(2,(a_1,a_2),\chi \neq 0)^0$ be the moduli space of (a_1,a_2)-semi-stable rank two vector bundles on X with Euler characteristic $\chi(\neq 0)$. Assume that $a_1\chi$ is not an integer. From the results of § 2, we see that $M(2,(a_1,a_2),\chi \neq 0)^0 = M_1^0 \cup M_2^0$, where M_1^0 (resp. M_2^0) is the moduli space of (a_1,a_2)-semi-stable vector bundles $\mathscr{F}$ on X such that

$$a_1\chi < \chi_{X_1}(\mathscr{F}|_{X_1}) < a_1\chi + 1$$

$$(\text{resp. } a_1\chi + 1 < \chi_{X_1}(\mathscr{F}|_{X_1}) < a_1\chi + 2)$$

and $a_2\chi + 1 < \chi_{X_2}(\mathscr{F}|_{X_2}) < a_2\chi + 2$ (resp. $a_2\chi < \chi_{X_2}(\mathscr{F}|_{X_2}) < a_2\chi + 1$).
Thus we have a morphism

$$(\det)'_1 : M_1^0 \to J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2)$$

$$(\text{resp. } (\det)'_2 : M_2^0 \to J_{\chi_1+1-(1-g_1)}(X_1) \times J_{\chi_2-1-(1-g_2)}(X_2))$$

where χ_1, χ_2 are integers satisfying

$$a_1\chi < \chi_1 < a_1\chi + 1, \quad a_2\chi + 1 < \chi_2 < a_2\chi + 2,$$

with $\chi_1 + \chi_2 - 2 = \chi$. Now using the isomorphism

$$J_{\chi_1+1-(1-g_1)}(X_1) \times J_{\chi_2-1-(1-g_2)}(X_2) \to J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2)$$

$$(\mathscr{L}_1, \mathscr{L}_2) \mapsto (\mathscr{L}_1 \otimes \mathscr{O}_{X_1}(-p), \mathscr{L}_2 \otimes \mathscr{O}_{X_2}(p)),$$

we get a morphism

$$(\det) : M(2,(a_1,a_2),\chi \neq 0)^0 \to J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2).$$

Now the arguments of the previous paragraph show that the morphism

$$(\det): M(2,(a_1,a_2),\chi \neq 0)^0 \to J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2)$$

can be naturally extended to a set theoretic map from the moduli space $M(2,(a_1,a_2),\chi \neq 0)$ of (a_1,a_2)- semi-stable rank two torsion free sheaves on X with Euler characteristics $\chi(\neq 0)$.

PROPOSITION 7.1

The map

$$(\det): M(2,(a_1,a_2),\chi \neq 0) \to J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2)$$

is a morphism.

Proof. The proof follows from the following two facts:
(a) $\mathscr{F}$ is a torsion free sheaf of rank two on X with Euler characteristic χ, then

$$\chi(\overset{2}{\wedge}/(\text{torsion})) = \chi - (1-g).$$

(This can be proved considering the associated triple and the Remark 2.7).
(b) Let $\mathscr{F}_0$ be a (a_1, a_2)-semi-stable rank two torsion free non-free sheaf on X with Euler characteristic $\chi(\neq 0)$. Let $\mathscr{G}$ be a coherent sheaf on $S \times X$ which is flat over the base scheme S (finite type over the field $\mathbb{C}$ of complex numbers) and such that for all $s \in S, \mathscr{G}|_{s\times X}$ is an (a_1,a_2)-semi-stable rank two torsion free sheaf on X with Euler characteristic $\chi(\neq 0)$ and for the closed point $s_0 \in S, \mathscr{G}|_{s_0\times X} \simeq \mathscr{F}_0$. Let T be the maximal subsheaf of modules of $\overset{2}{\wedge}\mathscr{G}$ with proper support. Then $\overset{2}{\wedge}\mathscr{G}/T$ is a pure sheaf of modules on $S \times X$. The natural surjection

$$(\overset{2}{\wedge}\mathscr{G}/T)|_{s\times X} \to \overset{2}{\wedge}(\mathscr{G}|_{s\times X})$$

induces an isomorphism

$$(\overset{2}{\wedge}\mathscr{G}/T)|_{s\times X} \simeq (\overset{2}{\wedge}(\mathscr{G}|_{s\times X}))/(\text{torsion}),$$

for all $s \in S$. Thus $\overset{2}{\wedge}\mathscr{G}/T$ is a flat family of torsion free sheaves of rank one and Euler characteristic $\chi - (1-g)$ with the property

$$(\overset{2}{\wedge}\mathscr{G}/T)|_{s_0\times X} \simeq \overset{2}{\wedge}(\mathscr{G}|_{s_0\times X})/(\text{torsion}).$$

PROPOSITION 7.2

The fibres of the morphism

$$(\det): M(2,(a_1,a_2),\chi \neq 0) \to J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2)$$

are all reduced.

Proof. Note that any $[\mathscr{F}] \in S((a_1,a_2),(2,2),\chi \neq 0)$ is represented by a triple $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ with the property $a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 2, a_2\chi < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2, \mathrm{rk}(\vec{A}) \geqslant 1$ and $\chi_{X_1}(\mathscr{F}_1) + \chi_{X_2}(\mathscr{F}_2) - 2 = \chi$. Let $L_1 \in J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2)$ be represented by $(\mathscr{L}_1, \mathscr{L}_2, \vec{\lambda})$. Let $\mathscr{F} \in S((a_1,a_2),(2,2),\chi \neq 0)$ be such that $a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1$, $a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$ and 'det'$(\mathscr{F}) = L_1$. Then note that for all such $\mathscr{F}$ there exists a fixed integer n such that $\mathscr{F}(n)$ admits a trivial line subbundle. If $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$

is the triple associated to such a torsion free sheaf $\mathscr{F}$, then we see that

$$\mathscr{F}_1 \in \mathrm{Ext}^1_{\mathcal{O}_{X_1}}(\mathscr{L}_1(na), \mathcal{O}_{X_1}(-na)), \mathscr{F}_2 \in \mathrm{Ext.}^1_{\mathcal{O}_{X_2}}(\mathscr{L}_2(n(b-a)), \mathcal{O}_{X_2}(-n(b-a))).$$

Thus $(\mathscr{F}_1, \mathscr{F}_2, \vec{A})$ is isomorphic to a triple given by a point of the rank four vector bundle $\mathrm{Hom}\,(p_1^*(F_1), p_2^*(F_2))$ on

$$\mathrm{Ext}^1_{\mathcal{O}_{X_1}}(\mathscr{L}_1(na), \mathcal{O}_{X_1}(-na)) \times \mathrm{Ext}^1_{\mathcal{O}_{X_2}}(\mathscr{L}_2(n(b-a)), \mathcal{O}_{X_2}(-n(b-a))),$$

where F_1 (resp. F_2) is the restriction to $\mathrm{Ext}^1_{\mathcal{O}_{X_1}}(\mathscr{L}_1(na), \mathcal{O}_{X_1}(-na)) \times p$ (resp. $\mathrm{Ext}^1_{\mathcal{O}_{X_2}}(\mathscr{L}_2(n(b-a)), \mathcal{O}_{X_2}(-n(b-a))) \times p$) of the universal rank two vector bundle on $\mathrm{Ext}^1_{\mathcal{O}_{X_1}}(\mathscr{L}_1(na), \mathcal{O}_{X_1}(-na)) \times X_1$ (resp. $\mathrm{Ext}^1_{\mathcal{O}_{X_2}}(\mathscr{L}_2(n(b-a))), \mathcal{O}_{X_2}(-n(b-a))) \times X_2$). Hence there is a surjective morphism from an open subvariety of the smooth variety $\mathrm{Hom}\,(p_1^*(F_1), p_2^*(F_2))$ to the variety of all $\mathscr{F} \in M(2, (a_1, a_2), \chi \neq 0)$ such that $a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1$, $a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$ and 'det'$(\mathscr{F}) = L_1$. Thus the variety of all $\mathscr{F} \in M(2, (a_1, a_2), \chi \neq 0)$ such that $a_1\chi < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 1, a_2\chi + 1 < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 2$ and 'det'$(\mathscr{F}) = L_1$, is reduced and irreducible. Similarly, the variety of all $\mathscr{F} \in M(2, (a_1, a_2), \chi \neq 0)$ such that $a_1\chi + 1 < \chi_{X_1}(\mathscr{F}_1) < a_1\chi + 2, a_2\chi < \chi_{X_2}(\mathscr{F}_2) < a_2\chi + 1$ and 'det'$(\mathscr{F}) = L_1$ is reduced and irreducible. This proves that the fibres of 'det' morphism are all reduced.

Remark 7.1. The 'det' morphism has good specialization properties in the folowing sense:
Let $S = \mathrm{Spec}(R)$ be a spectrum of a complete discrete valuation ring R with residue field $\mathbb{C}$ and x_0 be its closed point. Let $\mathscr{X} \to S$ be a flat family of curves with χ a two dimensional regular scheme and $\mathscr{X}|_{x_0} \simeq X$. Choose a relative ample line bundle $\mathcal{O}_\chi(1)$ on $\chi \to S$ such that its restriction to the special fibre induces the polarization of type (a_1, a_2). Then the moduli scheme

$$\mathscr{M}(2, \mathcal{O}_{\mathscr{X}}, \chi \neq 0) \to S$$

of rank two torsion free sheaves along the fibres with Euler characteristic χ and semi-stable with respect to $\mathcal{O}(1)$, exists as a flat scheme over S (see Theorem 4.2). By the results of §4, we see that $\mathscr{M}(2, \mathcal{O}_{\mathscr{X}}, \chi \neq 0)$ is an integral scheme flat over S and regular if χ is odd. Moreover,

$$\mathscr{M}(2, \mathcal{O}_{\mathscr{X}}, \chi \neq 0)|_{x_0} \simeq \mathscr{M}(2, (a_1, a_2), \chi \neq 0).$$

Similarly, there exists a flat family

$$J(\mathscr{X}) \to S$$

of Jacobians, the fibres of $\mathscr{X} \to S$ of Euler characteristics $\chi - (1-g)$ and

$$J(\mathscr{X}|_{x_0}) \simeq J_{\chi_1 - (1-g_1)}(X_1) \times J_{\chi_2 - (1-g_2)}(X_2).$$

Using the results proved above we see that there is a natural morphism

$$(\det)/S : \mathscr{M}(2, \mathcal{O}_{X_1}, \chi \neq 0)/S \to J(\mathscr{X})/S$$

of S schemes such that

$$(\det)/S|_{x_0} : \mathscr{M}(2, \mathcal{O}_{\mathscr{X}}, \chi \neq 0)/S|_{x_0} \to J(\mathscr{X})/S|_{x_0}$$

is same as the morphism

$$(\det): \mathcal{M}(2,(a_1,a_2),\chi \neq 0) \to J_{\chi_1-(1-g_1)}(X_1) \times J_{\chi_2-(1-g_2)}(X_2).$$

Remark 7.2. It follows from the above remark that if we have a family such that a smooth curve Y of genus g specializes to the curve X and a line bundle $\mathcal{L}$ (with $\chi(\mathcal{L}) \neq g-1$) on Y specializes to a line bundle L, then the moduli space of rank two vector bundles on Y with fixed determinant $\mathcal{L}$ specializes to the moduli space of rank two torsion free sheaves on X with fixed determinant L. Moreover, this specialization carries the nice properties of the specializations of moduli space of vector bundles (without fixing the determinant) on Y to the moduli space of torsion free sheaves (without fixing the determinant) on X.

8. Fixed determinant moduli space over an irreducible nodal curve

Throughout this chapter, we let X denote an *irreducible* projective nodal curve with exactly one node. Let p be the node of X and $m_{X,p}$, the maximal ideal of X at p.

The notion of a triple in this paper can be suitably modified so that giving a torsion free sheaf $\mathcal{F}$ on X is equivalent a couple $(V,\vec{\phi})$, where V is a vector bundle on the normalization $\tilde{X}$ and $\vec{\phi}: V(p_1) \to V(p_2)$ is a homomorphism of the vector spaces, and $p_1, p_2 \in \tilde{X}$ are the points above the node $p \in X$ and $V(p_1)$, $V(p_2)$ are fibres of V.

Remark 8.1. Let $\mathcal{F}$ be a torsion free, but not free sheaf of rank n and degree d on X and let $(V, \vec{\phi})$ be the associated couple. Note that in this case $\det(\phi) = 0$.

(a) If rank $(\vec{\phi}) = n-1$ (i.e., locally at p, $\mathcal{F}$ is of the form $(\mathcal{O}_{X,p}^{n-1} \oplus m_{X,p})$), then it is easy to see that

$$\overset{n}{\wedge}\mathcal{F}/(\text{torsion}) \simeq L_0$$

where L_0 is the torsion free (but not free) sheaf of rank one of degree d on X associated to the couple $(\overset{n}{\wedge}V, \vec{0})$.

(b) If rank $(\vec{\phi}) < n-1$ (i.e., locally at p, $\mathcal{F}$ is of the form $(\mathcal{O}_{X,p}^{n-r} \oplus m_{X,p}^r)$, $(r \geqslant 2)$), then it is easy to see that for every linear map $\lambda: \overset{n}{\wedge}V(p_1) \to \overset{n}{\wedge}V(p_2)$ we have

$$\overset{n}{\wedge}\mathcal{F}/(\text{torsion}) \subset L_\lambda,$$

where L_λ is the torsion free sheaf of rank one and degree d on X associated to the couple $(\overset{n}{\wedge}V, \vec{\lambda})$.

Let Y be a non-singular projective curve which specializes to X and $n \geq 1$ be an integer. Let $M(Y,n,d)$ (resp. $M(X,n,d)$) denote the moduli space of vector bundles (resp. torsion free sheaves) of rank n and degree d on Y (resp. X) (see [8, 11]). One knows that $M(Y,n,d)$ specializes to $M(X,n,d)$ (for example this follows from the general theory of [13]). If $\mathcal{F}$ is a vector bundle of rank n and degree d on Y (resp. X), then

$$\mathcal{F} \mapsto \det(\mathcal{F}) := \overset{n}{\wedge}\mathcal{F}$$

induces a morphism $\det: M(Y,n,d) \to M(Y,1,d) =: J^d(Y)$ (resp. $\det: M(X,n,d)^0 \to M(X,1,d)^0 =: J^d(X)$, where $M(X,n,d)^0$ denotes the moduli space of vector bundles of rank n and degree d on X, which is an open subvariety of $M(X,n,d)$). For $\mathcal{L} \in J^d(Y)$, set

$$M(Y,n,\mathcal{L}) = \det{}^{-1}(\mathcal{L})$$

and for $L \in J^d(X)$, set

$$M(X,n,L)^0 = \det^{-1}(L)$$

DEFINITION 8.1

Let L be a torsion free sheaf of rank one and degree d on X. Define $M(X,n,L)$ as the set

$$\{[\mathscr{F}] \in M(X,n,d) | (\overset{n}{\wedge}\mathscr{F})/(\text{Torsion}) \subset L \quad \text{and} \quad m_p^n L \subset (\overset{n}{\wedge}\mathscr{F})/(\text{Torsion})\},$$

where (Torsion) denotes the maximal subsheaf of modules of $\overset{n}{\wedge}\mathscr{F}$ with proper support and m_p is the ideal sheaf of the point $p \in X$.

Remark 8.2. In fact we can define a subfunctor of the moduli functor so that this subfunctor is representable and is represented by a subscheme whose support is the set $M(X,n,L)$. Thus it is easy to see that $M(X,n,L)$ can be defined as a closed subscheme of $M(X,n,d)$.

Conjecture. (a) If L is a line bundle on X, then $M(X,n,L)$ is the closure of $M(X,n,L)^0$ in $M(X,n,d)$.
(b) Let $\mathscr{L}_0$ (resp. L_0) be a line bundle (resp. torsion free sheaf of rank one) of degree d on Y (resp. X). Assume that $\mathscr{L}_0$ specializes to L_0 as Y specializes to X. Then $M(X,n,L_0)$ is the specialization of $M(X,n,\mathscr{L}_0)$.

Below we provide some evidence to the conjecture.

Evidence. For simplicity, let n be equal to two. Let $\mathscr{F}$ be a rank two torsion free sheaf of degree d on X. If $\mathscr{F}$ is a vector bundle (resp. torsion free sheaf of type $(\mathscr{O}_{X,p} \oplus m_{X,p})$), then clearly $(\overset{n}{\wedge}\mathscr{F})$(resp. $(\overset{n}{\wedge}\mathscr{F})$/(torsion)) is a line bundle (resp. torsion free sheaf of rank one) of degree d on X. If $\mathscr{F}$ is a torsion free sheaf of type $(m_{X,p} \oplus m_{X,p})$ then $(\overset{n}{\wedge}\mathscr{F})$/(torsion) is a torsion free (but not free) sheaf of rank one and degree $d-1$. Thus for any torsion free sheaf L of rank one and degree d such that $(\overset{2}{\wedge}\mathscr{F})/(\text{torsion}) \subset L$, we see $\mathscr{F} \in M(X,n,L)$: let $\mathscr{F}$ be a torsion free sheaf of type $m_{X,p} \oplus m_{X,p}$ of degree d and let L be a torsion free sheaf of rank one and degree d such that $(\overset{2}{\wedge}\mathscr{F})/(\text{torsion}) \subset L$. Let R be a d.v.r. and let $\mathbf{F}$ be a flat family of torsion free sheaves of rank two and degree d on X parametrized by $S = \mathrm{Spec}(R)$. Assume that for the generic point of $\eta \in S$, $(\overset{2}{\wedge}\mathbf{F}|_{\eta \times X}) \simeq L \otimes Q(R)$ and for the closed point $s \in S$, $\mathbf{F}|_{s \times X} = \mathscr{F}$ where $Q(R)$ is the quotient field of R. Now if $\mathbf{L} = (\overset{2}{\wedge}\mathbf{F})/(\text{Torsion})$, where (Torsion) denotes the maximal subsheaf of modules of $\overset{2}{\wedge}\mathbf{F}$ with proper support, then $\mathbf{L}$ is a flat family of sheaves of rank one and degree d on X and there is an injective homomorphism

$$\mathbf{L} \to \pi_2^*(L),$$

where $\pi_2: S \times X \to X$ is the second projection. Since the surjection $\mathbf{L}|_{s \times X} \to (\overset{2}{\wedge}\mathscr{F})/$(torsion) induces an isomorphism

$$(\mathbf{L}|_{s \times X})/(\text{torsion}) \simeq (\overset{2}{\wedge}\mathscr{F})/(\text{torsion}),$$

it follows that $\mathscr{F} \in M(X,n,L)$, QED.

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References

[1] Artin M, *Deformations of singularities*, Lecture Notes TIFR, Bombay (1976)

[2] Bhosle Usha N, Generalised parabolic bundles and applications to torsionfree sheaves on nodal curves, *Arkiv för matematik*. **30** (1992)

[3] D'Souza C, Compatification of generalised Jacobians, *Proc. Indian Acad. Sci.* **88** (1979) 419–457

[4] Gieseker D, A degeneration of the moduli space of stable bundles, *J.Diff. Geometry* **19** (1984) 173–206

[5] Mehta V B and Seshadri C S, Moduli of vector bundles on curves with parabolic structures, *Math. Ann.* **248** (1980) 205–239

[6] Mumford D, *Geometric Invariant Theory* (Ergebnisse: Springer-Verlag (1965)

[7] Narashiman M S and Ramadas T R, Factorisation of generalised theta functions–I, *Invent. Math* **114** (1993) 565–623

[8] Newstead P E, *Introduction to moduli problems and orbit spaces*, TIFR, Bombay, lecture notes (1978)

[9] Oda T and Seshadri C S, Compactifications of the generalised Jacobian variety. *Trans. Am. Math Soc.* **253** (1979) 1–90

[10] Schessinger M, Functors of Artin rings, *Trans. Am. Math. Soc.* **130** (1968) 208–222

[11] Seshadri C S, Fibrés vectoriels sur les courbes algébriques, *Astérisque* **96** (1982)

[12] Seshadri C S, Space of unitary vector bundles on a compact Riemann surface, *Ann. Math.* **85** (1967) 303–336

[13] Seshadri C S, Geometric reductivity over arbitrary base, *Advent Math.* **26** (1977) 225

[14] Xia Huashi, Degenerations of moduli of stable bundles over algebraic curves, *Compos. Math.* **98** (1995) 305–330