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Limits and models in fluid mechanics*

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Abstract. A framework is presented for examining the effectiveness of limiting and modelling arguments used in the analysis of fluid flows. It consists of examining the consequences of the arguments on the flow problem as a whole and breaking down the limiting/modelling process into a sequence of steps and associated sequence of flow problems, termed here as 'lidels'. The notion of validity of lidels is given. Several examples are given to explain the present approach.

Keywords. Limit; flow model; modelling; fluid mechanics; asymptotic expansions.

1. Introduction

Limiting and modelling arguments are probably the most fascinating elements in the wizardry used in the analyses of flow problems. These flexible instruments enable fluid dynamicists to apply physical and mathematical ingenuity to highlight key features of the problem at hand and to strip it of essentially secondary features. The resulting simplification often facilitates analysis and computation and gives valuable insight into the nature of fluid flows. Occasionally, however, the results are intriguing and it may not be evident whether we can accept them as manifestations of real flows.

A typical limiting argument applied to a governing equation or initial or boundary condition seeks to obtain a statement that is exact in a chosen limit. The statement is also expected to be a good approximation when the parameters are close to the limiting values. The argument can in general be executed formally and systematically by using asymptotic expansions to indicate the conditions under which the resulting statement is a good approximation or to obtain improved approximations. A modelling argument, on the other hand, relies on physical or mathematical judgement, experimental information, and, in some cases, on analogies, and it leads to a statement that is taken as a part of the problem. Unlike limiting arguments, there is usually no indication of conditions under which it is exactly or approximately correct. Also, there are no general systematic ways of constructing improved models. Furthermore, some of the resulting statements may contain parameters or functions that are chosen subsequently depending on the

Invited general lecture at 3rd ACFM, Tokyo, September 1986 particular class of problems one is dealing with. Thus there is generally much greater understanding and agreement about the limiting arguments and the range over which they provide useful approximations, although there have been quite a few surprises when apparently straightforward limiting arguments have been applied to seemingly innocuous problems.

Although these arguments are qualitatively different, it is proposed to consider them together in this work for several reasons. First, they often alter the nature of the problem as a whole and their effectiveness is best judged by the information provided by the modified problem. Second, there are instances when we have to use both limiting and modelling arguments (e.g. turbulent boundary layer) and a common methodology of examining the consequences would be useful. Third, there seems to be enough scope for extending certain ideas such as validity, which have been fruitfully applied to limiting arguments, to the modelling arguments. The last reason is the main motivating factor of the present approach. which, one hopes, would lead to a sharpening of the acceptance standards of modelling arguments.

Let us recall a few classical examples. The low Reynolds number analysis of Stokes, based essentially on a limiting argument, led to the formula for the drag of a spherical particle that has been widely used. However, when a similar argument was applied to a circular cylinder, it led to nonexistence of a solution. This type of difficulty was resolved much later by using methods of singular perturbations inspired by boundary layer analysis.

The problem of steady attached flow past a streamlined body for an incompressible fluid at large Reynolds number was historically first attacked by solving the corresponding inviscid flow problem in two dimensions. An additional condition, known as the Kutta condition, has to be imposed to determine the circulation around the aerofoil. This could be interpreted as the application of limiting arguments coupled with a modelling argument. One could advance limiting arguments to obtain the Kutta condition as a conclusion, although such a systematic argument is not known to the author.

While the correctness of the Kutta condition has never been seriously questioned for the original class of problem for which it was proposed, it should be noted that neither is a higher order Kutta condition available nor is it clear what should be a similar argument or statement when the aerofoil is oscillating or when there is an oscillating flow about a stationary aerofoil. Flow visualisation studies seem to show that the stagnation point, under certain conditions, is not at the trailing edge, but near it. Yet inviscid analysis based on the Kutta condition has been immensely useful in aeronautics and indeed it has been widely applied to cases where the aerofoil has a small base or when there is a small separated region near the trailing edge.

The well-known analysis of von Karman for a vortex street deals with a model problem in which not only the viscous diffusion of vorticity is not considered, but also vorticity is taken to be concentrated, illustrating that not only governing equations but also initial or boundary conditions may be modified by these arguments.

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*Limiting arguments are related to so-called rational approximations (Van Dyke 1964, p. 3) which can be distinguished from other, irrational, approximations. These approximations can be distinguished from 'models' in several ways as was done by Ojhi (1982, p. 105) for turbulent flows.

Free streamline models, beginning with the Kirchoff model, specify the displacement effect of the wake in a separated flow past a body usually in terms of a boundary condition on the separating streamline. This family of models illustrates another rather common feature of model problems, namely incompleteness, as one quantity, typically base pressure, has to be specified. This type of input is problem-specific. Of course, there are well-known cases of incompleteness, which call for general inputs. The problem of calculating mean flow and moments of low order in stationary turbulent flows is bedevilled by the absence of a physically sound and mathematically sufficiently general closure hypothesis, although several well-known closure hypotheses have been useful in practical applications and also in clarifying some features of the flow.

As explained above, we are concerned here with examining limiting arguments and model arguments together on the basis of their consequences on the problem as a whole. It is expedient to call the modified problem a *lidel* (a word coined by the combination of 'limit' and 'model') to distinguish the argument from the resulting problem and to emphasise a common framework for considering both types of arguments. The plan is to present first a general framework, which is followed by the notion of validity and a classification scheme to handle similarities of structures in a systematic way. The general methodology of examining the consequences of the arguments in several stages is finally illustrated by several examples.

2. Framework

The main idea is to consider arguments of limiting or modelling character together and to examine the resulting modified problem, which is termed as a 'lidel' or a 'lidel problem'. The arguments which make assumptions about solutions of the original problem are also included. For instance, if it is argued that the flow, which is subjected to time-independent boundary conditions, and which starts from a given initial condition, approaches a steady flow at large times, the resulting steady flow problem is considered a lidel for large time behaviour. One case study is given later to indicate several types of large time behaviour that can arise in flow problems. Similarly, if a boundary layer for given initial conditions and a suitable pressure distribution tends to become similar, the problem of obtaining a similarity solution is considered as a lidel for large downstream distances.

Classical hydrodynamic stability problems of boundary layers formulated for early stages of transition employ several ingredients such as undisturbed parallel flow, small disturbance of assumed form, modelling of spatial growth by temporal growth, which can be considered as limiting or modelling arguments; the stability problem is then regarded as a lidel for a certain class of disturbances and a certain range of downstream distance. When the analysis is applied to the flow resulting from a vibrating ribbon, an additional ingredient which changes a forced oscillation problem into a free oscillation problem is needed, which can also be regarded as a modelling argument.

Since the limiting modelling process frequently leads to incompleteness, hypotheses or statements of a general nature that are added to complete the problem are treated as parts of the lidel. On the other hand, information about numerical values of certain parameters, which are problem-specific, or which are not obtained by the argument but by comparison of the results with experimental data or other solutions, is termed as auxiliary input. Some inputs are matters of convenience (e.g. wall functions in certain turbulent flow computations) and some are such that the final results are not supposed to be sensitive to the chosen values provided they are in certain ranges (e.g. velocity profiles in integral models of boundary layers). This type of information which is necessary but which is not central to the lidel problem is also treated as auxiliary input (see figure 1). Arguments used in replacing partial differential equations and the initial and the boundary conditions by finite difference statements are also considered here as limiting/modelling arguments. Grid size, coefficients of terms added for damping out certain effects (that is, numerical viscosity), and relaxation factor etc. are examples of auxiliary inputs.

A few points have to be noted about the present approach. First, there is no restriction on the mathematical form of the lidel problem. Second, the expected correspondence between the solutions of the lidel and the original problem can vary from being exact for certain conditions, certain values of parameters, or in certain limits, to mere qualitative similarity. Since this range of correspondence may appear too broad to some readers, the need for such flexibility is shown by an example. The problem of a two-dimensional inviscid layer of constant vorticity with a slip at the wall responding to a small initial disturbance at the surface of the rotational region was formulated (e.g. Pullin 1981) to simulate dominant features of a turbulent boundary layer at large Reynolds number. Clearly, the correspondence between the results of this problem and real boundary layers can at best be qualitative. We regard this problem as a lidel of real turbulent boundary layers for large Reynolds numbers. Third, a clear distinction is made between the statement of a lidel and a method for *its* solution. However, it should be pointed out that some arguments which change the nature of the problem are conventionally regarded as

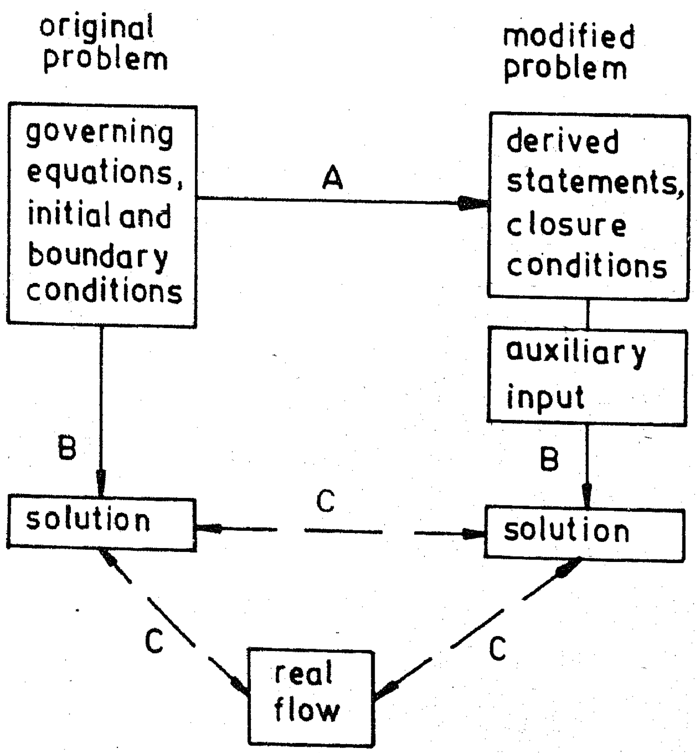


Figure 1. General framework for examining consequences of limiting and modelling arguments. A arguments, B analysis or computation, C correspondence.

a part of the method of solution of the *original* problem. For instance, there is a class of methods of obtaining approximate solutions of boundary layers in which the partial differential equations are first replaced by a few integral relations, then some additional statements are added and then one proceeds to solve ordinary differential equations. We first note that what is actually solved is a substitute problem and not the original problem. We regard the first two parts as not really part of the method of solution of the original problem governed by partial differential equations but as modelling arguments that replace the original problem by a substitute problem. It should be emphasised that the change in the mathematical structure is evidently accompanied by the change in the nature of information. For instance, Newton's laws and energy conservation would hold for the layer as a whole but not for its individual parts.

The next major idea is to split the limiting/modelling process into a sequence of steps or stages and to examine the solutions of problems before and after a step. Although such splitting may not always be possible, it can give useful understanding of the limiting/modelling process. In some instances, examination of the solution of the problem prior to a limiting/modelling stage reveals conditions when the model is likely to be valid. For instance, the problem of steady incompressible inviscid flow past an aerofoil can be obtained in two steps or stages. First, one considers the large-time behaviour for a viscous fluid initially at rest and takes a large-time limit. Second, one allows the viscosity to approach zero or the Reynolds number to approach infinity without stretching coordinates and one adds the Kutta condition. The problem obtained at the end of the first step is a lidel that is capable of describing viscous effects in the boundary layer and the wake in the absence of unsteadiness. The problem obtained at the end of the second step, that is, the inviscid flow problem, is also a lidel in the present context. In order to understand what the second step does, one can compare the problems at the end of the first and the second stage.

Consider the discrete potential vortex model classically used in the computational studies of the stability of a mixing layer. One can visualise three steps of the limiting/modelling process starting from an initial value problem of a viscous layer of finite thickness. First, viscosity is allowed to approach zero. Second, the thickness of the layer is allowed to approach zero, so that the thin layer is replaced by a velocity discontinuity, a vortex sheet. Third, the vortex sheet is replaced by a set of initially equidistant discrete (potential) vortices of equal strength. The problem at the end of the second step has no length scale, while a length scale, the initial distance between neighbouring discrete vortices, is introduced in the last step. Little wonder that the vortex sheet is unstable to disturbances of any wave length (Kelvin-Helmholtz instability) and the discrete vortex calculation, if performed without numerical fixes, shows that the layer crosses itself after some time.

In some instances, when the model is thought to be fully satisfactory, one learns from such examination about some features that the modelling process might have filtered out. One has to keep in mind that the order of steps may be important and that there might be more than one acceptable sequence. A few examples are given later to illustrate the possibilities.

There is another type of splitting that can be thought of. When the incompleteness of the lidel problem is remedied by closure conditions, one can, in some cases, compare them with corresponding experimental results or results of problems that are upstream in the limiting/modelling process. Such direct comparison can tell us about the range of applicability of the closure conditions. One case study is given later to illustrate this type of comparison.

Before proceeding further, we need to examine the notion of the validity of a lidel.

3. Validity

In oıder to discuss conditions under which a lidel serves the intended purpose, it seems desirable to introduce the notion of the validity of a lidel. It is essentially a generalisation of the idea of validity of an asymptotic expansion. If a standard is specified for the solution of a lidel and a criterion is specified for the difference between the standard and the solution of the lidel, we can say that the lidel is valid if it meets the criterion on comparison with the standard. The standard can be the solution of the original problem or a problem that is upstream in the limiting/modelling process or the results of a carefully chosen experiment. If the correspondence is exact for some conditions or quantitative, but approximate in some sense, it is easy to see how the criterion can be specified. On the other hand, it is difficult to see how qualitative correspondence can be translated into a criterion. For example, it is difficult to specify when flow visualisation pictures have the desired correspondence with discrete vortex model prediction. However, it turns out that it is clear when the lidel solution differs qualitatively from the standard. For example, if a lidel consisting of closely spaced discrete vortices for a vortex sheet leads to the sheet crossing itself after a certain time, it is clear that the lidel cannot be valid for such large times.

We need to bear in mind that the validity of a lidel is usually restricted to a certain range of parameters or regions in space and time. Also, if the lidel needs auxiliary input, the validity is subject to the values of adjustable parameters being chosen in a certain range.

While limiting and modelling arguments might give some information on or indication of the range of validity, experience has shown it to be quite incomplete. Therefore, an a posteriori validation exercise is necessary if we do not wish to run the risk of being misled by some features of the lidel flows into believing that real flows have those features.

4. Classification of lidels

The lidels can be classified in several ways. For instance, one can differentiate them on the basis of the nature of correspondence expected, or on the nature of arguments used in formulating the lidel problems, or on the degrees of freedom of the lidels as indicated by the number of adjustable parameters. The scheme given here for flow problems is on the basis of the mathematical form of the lidel problem. Table 1 gives the classes with illustrative examples, which are largely self-explanatory.

Table 1. Classification of lidels

Class	Example
Differential	Attached potential flow with Kutta condition Free streamline models Karman vortex street Discrete vortex models Flows governed by Euler, boundary layer or parabolised Navier–Stokes equations Large Reynolds number approximations of internal/external separated flows Stokes flow Hydrodynamic stability problems Reynolds averaged equation models with closure hypotheses Lorenz convection model
Integral & integro-differential	Control volume analysis Integral models of laminar/turbulent boundary layers Integral relations for laminar boundary layer Contour dynamics analysis of flow with distributed vorticity
Numerical	Panel & vortex lattice models Finite difference models of potential flow, Euler equations, boundary layer equations, parabolised or full Navier–Stokes equations
Mapping	Poincaré maps of certain flow problems
Hybrid	Viscous-inviscid interaction Chapman–Korst model Large eddy simulation models

5. Examples

A few case studies are given to illustrate and elaborate the ideas in the earlier sections.

5.1 *Vortex interactions*

The spatial organisation of rotational regions has been of intense interest since the early seventies. In nominally 2D mean flow fields, these regions, called "vortices" for brevity hereafter, undergo striking motions which have been termed pairing, merging etc. Several investigations model the motion as two-dimensional and inviscid, as some of the observed features of flows are believed to be governed by largely two-dimensional and inviscid processes.

The motion at a point in the interior of such a rotational region can be decomposed into two parts, one owing to vorticity in that region and the other to the rest. The second part essentially convects the region and imposes an irrotational straining velocity field.

The first basic model problem A consists of a single vortex, having uniform vorticity in a region of elliptic shape. The flow outside the region is irrotational. Also, there is a velocity field with uniform strain rate imposed by boundary conditions at large distances. The flow field is termed as an elliptic vortex in a uniform straining field. Following the steady solution studied by Moore & Saffman (1975), Kida (1981) and Neu (1984) have obtained a class of exact solutions of Euler equations which are illuminating. We first describe certain features of solutions before taking up modelling issues.

As the boundarv of the rotational region retains its shape and its area is constant, the velocity field at an instant is characterised by two quantities, namely, the ratio of the major to the minor axis of the elliptical boundary and the angle made by the major axis with one principal direction of the strain rate, the parameters of the problem being vorticity ω and the principal value of strain rate γ. The problem can be transformed into that of a Hamiltonian system and the phase space trajectory diagrams are given in figures 2a–e. The radial coordinate r and the angular coordinate θ of a point indicate the ratio of the major to the minor axis and the angle made by the major axis with the principal axis at any instant. An isolated point (nodal point) or an intersection of trajectories is a steady state, which may be stable or unstable depending on whether none or some of the neighbouring trajectories move away from it. A closed trajectory that goes to all quadrants indicates a periodic motion in which the major axis rotates around the vortex centre. A closed trajectory that does not go to all quadrants indicates that the vortex undergoes angular oscillations. A trajectory that moves indefinitely away from the origin indicates indefinite increase of the ratio of the major to the minor axis or flattening out of the ellipse. We thus see from the figures 2a–e that there are many different types of large-time behaviour that can arise depending on the value of strain rate to vorticity ratio and the initial condition. We summarise the ranges in which these types of behaviour can occur.

Steady vortex	$0 < \gamma/\omega < 0{\cdot}15$
Oscillating vortex	$0 < \gamma/\omega < 0{\cdot}15$
Rotating vortex	$0 \leq \gamma/\omega < 0{\cdot}1227$
Indefinite flattening of vortex	$0 < \gamma/\omega$

Thus an elliptic vortex can undergo simple motions, only if the strain rate is sufficiently small. If the strain rate is sufficiently large, the vortex invariably flattens out.

The second basic model problem B consists of two interacting vortices, initially of circular cross-section and of uniform vorticity, both having equal circulation and area. Two models B1 and B2 obtained from B are based on the contour dynamics method (Zabusky *et al* 1979) and the discrete vortex model in which distributed vorticity is modelled by a cluster of concentrated vorticity. Figures 3a & b show that merging occurs if the initial distance between vortices to diameter ratio is less than a critical value and simple relative motions occur otherwise. The results of the two models B1 and B2 are qualitatively similar, and the critical values obtained by them are about 1·702 and 1·7.

One may link the problems A and B by arguing that one vortex influences the other vortex by imposing a rigid body translation and rotation, and a straining velocity field. In the simplest case, the strain rate can be taken to be uniform. So the problem A can be considered as a lidel having qualitative correspondence with the problem B. Then one expects a correspondence between the parameters of the two models. The initial strain rate (at the vortex centre) to vorticity ratio is

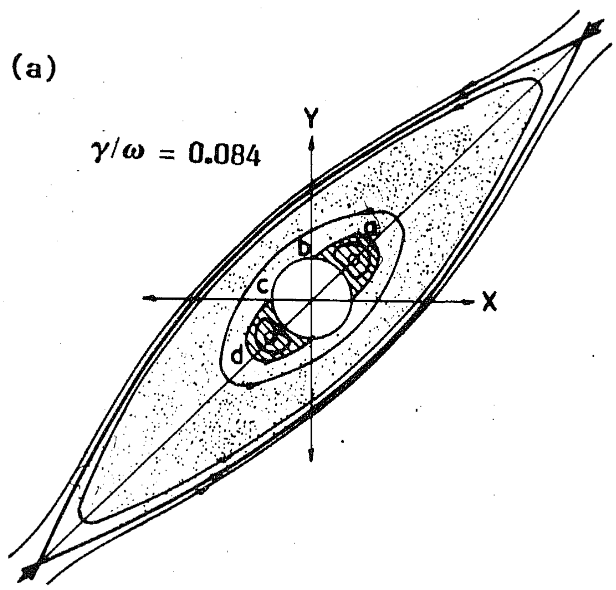
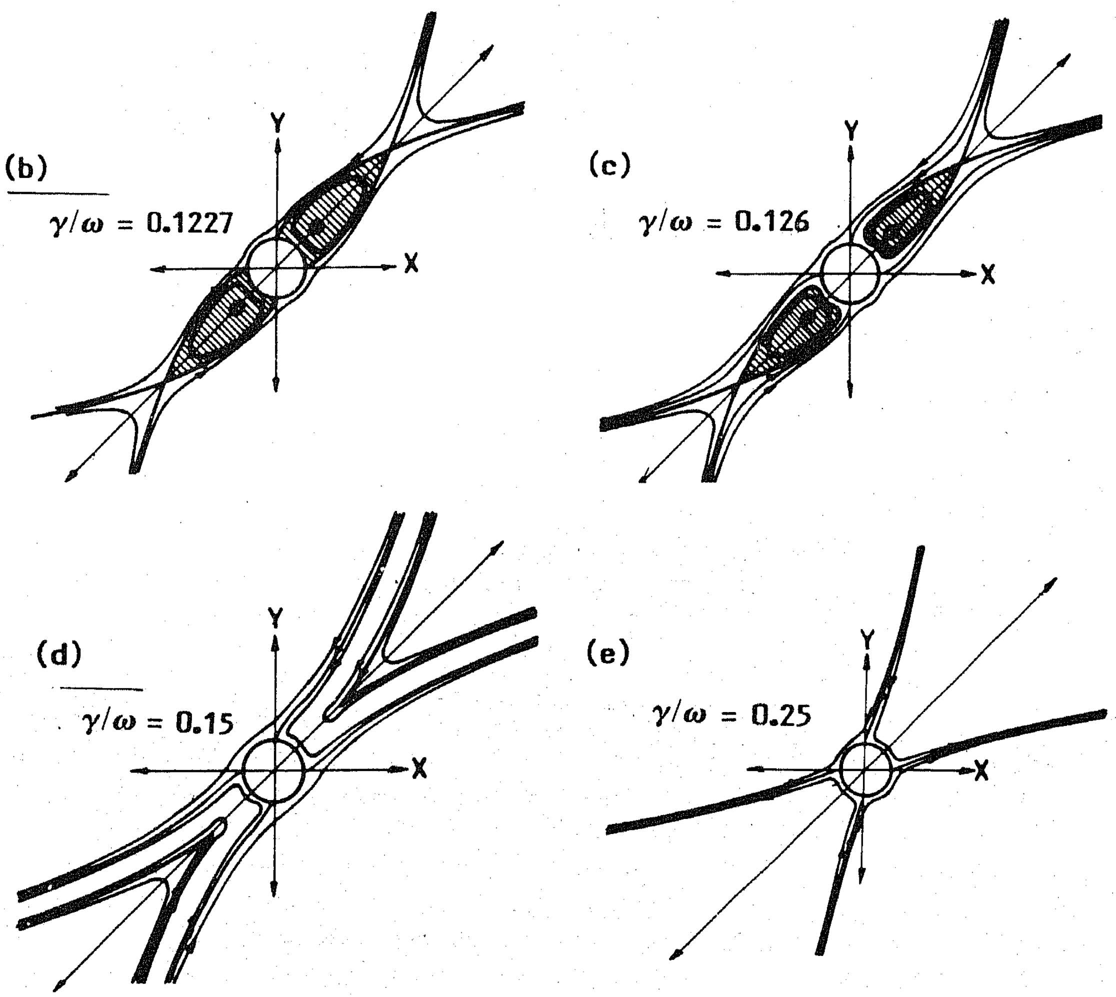


Figure 2. Phase space polar plot for elliptic vortex in a uniform strain rate field (Neu 1984). Radial and angular coordinates (r, θ) indicate ratio of major to minor axis and the angle made by the major axis to one principal direction of strain rate. Strain rate/vorticity (**a**) below the first critical value (0·1227); (**b**) equal to the first critical value (0·1227); (**c**) between the first and the second critical values (0·1227 and 0·15); (**d**) equal to the second critical value (0·15); (**e**) larger than the second critical value.



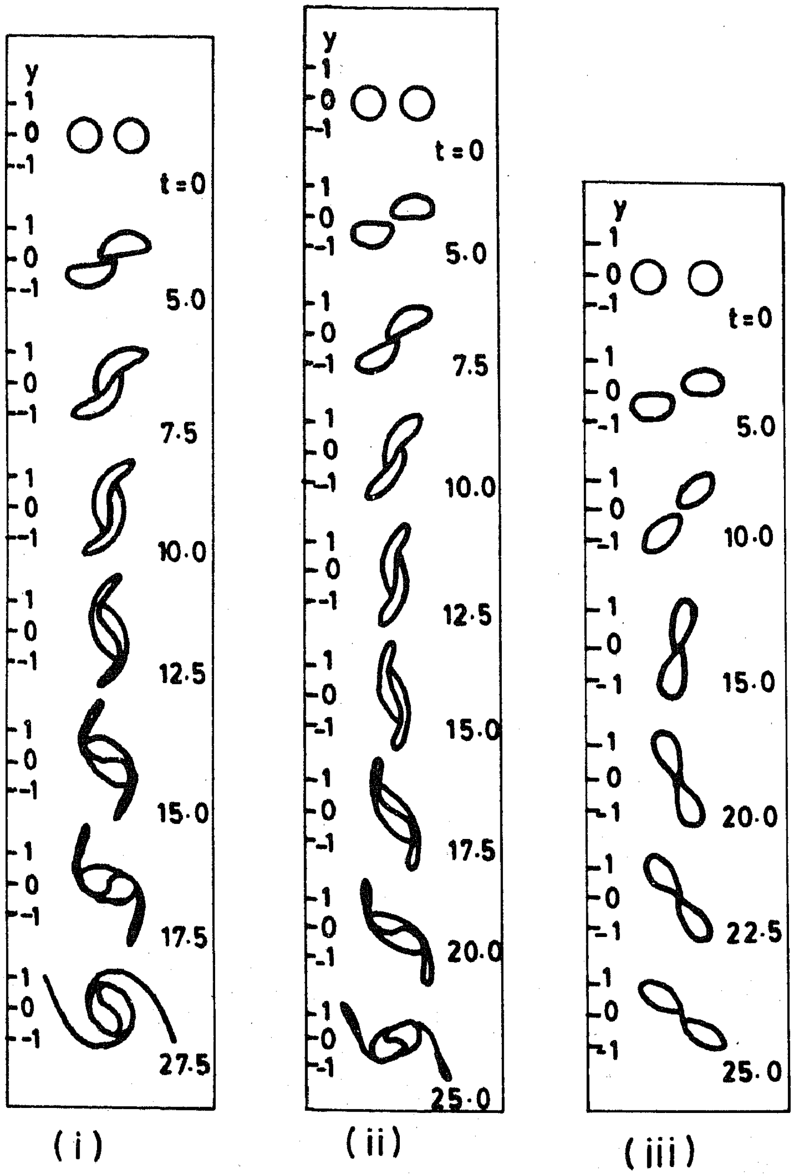


Figure 3(a). Interaction of two vortices of equal strength and of initial circular shape of equal diameter. Results of contour dynamics (Jacob & Pullin 1985). Initial distance between vortex centres/diameter has values 1·3293, 1·5066 and 1·7016 in cases (i), (ii) and (iii), respectively.

inversely proportional to the square of the distance to the diameter ratio. The critical value of strain rate/vorticity that allows the elliptic vortex to retain its shape and rotate is 1·227. This would correspond to the critical value of the initial distance between vortices to the diameter ratio of 1·01, which is much smaller than 1·7. There is a qualitative corresondence between the two models in the sense that when the strain rate is smaller than a certain critical fraction of vorticity, or when the distance between the two vortices is greater than a particular critical multiple of vortex diameter, each vortex is able to undergo simple motions with limited deformation. Limitations on the correspondence arise from non-uniform and time

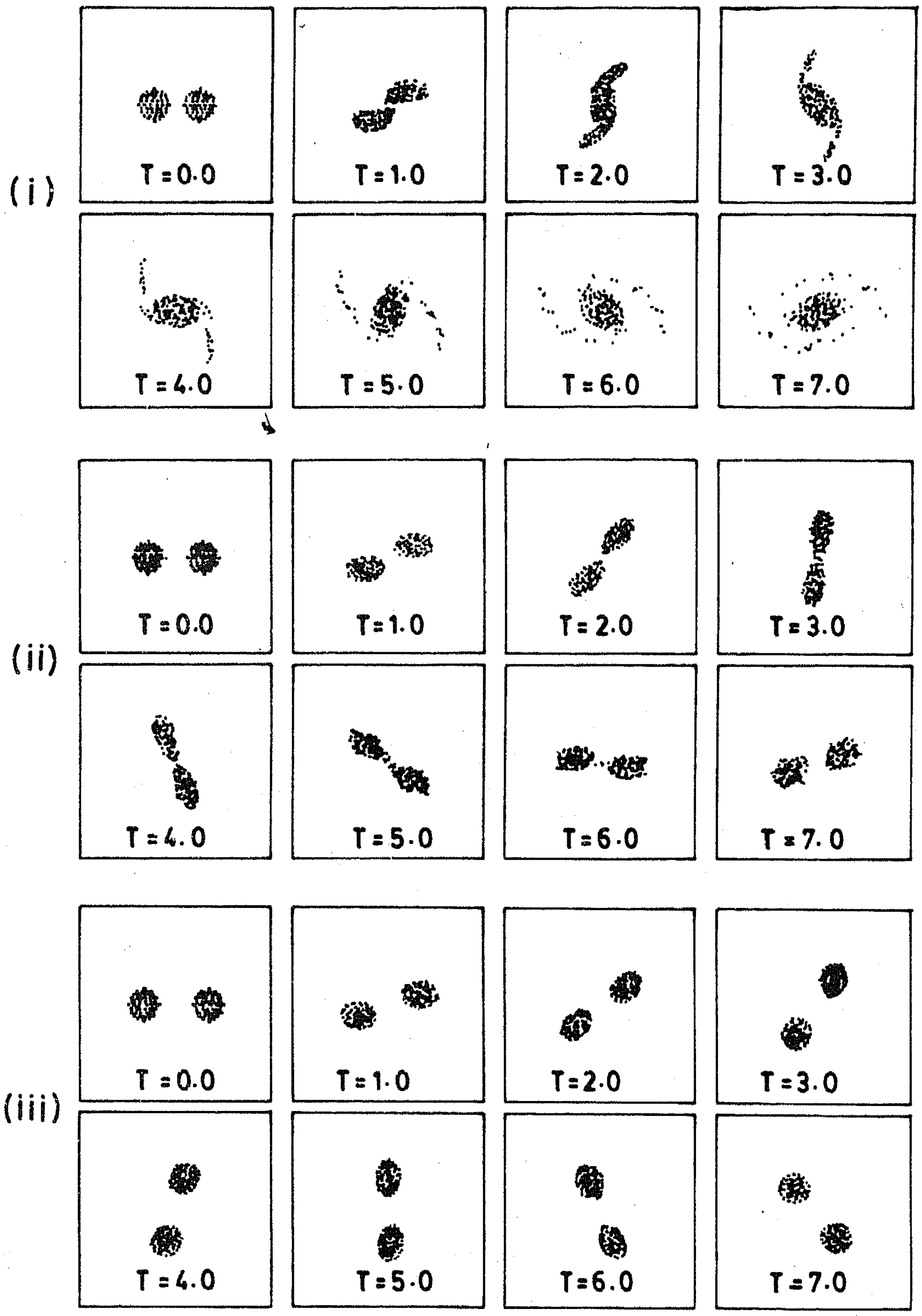


Figure 3(b). Results of discrete vortex model (Tsuboi & Oshima 1985). Distance/diameter has values 1·4, 1·7 and 2 in cases (i), (ii) and (iii), respectively.

dependent strain rate fields imposed by the neighbouring vortex and subsequent non-elliptic shapes in the model B.

There is another aspect of the results of model A which is of interest. The steady flow solution can be considered to be a candidate lidel for large-time behaviour for time-independent boundary conditions. Frequently one argues that, for time independent boundary conditions, the flow may be expected to approach a steady state or a similarity solution. Sometimes longitudinal distance plays a role similar to time and one seeks fully developed flow. In turbulent flow investigations, one considers stationary flow, self-preserving flow, or equilibrium flow. The results of the model A show that if such large-time (or large-streamwise distance) lidels are used, the following possibilities could well arise.

(a) The steady flow solution may exist only for a range of parameters.

(b) In this range, there could be more than one steady flow solution (including stable and unstable cases).

(c) If the parameter is within the range required for steady flow, initial conditions for the flow at large-time to approach the steady flow may be restricted to some range.

The above possibilities have to be kept in mind in interpreting results of simple large-time lidels, especially for flows with distributed vorticity.

5.2 *Attached flow past a body*

Attached steady inviscid flow with the Kutta condition as applied to air-flow past an aerofoil can be considered a lidel obtained in a sequence of three steps, namely (a) small compressibility effect, (b) large-time after start-up from rest, and (c) small viscosity. The Kutta condition is needed in the third step for 2D problems. Another acceptable sequence of steps would be (b) and (c) followed by (a). Several calculations of the upstream model using this sequence (that is, allowing for compressibility, with Euler equations) suggest that the problem of incompleteness does not appear at this stage. Even if the freestream speed is small, the flow near the body would need to pass through a supersonic pocket, and a shock, to go around the sharp trailing edge with accompanying drop in total pressure and the resulting pressure discontinuity near the rear stagnation point on the smooth surface. This is an example of how one learns about flows as well as the limiting/modelling processes by splitting into limiting/modelling steps.

5.3 *Integral lidels of laminar boundary layers*

A large family of methods were developed in the early stages of boundary layer theory, which model dynamics of the layer as a whole by using momentum, and in some cases, energy integrals, and additional inputs. The methods were intended to provide quick, reasonably good approximations. Despite their simplicity and gross nature of additional information, they ended up providing rather accurate information on integral parameters. Several variants for turbulent boundary layers have also enjoyed success and sustained use in applications.

From the present viewpoint, an integral model of a boundary layer consists of integral relations like momentum and energy integrals, additional inputs in the form of velocity profile families or relations amongst integral parameters and possibly auxiliary inputs.

One can visualise obtaining such a lidel in three steps. The first one would be to obtain an integral formulation of the boundary layer problem. There is an integral transform formulation of the boundary layer problem (Yajnik 1984) that is suitable for the present purpose. The second step would consist of expansion in the transform variable, which generates a sequence of integral relations. One truncates after a few terms. Hence, this second step is essentially based on limiting arguments. The third step consists of imposing additional conditions. This is viewed as a modelling step. Traditional integral methods employ the first relation, that is, the momentum integral, and, at most, the second relation, which is the energy integral. There is a considerable variety of additional conditions that have been used (see, for example, Rosenhead 1963, pp. 292–317). Interestingly, if one retains a larger number of terms of the expansion, one needs additional conditions that can provide for such flexibility. The method of weighted residuals (e.g. Dorodnitsyn 1962, Abbott & Bethel 1968) provides for the use of certain polynominal expressions for this purpose. The above interpretation of integral methods provides improved understanding. The first step interestingly does not amount to any loss of information or involve any approximation, as it has been used to obtain a few exact solutions of the boundary layer problem. The transform formulations have some additional conceptual interest, as the basic nonlinear transform used in the formulation is a generalisation of boundary layer thickness and its expansion in the transform variable generates displacement, momentum and energy thickness, and similar 'higher-order' thicknesses! This example illustrates that examination of problems upstream in the limiting/modelling process tend to enhance our understanding.

5.4 *Modelling of turbulent boundary layers*

The classical modelling process consists of applying order of magnitude arguments to Reynolds averaged equations and introducing closure conditions of different kinds. If one carries out the process in two parts, namely, using limiting arguments of the boundary layer type, and subsequently invoking various closure hypotheses, there are some advantages (Yajnik 1970). First, the results of the first step are independent of any particular closure condition. Second, the results can be considered to be exact in the limit of large Reynolds numbers. Third, surprisingly asymptotic forms similar to well-known empirical laws can be obtained without making any closure hypotheses. This approach has been extended by many workers to account for heat transfer, compressibility, spectral domain, moderately large Reynolds number, non-stationary case etc. This example shows that it is advantageous to carry out limiting argument stages before modelling stages.

5.5 *Wake-boundary layer interaction*

Two Stanford conferences have shown how evaluation of calculation methods for turbulent shear flows needs highly coordinated exercises. The methodology of evaluating a method consisting of derived equations, closure hypotheses and an integration procedure certainly shows the effectiveness of the package. In some cases, however, a simpler exercise aimed at testing components of lidels can be quite instructive. For instance, in a recent study of the interaction of the turbulent wake of an aerofoil and a wall boundary layer by Sundaram & Yajnik (1986) one finds that experimental observations as far downstream as four chords have a trend similar to the Cebeci-Smith eddy viscosity hypothesis, but the scales are different (figure 4). The inference is that when simple closure conditions are applied to complex flows arising from the merging of flows of two different types, some modifications in the closure conditions are called for.

Although here we have split the model problem rather than the modelling process into components, the basic idea is to break down a complex structure into

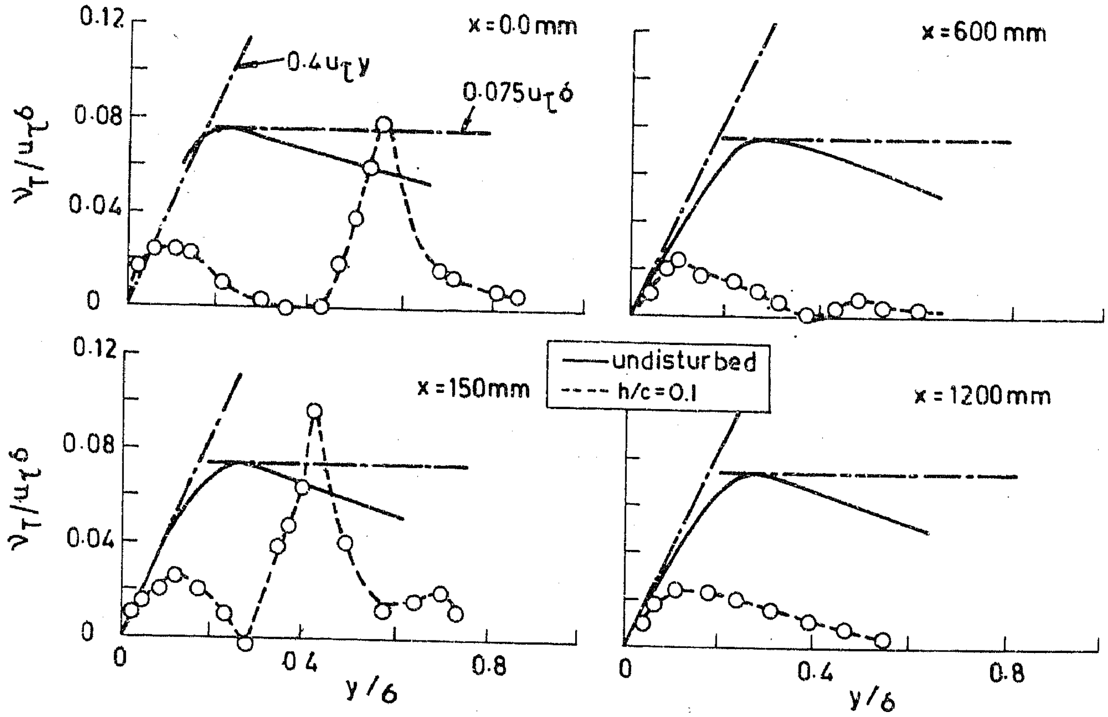


Figure 4. Eddy viscosity distribution in an undisturbed boundary layer in nominally zero pressure gradient (——), and in an aerofoil wake merging with a boundary layer (– o –) in experiments of Sundaram & Yajnik (1986); (—·—) Cebici-Smith model. h, height of aerofoil (NACA 0012) above wall, c, aerofoil chord, ν_T eddy viscosity, u_τ, skin friction velocity, δ, boundary layer thickness, x, longitudinal distance.

parts to facilitate investigation and to enhance our understanding of flows as well as the reasoning used in the analysis of flows.

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