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BOUNDED VECTORS FOR UNBOUNDED REPRESENTATIONS AND STANDARD REPRESENTATIONS OF POLYNOMIAL ALGEBRAS

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Abstract. Let A be a unital commutative *-algebra. Let π be a hermitian representation of A into (not necessarily bounded) Hilbert space operators. Analytic vectors and bounded vectors for π are investigated; and are used to show that π is a direct sum of bounded (operator) representations iff π admits a core consisting of bounded vectors. This, in turn, is used to show that if A is either of the polynomial algebras $\mathcal{P}(x)$ or $\mathcal{P}(x, y)$ in one or two commuting hermitian generators then π is standard iff π is a direct sum of bounded representations. Various selfadjointness and standardness criteria for representations of these polynomial algebras are developed, highlighting the difference between the representation theory of these two algebras, and supplementing known results.

1. Introduction and Preliminaries

Let $T: D(T)\subset H\rightarrow H$ be a linear operator (not necessarily bounded) defined on a dense subspace $D(T)$ (Domain of T) of a Hilbert space H. Then T is *formally normal* if

$$D(T)\subset D(T^*) \quad \text{and} \quad \|T\xi\|=\|T^*\xi\| \qquad \text{for all } \xi\in D(T)\,.$$

A formally normal operator T is *normal* if T is closed and $D(T)=D(T^*)$. The spectral theorem [14, §7.5] represents a normal operator as

$$T=\int\lambda dE(\lambda)$$

for a spectal measure E. Following [8], [12], given an operator T, a vector ξ in

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$$C^{\infty}(T):=\bigcap_{n=1}^{\infty} D(T^n) \ (=C^{\infty}\text{-}vectors\ for\ T)$$

is a *bounded vector* (resp. *an analytic vector*) if there exist $a>0$, $c>0$ such that $\|T^n\xi\|\leqq ac^n$ for all $n\in N$ (resp. if there exists $t>0$ such that $\sum_{n=1}^{\infty} t^n\|T^n\xi\|/n!<\infty$). Bounded vectors are analytic, but not conversely. The space of all bounded (resp. analytic) vectors will be denoted by $B(T)$ (resp. $D_{\omega}(T)$). Now a closed formally normal operator T is normal iff there exists a subspace (linear manifold, not necessarily closed) $X\subset D(T)$, $TX\subset X$, $T^*X\subset X$ such that $B(T)\cap X$ is dense in H [12] iff T is a direct sum of bounded normal operators (a consequence of the spectral Theorem) iff for a dense subspace X of $D(T)$ satisfying $TX\subset X$, $T^*X\subset X$, $D_{\omega}(T)\cap X$ is dense in H [8, Lemma 3.2]. The purpose of this paper is to investigate the representation theoretic analogous of these in the framework of unbounded representations [5], [6], [9]; and to apply them to the representations of the polynomial algebras $\mathcal{P}(x)$ and $\mathcal{P}(x, y)$ in one and two commuting hermitian generators, thereby refining the main results in [5].

Let A be a $*$-algebra viz. a linear associating involutive algebra over complex scalars and having identity 1. A *hermitian representation* (or $*$-representation) $(\pi, D(\pi), H)$ of A on a Hilbert space H is a mapping π of A into linear operators (not necessarily bounded) all defined on a subspace $D(\pi)$ dense in H such that for all x, y in A, all scalars α, β and all vectors ξ, η in $D(\pi)$, the following hold:

(i) $\pi(\alpha x+\beta y)\xi=\alpha\pi(x)\xi+\beta\pi(y)\xi$,
(ii) $\pi(x)D(\pi)\subset D(\pi)$, and $\pi(x)\pi(y)\xi=\pi(xy)\xi$,
(iii) $\pi(1)=1$, the identity operator,
(iv) $\pi(x^*)\subset\pi(x)^*$ (: operator adjoint), i.e., $\langle\pi(x)\xi, \eta\rangle=\langle\xi, \pi(x^*)\eta\rangle$.

It follows that for each normal element x of A, $x=h+ik$ with $h=h^*$, $k=k^*$, $hk=kh$, the operator $\pi(x)$ is formally normal. Unbounded representations have been investigated in the contexts of Wightmann quantum field theory, representations of Lie algebras, representation theory of non-normed topological $*$-algebras and algebras of unbounded operators. There is an analogy between unbounded symmetric operators and unbounded hermitian representations [9]. The *closure* $(\bar{\pi}, D(\bar{\pi}), H)$ of a hermitian representation $(\pi, D(\pi), H)$ of A is the $*$-representation $\bar{\pi}$ defined on $D(\bar{\pi})=\cap\{D(\overline{\pi(x)})\mid x\in A\}$ as

$$\bar{\pi}(x)=\overline{\pi(x)}\,|_{D(\bar{\pi})},$$

$\overline{\pi(x)}$ denoting the closure of the operator $\pi(x)$; and π is *closed* if $\pi=\bar{\pi}$, i.e. $D(\pi)=D(\bar{\pi})$. *Hermitian adjoint* $(\pi^*, D(\pi^*), H)$ of a $*$-representation $(\pi, D(\pi), H)$ of A is the representation (not necessarily satisfying (iv) above) with domain

$D(\pi^*)=\cap\{D(\pi(x^*)^*)\mid x\in A\}$ defined as

$$\pi^*(x)=\pi(x^*)^*\mid_{D(\pi^*)};$$

and π is *selfadjoint* if $\pi=\pi^*$, i. e. $D(\pi)=D(\pi^*)$. Note that given a hermitian representation π, the symmetric operator $\pi(h)$, for $h=h^*$ in A, need not be essentially selfadjoint, even if π is selfadjoint. In fact, π is *standard* [5], [6], [9] if $\pi(x)^*=\overline{\pi(x^*)}$ for all $x\in A$. We say that a representation π is *bounded* if each $\pi(x)$, $x\in A$, is a bounded operator. A π-invariant subspace M of $D(\pi)$, for a hermitian representation $(\pi, D(\pi), H)$ of A, defines a subrepresentation π_M with domain $D(\pi_M)=M$ on the Hilbert space $\overline{M}$ (closure in H) as

$$\pi_M(x)=\pi(x)\mid_M \qquad (x\in A).$$

We shall call a π-invariant subspace X of $D(\pi)$ a *core for* π if $\bar{\pi}=\bar{\pi}_X$ $(=(\pi\mid_X)^-)$.

Definition 1.1. Let $(\pi, D(\pi), H)$ be a hermitian representation of a $*$-algebra A. The *analytic vectors for* π are the vectors in

$$D_\omega(\pi)=\bigcap_{x\in A} D_\omega(\pi(x));$$

and the *bounded vectors for* π are the vectors in

$$B(\pi)=\bigcap_{x\in A} B(\pi(x)).$$

It is shown in Section 2 that for a closed $*$-representation π of a commutative $*$-algebra A, π is a direct sum of bounded representations iff π is a closed linear span (in an appropriate sense) of bounded representations iff $B(\pi)$ contains a core for π. In this case, π is standard. This is used in Sections 3 and 4 to discuss standardness criteria for representations of polynomial algebras $\mathcal{P}(x)$ and $\mathcal{P}(x, y)$ in commuting hermitian generators x and y, highlighting the essential differences between the representation theory of these two algebras. Given a representation π of either $\mathcal{P}(x)$ or $\mathcal{P}(x, y)$, it is shown that π is standard iff it is a direct sum of bounded representations. Standardness and selfadjointness are equivalent for $\mathcal{P}(x)$; but not for $\mathcal{P}(x, y)$. A selfadjoint representation π of $\mathcal{P}(x, y)$ is standard iff $\pi(x^2+y^2)$ is essentially selfadjoint. We also discuss the problems of selfadjoint extensions for representations of these polynomial algebras.

2. Representations and bounded vectors

Theorem 2.1. *Let $(\pi, D(\pi), H)$ be a closed hermitian representation of a commutative $*$-algebra A. The following are equivalent:*

(1) *π is a direct sum of bounded hermitian representations of A.*

(2) *π is a closed linear span of bounded hermitian representations of A.*

(3) *There exists a subspace X of $B(\pi)$ which forms a core for π.*

Under any of these conditions, π is a standard representation; and for x, y in A, the normal operators $\overline{\pi(x)}$ and $\overline{\pi(y)}$ have mutually commuting spectral projections.

Let $\{(\pi_i, D(\pi_i), H_i) \mid i \in I\}$ be a family of hermitian representations of A. The *direct sum* $\pi = \sum \pi_i$ is the representation $(\pi, D(\pi), H)$ of A defined on the Hilbert space $H = \bigoplus_i H_i$ with domain

$$D(\pi) = \{\xi = (\xi_i) \in H \mid \xi_i \in D(\pi_i)\ (i \in I),\ \ \sum \|\pi_i(x)\xi_i\|^2 < \infty\ (x \in A)\}$$

defined as

$$\pi(x)\xi = \sum \pi_i(x)\xi_i\,.$$

The closed linear span of bounded representations defined below is the representation theoretic analogue of closed linear span of bounded operators [12].

Let $(D, \leqq)$ be a directed set. By $H_\alpha \uparrow$ is meant a family $\{H_\alpha \mid \alpha \in D\}$ of Hilbert spaces H_α such that for $\alpha \leqq \beta$ in D H_α is a closed subspaces of H_β, the embedding $H_\alpha \hookrightarrow H_\beta$ being isometric. By $\pi_\alpha \uparrow$ is meant a family $\{(\pi_\alpha, D(\pi_\alpha), H_\alpha) \mid \alpha \in D\}$ of hermitian representations π_α of A such that

(i) $H_\alpha \uparrow$,

(ii) for each α, $D(\pi_\alpha) = H_\alpha$ and π_α is bounded,

(iii) for each $x \in A$, $\alpha \leqq \beta$ in D, $\pi_\alpha(x) \subset \pi_\beta(x)$.

Given $\pi_\alpha \uparrow$, define $\cup \pi_\alpha$ by taking $D(\cup \pi_\alpha) = \cup H_\alpha$, and

$$(\cup \pi_\alpha)(x) = \cup \pi_\alpha(x)\,,$$

i. e., for each $\xi \in H_\alpha$, $(\cup \pi_\alpha)(x)\xi = \pi_\alpha(x)\xi$. It is easily seen that this defines a hermitian representation of A on the Hilbert space $\vee H_\alpha$=closed linear span (in this case, completion) of $\cup H_\alpha$ with the inner product that is naturally defined. Let $\vee \pi_\alpha$ be the *closure* of $\cup \pi_\alpha$ having domain $D(\vee \pi_\alpha) = \cap \{D(\overline{(\cup \pi_\alpha)(x)}) \mid x \in A\}$,

$$(\vee \pi_\alpha)(x)\xi = \overline{(\cup \pi_\alpha)(x)}\xi\,.$$

We call $\vee \pi_\alpha$ a *closed linear span of bounded hermitian representations*. In view of Nelson's Theorem [14, Th. 8.31, p. 261], $\vee \pi_\alpha$ is a standard representation, admitting a dense set of bounded vectors H_α forming a core for π. Theorem 2.1 establishes the converse.

Let $L^+(D(\pi))$ denote the $*$-algebra of all linear operators $T: D(\pi) \to D(\pi)$ such that $T^*D(\pi) \subset D(\pi)$. The (formal) *unbounded commutant* of π is

$$\pi(A)^c = \{T \in L^+(D(\pi)) \mid T\pi(x)\xi = \pi(x)T\xi\ (x \in A,\ \xi \in D(\pi))\}\,.$$

Lemma 2.2. *Let $(\pi, D(\pi), H)$ be a hermitian representation of a commutative $*$-algebra A. Then the following hold.*

(1) *Each of $B(\pi)$ and $D_\omega(\pi)$ is a π-invariant subspace of $D(\pi)$; and*

$$B(\pi)=\cap\{B(\pi(h))\,|\,h=h^* \text{ in } A\}\subset D_\omega(\pi)=\cap\{D_\omega(\pi(h))\,|\,h=h^* \text{ in } A\}.$$

(2) $\pi(A)^c D_\omega(\pi)\subset D_\omega(\pi)$, $\pi(A)^c B(\pi)\subset B(\pi)$.

(3) *If there exists $Q\subset D(\pi)$ such that the linear span of $\pi(A)^c Q$ is dense in H, and if π is closed, then π is standard; and for x, y in A, the operators $\overline{\pi(x)}$ and $\overline{\pi(y)}$ are normal, having mutually commuting spectral projections.*

(4) *Let $D=\boldsymbol{N}^A$=the set of all functions $f: A\to\boldsymbol{N}$. For $f\in D$, let*

$$B_f(\pi)=\{\xi\in D(\pi)\,|\text{ for each } x\in A \text{ there exists a positive number } a_{x,\xi} \text{ satisfying } \|\pi(x^n)\|\leqq a_{x,\xi}(f(x))^n\ (n\in\boldsymbol{N})\}.$$

Then $B(\pi)=\cup\{B_f(\pi)\,|\,f\in D\}$, and each $B_f(\pi)$ is a π-invariant subspace of $D(\pi)$.

Sublemma 2.3. *Let N be a formally normal operator in a Hilbert space H with dense domain $D(N)$ such that $ND(N)\subset D(N)$, $N^*D(N)\subset D(N)$. Let $\mathrm{Re}\,N$ and $\mathrm{Im}\,N$ denote respectively the real and imaginary parts of N. The following hold.*

(1) $D_\omega(N)=D_\omega(\mathrm{Re}\,N)\cap D_\omega(\mathrm{Im}\,N)$, $B(N)=B(\mathrm{Re}\,N)\cap B(\mathrm{Im}\,N)$,

(2) $\{N, N^*\}^c D_\omega(N)\subset D_\omega(N)$, $\{N, N^*\}^c B(N)\subset B(N)$.

Proof of Sublemma. (1) One has $\mathrm{Re}\,N=(N+N^*)/2$, $\mathrm{Im}\,N=(N-N^*)/2i$, so that $N=\mathrm{Re}\,N+i\,\mathrm{Im}\,N$. Also, $D(N)\subset D(N^*)$; and for any $\xi\in D(N)$, $\|N\xi\|=\|N^*\xi\|$, $N^*N\xi=NN^*\xi$. The assertion $D_\omega(N)=D_\omega(\mathrm{Re}\,N)\cap D_\omega(\mathrm{Im}\,N)$ has been noted in the remark on p. 34 following [13, Prop. 1]. This can also be verified by arguments similar to those for bounded vectors given below. Let $\xi\in B(N)$, so that for suitable $a>0$, $c>0$, $\|N^n\xi\|\leqq ac^n$ for all $n\in\boldsymbol{N}$. Then, for any $n\in\boldsymbol{N}$,

$$\begin{aligned}\|(\mathrm{Re}\,N)^n\xi\|&=\left\|\left(\frac{N+N^*}{2}\right)^n\xi\right\|=\left\|\sum_{k=0}^n\binom{n}{k}N^k(N^*)^{n-k}\xi\right\|\\&\leqq\sum_{k=0}^n\binom{n}{k}\|N^k(N^*)^{n-k}\xi\|=\|N^n\xi\|\sum_{k=0}^n\binom{n}{k}\\&\leqq 2^n ac^n=a(2c)^n.\end{aligned}$$

Thus $\xi\in B(\mathrm{Re}\,N)$. Similarly, $\xi\in B(\mathrm{Im}\,N)$. If follows that $B(N)\subset B(\mathrm{Re}\,N)\cap B(\mathrm{Im}\,N)$. Conversely, let $\xi\in B(\mathrm{Re}\,N)\cap B(\mathrm{Im}\,N)$. Choose $a>0$, $c>0$ such that for all $n\in\boldsymbol{N}$, $\|(\mathrm{Re}\,N)^n\xi\|\leqq ac^n$, $\|(\mathrm{Im}\,N)^n\xi\|\leqq ac^n$. Then, using Cauchy-Schwarz inequality, we get, for each $n\in\boldsymbol{N}$,

$$\begin{aligned}\|N^n\xi\| &= \|(\operatorname{Re} N+i\operatorname{Im} N)^n\xi\| \\ &= \left\|\sum_{k=0}^{n}\binom{n}{k}(\operatorname{Re} N)^n(i\operatorname{Im} N)^{n-k}\xi\right\| \\ &\leqq \sum\binom{n}{k}\|(\operatorname{Re} N)^n(\operatorname{Im} N)^{n-k}\xi\| \\ &\leqq \sum\binom{n}{k}(\|(\operatorname{Re} N)^n\xi\|\,\|(\operatorname{Im} N)^{n-k}\xi\|)^{1/2} \\ &\leqq \sum\binom{n}{k}(ac^n ac^n)^{1/2}=a(2c)^n\,.\end{aligned}$$

Thus, $\xi\in B(N)$, and $B(\operatorname{Re} N)\cap B(\operatorname{Im} N)\subset B(N)$.

(2) The assertion concerning analytic vectors follows from [13, Prop. 2]. Let $\xi\in B(N)$ satisfying $\|N^n\xi\|\leqq ac^n$ for all $n\in\boldsymbol{N}$. Let $T\in\{N, N^*\}^c$. Then, for any $n\in\boldsymbol{N}$,

$$\begin{aligned}\|N^nT\xi\|^2 &= \langle N^nT\xi,\ N^nT\xi\rangle=\langle (N^*)^nN^n\xi,\ T^*T\xi\rangle \\ &\leqq \|(N^*)^nN^n\xi\|\,\|T^*T\xi\| \\ &\leqq \|T^*T\xi\|\,\|N^{2n}\xi\| \\ &= (\sqrt{\|T^*T\xi\|})^2ac^{2n}=t^2c^{2n}\,.\end{aligned}$$

It follows that $T\xi\in B(N)$.

Proof of Lemma 2.2. Since A is commutative, $\pi(A)\subset\pi(A)^c$. Thus the assertions (1) and (2) follow from the sublemma. If there exists a subset $Q\subset D_\omega(\pi)$ such that $\pi(A)^cQ$ is linearly dense in H, then [13, Th. 1] implies that the formally normal operators $\pi(x)$, $x\in A$, are all essentially normal. Thus, for each $h=h^*$ in A, $\overline{\pi(h)}$ is selfadjoint. This, by [9(I), Th. 7.1], implies that $\overline{\pi(x^*)}=\pi(x)^*$ for all x. Thus π is standard. Futher, [13, Th. 2] also implies that for x, y in A the normal operators $\overline{\pi(x)}$ and $\overline{\pi(y)}$ have mutually commuting spectral projections. This gives (3). For (4), it is obvious that each $B_f(\pi)$ is a subspace of $B(\pi)$, and $B(\pi)=\cup\{B_f(\pi)\mid f\in D\}$. We show that $\pi(A)B_f(\pi)\subset B_f(\pi)$. Let $f\in D$, $\xi\in B_f(\pi)$, $y\in A$. Then, for each $x\in A$, $n\in\boldsymbol{N}$,

$$\begin{aligned}\|\pi(x)^n\pi(y)\xi\|^2 &= \langle\pi((x^*x)^n)\xi,\ \pi(y^*y)\xi\rangle \\ &\leqq \|\pi(y^*y)\xi\|\,\|\pi((x^*x)^n)\xi\| \\ &= \|\pi(y)^*\pi(y)\xi\|\,\|(\pi(x^n))^*(\pi(x)^n)\xi\| \\ &\leqq \|\pi(y)^2\xi\|\,\|\pi(x)^{2n}\xi\| \\ &\leqq a_{y,\xi}f(y)^2a_{x,\xi}f(x)^{2n}\end{aligned}$$

for suitable $a_{y,\xi}>0$, $a_{x,\xi}>0$. It follows that $\pi(y)\xi\in B_f(\pi)$.

Proof of Theorem 2.1. That (2) implies (3) follows immediately from the definition of closed linear span of bounded representations. Conversely, assume (3). Let $D=N^A$ be directed by the partial order $f\leqq g$ if $f(x)\leqq g(x)$ for all $x\in A$. For $\alpha\in D$, let $X_\alpha=B_\alpha(\pi)\cap X$, a π-invariant subspace by Lemma 2.2. Let $H_\alpha=\bar{X}_\alpha$, closure in H. Clearly $H_\alpha\uparrow$. Define a hermitian representation π_α of A on H_α with domain $D(\pi_\alpha)=X_\alpha$ by $\pi_\alpha(x)=\pi(x)|_{X_\alpha}$, $x\in A$. Then, for all $\xi\in X_\alpha$, $x\in A$, $n\in N$, one has

$$\|\pi_\alpha(x^n)\xi\|\leqq a_{x,\xi}(\alpha(x))^n .$$

Now

$$\begin{aligned}\|\pi_\alpha(x^n)\xi\|^2&\leqq\|\xi\|\|(\pi_\alpha(x^n))^*\pi_\alpha(x)^n\xi\|\\&=\|\xi\|\|\pi_\alpha(x)^{2n}\xi\| .\end{aligned}$$

Hence, by iterations, one gets, for all $n\in N$,

$$\begin{aligned}\|\pi_\alpha(x)\xi\|^2&\leqq\|\xi\|^{2(1-1/2^n)}\|\pi_\alpha(x)^{2^n}\xi\|^{2/2^n}\\&\leqq\|\xi\|^{2(1-1/2^n)}[a_{x,\xi}\alpha(x)^{2^n}]^{2/2^n} .\end{aligned}$$

Hence $\|\pi_\alpha(x)\xi\|\leqq\alpha(x)\|\xi\|$ ($\xi\in D(\pi_\alpha)$, $x\in A$), showing that π_α is bounded; $\|\pi_\alpha(x)\|\leqq\alpha(x)$ ($x\in A$). Hence, $D(\bar{\pi}_\alpha)=H_\alpha$, $\bar{\pi}_\alpha\uparrow$, $\bar{\pi}_\alpha$ is bounded, and $\pi_\alpha\subset\bar{\pi}_\alpha\subset\bar{\pi}=\pi$. Thus $\pi|_X\subset\bigcup_\alpha\pi_\alpha\subset\pi$. This gives, by assumption about X, that $\bar{\pi}=(\pi|_X)^-\subset(\bigcup\bar{\pi}_\alpha)^-=\vee\bar{\pi}_\alpha\subset\bar{\pi}=\pi$ showing that $\pi=\vee\bar{\pi}_\alpha$. Thus (3) implies (2). Now assume (1), say $\pi=\sum\pi_i$, with each π_i a bounded hermitian representation with domain $D(\pi_i)=H_i$. Then $H_i\subset B(\pi)$ for each i, and it follows by the definition of direct sum, that $B(\pi)$ is a core for π. Thus (1)$\Rightarrow$(3)$\Rightarrow$(2). Finally, assume (2), so that $\pi=\vee\{\pi_\beta|\beta\in J\}$, $\pi_\beta\uparrow$, each π_β bounded, and $D(\pi_\beta)=H_\beta$. Then $\bigcup_\beta H_\beta\subset B(\pi)\subset D(\pi)$; and since π is closed, $D(\pi)=\overline{B(\pi)}^{t_\pi}$ closure in the locally convex topology t_π on $D(\pi)$ defined by seminorms $\xi\to\|\pi(x)\xi\|$, $x\in A$. (Note that π being closed, $(D(\pi), t_\pi)$ is complete). Thus, for each $h=h^*$ in A,, $D_\omega(\pi(h))$ is dense in H. It follows from Nelson's Analytic Vector Theorem [14, Th. 8.31, p. 261] that $\pi(h)$ is essentially selfadjoint. Hence by [9(I), Th. 7.1], π is standard, and hence selfadjoint. Let $B(H)$ denote the C^*-algebra of all bounded linear operators on H. Selfadjointness of π implies [9(I), Th. 4.7] that the weak bounded commutant $\pi(A)'_W=\{T\in B(H)|\langle T\pi(x)\xi, \eta\rangle=\langle T\xi, \pi(x^*)\eta\rangle$ for $x\in A$, ξ, $\eta\in D(\pi)\}$ is a von Neumann algebra; and there is a one-one correspondence between orthogonal projections in $\pi(A)'_W$ and selfadjoint π-invarient subspaces of $D(\pi)$. Now let $\xi\in B(\pi)$, $M=\pi(A)\xi$. Then M is π-invarient, $M\subset B(\pi)$. Let $H_\xi=\bar{M}$ (norm closure in H), $M_\xi=\bar{M}^{t_\pi}$ (closure in $(D(\pi), t_\pi)$). Let $D(\pi_\xi)=M_\xi$; and $\pi_\xi(x)=\pi(x)|_{M_\xi}$ ($x\in A$). Then $(\pi_\xi, D(\pi_\xi), H_\xi)$ is a bounded closed hermitian representation of A, hence is selfadjoint, with the result, the orthogonal projection $E_\xi: H \to H_\xi$ is in $\pi(A)'_W$. Thus $(1-E_\xi) \in \pi(A)'_W$, $(1-E_\xi)D(\pi) \subset D(\pi)$. Let $N = (1-E_\xi)D(\pi)$, $H_1 = H_\xi^\perp = \bar{N}$. Let $D(\pi_1) = \bar{N}^{t_\pi}$, $\pi_1 = \pi|_{D(\pi_1)}$. Then π_1 is a selfadjoint representation of A, and there exists $\eta \in B(\pi) \cap H_1$. As above, this would define a bounded closed hermitian representation $(\pi_\eta, D(\pi_\eta), H_\eta)$ such that $H_\xi \perp H_\eta$. Now, by Zorn's Lemma, there exists a maximal family $(\xi_i | i \in I)$ of vectors in $D(\pi)$ such that each $H_i = \overline{\pi(A)\xi_i} \subset D(\pi)$, $H_i \perp H_j$ $(i \neq j)$; and $\pi_i = \pi|_{D(\pi_i)}$, with $D(\pi_i) = (\overline{\pi(A)\xi_i})^{t_\pi}$, defines a bounded closed hermitian representation of A. The maximality and the assumption that π is closed imply that $\pi = \sum \pi_i$; and (1) follows. This completes the proof of Theorem 2.1.

Corollary 2.4. *Let $(\pi, D(\pi), H)$ be a selfadjoint representation of a commutative $*$-algebra A. Then $\pi = \pi_c \oplus \pi_d$, where π_c is a closed linear span of bounded hermitian representations and π_d is completely unbounded in the sense that π_d admits no nonzero bounded vector. Thus, if π is irreducible, then either π is one dimensional or it admits no nonzero bounded vector.*

Remark 2.5. Let $(\pi, D(\pi), H)$ be a selfadjoint representation of a commutative $*$-algebra A. By Lemma 2.2, $\pi(A)D_\omega(\pi) \subset D_\omega(\pi)$. Let $D(\pi_\omega)$=closure of $D_\omega(\pi)$ in $(D(\pi), t_\pi)$. Let H_ω=norm closure of $D_\omega(\pi)$ in H. Let $\pi_\omega(x) = \pi(x)|_{D(\pi_\omega)}$ $(x \in A)$. Then $(\pi_\omega, D(\pi_\omega), H_\omega)$ is a closed hermitian representation admitting a dense set of analytic vectors. Hence π is standard, and so is selfadjoint. Thus the projection $E_\omega: H \to H_\omega$ is in $\pi(A)'_W$; and the complementary representation $(\pi_s, D(\pi_s), H_s)$, contains no nonzero analytic vector. Thus one gets an analytic decomposition of π as $\pi = \pi_\omega \oplus \pi_s$.

Example 2.6. Let Z be a measure space with positive measure M. Following [7], a dense subalgebra $\mathfrak{A}$ of $L^2(Z, \mu)$ is a *$*$-algebra in $L^2(Z, \mu)$* if $\mathfrak{A}$ is a $*$-algebra with pointwise operations and complex conjugation. Let A be any commutative $*$-algebra. Let ϕ be a $*$-homomorphism of A onto a $*$-algebra in $L^2(Z, \mu)$. Define a hermitian representation σ of A in the Hilbert space $L^2(Z, \mu)$ by $\sigma(x) = M_{\phi(x)}$, $M_{\phi(x)}g = \phi(x)g$, viz. the multiplication operator with maximum domain

$$D(M_{\phi(x)}) = \{g \in L^2(Z, \mu) | \phi(x)g \in L^2(Z, \mu)\}.$$

The domain of σ is $D(\sigma) = \cap \{D(M_{\phi(x)}) | x \in A\}$. Then σ is a standard representation of A. The following hold:

(i) $L_2^\omega(Z, \mu) = \bigcap_{2 \le p < \infty} L^p(Z, \mu)$ and $L^\omega(Z, \mu) = \bigcap_{1 \le p < \infty} L^p(Z, \mu)$ are $*$-algebras in $L^2(Z, \mu)$ [1].

(ii) If μ is finite, then $L^\infty(Z, \mu) \subset L^\omega(Z, \mu) = L_2^\omega(Z, \mu)$.

(iii) Any $*$-algebra in $L^2(Z, \mu)$ is contained in $L_2^\omega(Z, \mu)$.

(iv) $B(\sigma)=\left\{g\in D(\sigma)\,\middle|\,\begin{matrix}\text{for each } x\in A,\ \phi(x) \text{ is essentially}\\ \text{bounded on the support of } g\end{matrix}\right\}$.

3. Representations of $\mathcal{P}(x)$

Let $\mathcal{P}(x)$ be the free commutative algebra in one hermitian generator x. Thus $\mathcal{P}(x)$ is the $*$-algebra with identity consisting of all complex polynomials in $x=x^*$. Inoue and Takesu [5] have investigated selfadjoint representations of $\mathcal{P}(x)$. In what follows, the results in [5] are refined and supplimented using the results in Section 2.

Theorem 3.1. *Let $(\pi, D(\pi), H)$ be a closed hermitian representation of $\mathcal{P}(x)$. Then the following hold.*

(A) $B(\pi)=B(\pi(x))$

(B) *The following are equivalent*

(1) *π is standard*

(2) *π is a direct sum of bounded hermitian representations.*

Proof. (A) Let $\xi\in B(\pi(x))$. Choose $a>0$, $c>0$ such that $\|\pi(x)^n\xi\|\leqq ac^n$ for all $n\in \boldsymbol{N}$. Let $P(x)=\sum_{j=0}^{k} a_j x^j\in\mathcal{P}(x)$. Then $P(x)^n=\sum_{j_1,\cdots,j_n=0}^{k} a_{j_1}\cdots a_{j_n}x^{j_1+\cdots+j_n}$. Hence, for all $n\in\boldsymbol{N}$,

$$\|\pi(P(x))^n\xi\|\leqq \sum_{j_1,\cdots,j_n=0}^{k} a_{j_1}a_{j_2}\cdots a_{j_n}\|\pi(x)^{j_1+\cdots+j_n}\xi\|\leqq aP(c)^n .$$

Thus $\xi\in B(P(x))$; and (A) follows.

(B) By Theorem 2.1, (2)$\Rightarrow$(1). Assume that π is standard. Then $\overline{\pi(x)}$ is a selfadjoint operator; hence $B(\overline{\pi(x)})$ is dense in H. The spectral resolution

$$\overline{\pi(x)}=\int_{-\infty}^{\infty} dE(\lambda)$$

implies that

$$B(\overline{\pi(x)})=\bigcup_{n=1}^{\infty} E[-n, n] \quad H=B(\overline{\pi(x)})\cap C^{\infty}(\overline{\pi(x)})$$

[12, p. 365]. But since π is selfadjoint, [5, Th. 2.1] implies that $D(\pi)=C^{\infty}(\overline{\pi(x)})$. Hence

$$B(\pi)=B(\pi(x))=D(\pi)\cap B(\overline{\pi(x)})=B(\overline{\pi(x)}),$$

which is dense in $D(\pi)$; and $\pi=(\pi|_{B(\pi)})^{-}$, as $D(\pi|_{B(\bar{\pi})})$ is the completion of $B(\pi)$ in the induced topology t_π. Theorem 2.1 implies (2).

The following follows from Theorem 3.1, [5, Th. 2.1] and [3].

Corollary 3.2. *Let $(\pi, D(\pi), H)$ be a closed hermitian representation of $\mathcal{P}(x)$. The following are equivalent;*

(1) *π is standard.*

(2) *π is a direct sum of bounded hermitian representations.*

(3) *π is selfadjoint.*

(4) $\pi(\mathcal{P}(x))'_W D(\pi)=D(\pi)$.

(5) *$\pi(x)$ is essentially selfadjoint and $D(\pi)=C^\infty(\overline{\pi(x)})$.*

(6) *$\pi(x)^n$ is essentially selfadjoint for all $n\in\boldsymbol{N}$.*

(7) *$\pi(\mathcal{P}(x))'_W=\pi(\mathcal{P}(x))'_s$ and $D(\pi^*)=\pi(\mathcal{P}(x))'_W D(\pi)$.*

(8) $\pi(\mathcal{P}(x))'_W=\pi(\mathcal{P}(x))'_s$.

Recall [3] that $\pi(A)'_s=\{C\in\pi(A)' \mid CD(\pi)\subset D(\pi)\}$, the *strong commutant* for a hermitian representation π of a $*$-algebra A.

The following refines [9(I), Lemma 3.2] and [5, Th. 2.4].

Proposition 3.3. *Let $(\pi, D(\pi), H)$ be a closed hermitian representation of $\mathcal{P}(x)$. Then the following are equivalent.*

(1) *π^* is standard.*

(2) *π^* is hermitian.*

(3) *π^* is selfadjoint.*

(4) *$\pi(x)$ is essentially selfadjoint.*

(5) *$\pi(\mathcal{P}(x))'_W$ is an algebra.*

(6) $D(\pi^*)=C^\infty(\overline{\pi(x)})$.

Proof. In above, (2) iff (3) iff (1) follow from [9(I), p. 95] and Corollary 3.2; (3) iff (4) is [5, Th. 2.4] and (4) iff (5) is [9(I), Lemma 3.2]. We show (5)$\Rightarrow$(6)$\Rightarrow$(3). Assume (5). Let

$$\overline{\pi(x)}=\int_{-\infty}^{\infty}\lambda dE(\lambda)$$

be the spectral theorem for the selfadjoint operator $\overline{\pi(x)}$. Let $\xi\in C^\infty(\overline{\pi(x)})=\bigcap_{n=1}^{\infty} D(\overline{\pi(x)}^n)$. There exists a sequence (ξ_k) in $D(\pi)$ such that $\xi_k\to\xi$. For each $n\in\boldsymbol{N}$,

$$E_n=\int_{-n}^{n} dE(\lambda)$$

is in $\pi(\mathcal{P}(x))'_W$; and so, for all n, k,

$$E_n\xi_k\in\pi(\mathcal{P}(x))'_W D(\pi)\subset D(\pi^*)$$

[9(I), Lemma 4.5]. Now, for each n,

$$E_n\xi_k \longrightarrow E_n\xi, \qquad \pi(x)^m E_n\xi_k \longrightarrow \overline{\pi(x)}^m E_n\xi \quad (m\in\boldsymbol{N}),$$

$\overline{\pi(x)}^m$ being bounded on $E_nD(\overline{\pi(x)})$. Thus, for all $z\in\mathcal{P}(x)$, $\pi(z)E_n\xi_k\to\overline{\pi(z)}E_n\xi$ for all n. Since π^* is closed, $E_n\xi\in D(\pi^*)$ for all $n\in\boldsymbol{N}$. Also, as $n\to\infty$,

$$E_n\xi\longrightarrow\xi,\quad \overline{\pi(x)}^m E_n\xi\longrightarrow\overline{\pi(x)}^m\xi,\quad \pi^*(x)^m E_n\xi\longrightarrow\pi^*(x)^m\xi\quad (m\in\boldsymbol{N}).$$

As above, by the closedness of π^*, $\xi\in D(\pi^*)$. Thus $C^\infty(\overline{\pi(x)})\subset D(\pi^*)$. Also,

$$\begin{aligned} D(\pi^*)&=\cap\{D(\pi(y^*)^*)\,|\,y\in\mathcal{P}(x)\}\\ &\subset\cap\{D((\pi(x^n))^*)\,|\,n\in\boldsymbol{N}\}=\cap\{D((\pi(x)^n)^*)\,|\,n\in\boldsymbol{N}\}\\ &=\cap\{D((\pi(x)^*)^n)\,|\,n\in\boldsymbol{N}\}\quad\text{(by [5, Lemma 2.2])}\\ &=\cap\{D(\overline{\pi(x)}^n)\,|\,n\in\boldsymbol{N}\}=C^\infty(\overline{\pi(x)}), \end{aligned}$$

$\pi(x)$ being essentially selfadjoint. It follows that (5)$\Rightarrow$(6). Now assume (6). Observe that $\pi\subset\pi^*$, hence $\pi^{**}\subset\pi^*$. For any $z\in\mathcal{P}(x)$, $\pi^*(z)=\pi(z^*)^*|_{D(\pi^*)}$, hence $\pi^*(z)\subset\pi(z^*)^*$, and so $\pi(z^*)\subset(\pi^*(z))^*$. Therefore,

$$\begin{aligned} D(\pi^{**})&=\cap\{D(\pi^*(z)^*)\,|\,z\in\mathcal{P}(x)\}\\ &\supset\cap\{D(\pi(z^*)^{**})\,|\,z\in\mathcal{P}(x)\}=\cap\{D(\overline{\pi(z^*)})\,|\,z\in\mathcal{P}(x)\}\\ &=\cap\{D(\overline{\pi(z)})\,|\,z\in\mathcal{P}(x)\}=\cap\{D((\overline{\pi(x)})^n)\,|\,n\in\boldsymbol{N}\}=D(\pi^*) \end{aligned}$$

by (6). Thus $\pi^*\subset\pi^{**}$, and so $\pi^*=\pi^{**}$ showing that π^* is selfadjoint.

Proposiiton 3.4. *Let $(\pi, D(\pi), H)$ be a hermitian representation of $\mathcal{P}(x)$. The following are equivalent:*

(1) *π has a standard extension in H.*
(2) *π has a selfadjoint extension in H.*
(3) *$\pi(x)$ has a selfadjoint extension in H.*

Proof. That (1)$\Rightarrow$(2)$\Rightarrow$(3) follow from Corollary 3.2. Assume (3). Let T be a selfadjoint operator with dense domain $D(T)$ in H such that $D(\pi)\subset D(T)$ and $\pi(x)\subset T$. Then $T=(T|_{B(T)})^-=(T|_{D_\omega(T)})^-=(T|_{C^\infty(T)})^-$. Taking $D(\sigma)=C^\infty(T)$, define $\sigma(P(x))=P(T)|_{D(\sigma)}$ $(P(x)\in\mathcal{P}(x))$. The hermitian representation $(\sigma, D(\sigma), H)$ is an extension of π satisfying $\overline{\sigma(x)}=T$. Since $\overline{T^n}=(\overline{T})^n$ $(n\in\boldsymbol{N})$,

$$\begin{aligned} D(\bar\sigma)&=\cap\{D(\overline{\sigma(z)})\,|\,z\in\mathcal{P}(x)\}\\ &\subset\cap\{D(\overline{\sigma(x^n)})\,|\,n\in\boldsymbol{N}\}=\cap\{D((T^n|_{D(\sigma)})^-)\,|\,n\in\boldsymbol{N}\}\\ &\subset\cap\{D(\overline{T}^n)\,|\,n\in\boldsymbol{N}\}=\cap\{D(T^n)\,|\,n\in\boldsymbol{N}\}=D(\sigma), \end{aligned}$$

showing that σ is closed. Also,

$$D(\sigma)=\bigcap_{n=1}^{\infty}D(T^n)=\bigcap_{n=1}^{\infty}D(\overline{\sigma(x)}^n).$$

Hence σ is standard by Corollary 3.2.

It follows that π is selfadjoint iff π is closed and $\pi(x)$ admits a selfadjoint extension T in H such that $D(\pi)=\bigcap_{n=1}^{\infty} D(T^n)$.

4. Representations of $\mathcal{P}(x, y)$

Let $\mathcal{P}(x, y)$ be the free commutative $*$-algebra with 1 generated by two commuting hermitian generators x, y. In [5, Th. 3.2], various standardness criteria for representations of $\mathcal{P}(x, y)$ have been discussed. In this section, using Theorem 2.1, several other standardness criteria for π are discussed.

Theorem 4.1. *Let $(\pi, D(\pi), H)$ be a closed hermitian representation of $\mathcal{P}(x, y)$. Then the following hold:*

(A) $B(\pi)=B(\pi(x)+i\pi(y))=B(\pi(x))\cap B(\pi(y))$.

(B) *The following are equivalent.*

(1) *π is standard.*

(2) *π is a direct sum of bounded hermitian representations.*

Proof. (A) Sublemma 2.3 gives $B(\pi(x)+i\pi(y))=B(\pi(x))\cap B(\pi(y))$. Define

$$B(\pi(x), \pi(y)) = \{\xi\in D(\pi) \mid \exists a, c_1, c_2>0 \text{ s. t. } \|\pi(x)^k\pi(y)^m\xi\|\leqq a c_1^k c_2^m \ (k, m\in \boldsymbol{N})\}.$$

Let $\xi\in B(\pi(x), \pi(y))$ with a, c_1, c_2 as above. Let

$$P(x, y)=\sum_{i=0}^{k}\sum_{j=0}^{m} a_{ij}x^i y^j .$$

Then, for any $n\in \boldsymbol{N}$,

$$P(x, y)^n = \sum_{i_1,\cdots,i_n=0}^{k}\ \sum_{j_1,\cdots,j_n=0}^{m} a_{i_1j_1}a_{i_2j_2}\cdots a_{i_nj_n}x^{i_1+i_2+\cdots+i_n}y^{j_1+\cdots+j_n} .$$

Hence,

$$\begin{aligned}\|(\pi P(x, y))^n\xi\| &= \|P(\pi(x), \pi(y))^n\xi\| \\ &\leqq \sum a_{i_1j_1}a_{i_2j_2}\cdots a_{i_ni_n}\|\pi(x)^{i_1+\cdots+i_n}\pi(y)^{j_1+\cdots+j_n}\xi\| \\ &\leqq aP(c_1, c_2)^n .\end{aligned}$$

Thus $\xi\in B(\pi(P(x, y)))$ for all $P(x, y)$; and $B(\pi(x), \pi(y))\subset B(\pi)$. Conversely, let $\xi\in B(\pi)$. Let n, m be in $\boldsymbol{N}$. Then

$$\|\pi(x)^m\pi(y)^n\xi\|^2=\langle\pi(x^m y^n)\xi,\ \pi(x^m y^n)\xi\rangle=\langle\pi(x)^{2m}\xi,\ \pi(y)^{2n}\xi\rangle \qquad \cdots\cdots\cdots\cdots(*)$$
$$\leqq\|\pi(x)^{2m}\xi\|\,\|\pi(y)^{2n}\xi\|\leqq a_{x_1}c_{x_1}^{2m}a_{y_1}c_{y_1}^{2n}\leqq ac_1^{2m}c_2^{2n}$$

for suitable a, c_1 and c_2. Thus $\xi\in B(\pi(x),\ \pi(y))$, and $B(\pi)=B(\pi(x),\ \pi(y))$. Obviously, $B(\pi(x),\ \pi(y))\subset B(\pi(x))\cap B(\pi(y))$; whereas, above $(*)$ implies $B(\pi(x))\cap B(\pi(y))\subset B(\pi(x),\ \pi(y))$. This proves (A).

(B) Assume (1). By [5, Th. 3.2], $\overline{\pi(x)}$ and $\overline{\pi(y)}$ are selfadjoint operators with mutually commuting spectral projections, so that $N=\overline{\pi(x)}+i\overline{\pi(y)}$ is a normal operator. As noted in [7, p. 399], $D(N^n)=D((\overline{\pi(x)})^n)\cap D((\overline{\pi(y)})^n)$; hence $C^\infty(N)=\bigcap_{n=1}^{\infty}D(N^n)=D(\pi)$ by [5, Th. 3.2]. Let

$$N=\int\lambda dE(\lambda)$$

be the spectral resolution. Then $B(N)=\bigcup_{c>0}E(D_c)H$, where $D_c=\{z\in\boldsymbol{C}\mid |z|\leqq c\}$. Hence $B(N)\subset C^\infty(N)$. Thus

$$B(N)=B(N)\cap C^\infty(N)=B(N)\cap D(\pi)$$
$$=B(\overline{\pi(x)}+i\overline{\pi(y)})\cap D(\pi)=B(\pi(x)+i\pi(y))=B(\pi).$$

Thus, the π-invariant subspace $B(\pi)$ is dense in H; and $(\pi_{B(\pi)})^-=\pi$. By Theorem 2.1, (2) follows.

The following supplements [5, Th. 3.2]. For the sake of completeness and comparision with Corollary 3.2, relavant statements from this reference are included herein.

Theorem 4.2. *Let $(\pi,\ D(\pi),\ H)$ be a hermitian representation of $\mathcal{P}(x, y)$. Then the following are eqiuvalent.*

(1) *π is standard.*

(2) *π is closed; and it is a direct sum of bounded hermitian representations.*

(3) *$\pi(x)+i\pi(y)$ is essentially normal, and $D(\pi)=\bigcap_{n=1}^{\infty}\{D((\overline{\pi(x)})^n)\cap D((\overline{\pi(y)})^n)\}$.*

(4) *For each $x=1, 2, \cdots$, $\pi(x)^n$ and $\pi(y)^n$ are essentially selfadjoint; $\pi(x)+i\pi(y)$ has a normal extension in H; and π is closed.*

(4′) *For all $n=1, 2, \cdots$, $(\overline{\pi(x)})^n$ and $(\overline{\pi(y)})^n$ are selfadjoint with mutually commuting spectral projections.*

(5) *π is closed, and $\pi(x^2+y^2)$ is essentially selfadjoint.*

(6) *$\pi((x^2+y^2)^n)$ is essentially selfadjoint for all $n=1, 2, \cdots$.*

(7) *$\pi(x^2+y^2)'_W D(\pi)=D(\pi)$.*

(8) *π is selfadjoint, and $\pi(x)+i\pi(y)$ has a normal extension in H.*

(9) *$\pi(x)$ and $\pi(y)$ are essentially selfadjoint, $D(\pi)=\bigcap_{n=1}^{\infty}(D((\overline{\pi(x)})^n)\cap D((\overline{\pi(y)})^n))$ and $\pi(x)+i\pi(y)$ has a normal extension in H.*

(10) *π is closed, $\overline{\pi(x)}$ and $\overline{\pi(y)}$ are selfadjoint operators with mutually commuting spectral projections and $\pi(\mathcal{P}(x, y))'_W D(\pi)=D(\pi)$.*

Proof. The assertions (1) iff (8) iff (9) iff (10) constitute [5, Th. 3.2]; where (1) iff (2) follows from Theorem 4.1. Now assume (1). Then

$$\overline{\pi(x)}+i\overline{\pi(y)}=N=\int\lambda dE(\lambda)$$

is a normal operator. In fact, $N=\overline{\pi(x)+i\pi(y)}$. Indeed, let $\xi\in D(N)$. Then, for $c>0$,

$$\xi_c=E(\Delta_c)\xi \quad (\Delta_c=\{z\in \boldsymbol{C} \mid |z|\leqq c\})$$

is a bounded vector for N, and

$$\xi_c\in\bigcap_n D(N^n)=C^\infty(N)=\bigcap_n(D((\overline{\pi(x)})^n)\cap D((\overline{\pi(y)})^n))=D(\pi).$$

Also, $\xi_c\to\xi$ as $c\to\infty$, and $(\pi(x)+i\pi(y))\xi_c=N\xi_c$ form a Cauchy sequence $(c=1/n)$. Thus $\xi\in D(\overline{\pi(x)+i\pi(y)})$, and $N\subset\overline{\pi(x)+i\pi(y)}$. Since normal operators are maximal among formally normal operators, $N=\overline{\pi(x)+i\pi(y)}$. Thus (1)$\Rightarrow$(3). Conversely, assume that $N=\overline{\pi(x)+i\pi(y)}$ is normal, and $D(\pi)=\bigcap_{n=1}^{\infty}(D((\overline{\pi(x)})^n)\cap D((\overline{\pi(y)})^n))$, Then $D(\pi)=\bigcap_n D(N^n)=C^\infty(N)$. Thus, the formally normal operator $T=\pi(x)+i\pi(y)$ with domain $D(T)=D(\pi)$ contains a dense set of analytic vectors, and it satisfies

$$TD(T)\subset D(T), \qquad T^*D(T)\subset D(T).$$

One has

$$\|T\xi\|^2=\|\pi(x)\xi\|^2+\|\pi(y)\xi\|^2 \quad (\xi\in D(T)),$$

and $D(T)$ is also invariant for $\pi(x)$ and $\pi(y)$. By [4, Th. 2.1], one gets, for all $\xi\in D(T)$, $n\in\boldsymbol{N}$, that

$$\|\pi(x)^n\xi\|\leqq\|T^n\xi\|, \qquad \|\pi(y)^n\xi\|\leqq\|T^n\xi\|.$$

Thus, vectors in $D_\omega(T)$ are also analytic vectors for the hermitian operators $\pi(x)$ and $\pi(y)$. It follows from [16, Th. 3.1] that $\overline{\pi(x)}$ and $\overline{\pi(y)}$ are selfadjoint operators with mutually commuting spectral projections; and $\overline{\pi(x)}+i\overline{\pi(y)}$ is a normal operator. Thus (3)$\Rightarrow$(9)$\Rightarrow$(1). Next, we show that (1)$\Leftrightarrow$(4). Assume that π is standard. Then, as above, $\overline{\pi(x)+i\pi(y)}=\overline{\pi(x)}+i\overline{\pi(y)}$ is a normal operator; and for each $n=1, 2, 3, \cdots$ $\pi(x^n)$ and $\pi(y^n)$ are essentiallys elfadjoint operators with domain $D(\pi)$. Thus $\overline{\pi(x^n)}=(\overline{\pi(x)})^n$ and $\overline{\pi(y^n)}=(\overline{\pi(y)})^n$ are selfadjoint operators having mutually commuting spectral projections. Thus (1) implies each of (4) and (4'). Conversely, assume that π is closed, $\pi(x)^n$ and $\pi(y)^n$ are essentially selfadjoint for all $n\in\boldsymbol{N}$, and that there exists a normal operator N in H which is an extension of $\pi(x)+i\pi(y)$. Then, for all n, $\overline{\pi(x)^n}$

$=(\overline{\pi(x)})^n$, $\overline{\pi(y)^n}=(\overline{\pi(y)})^n$; and by [5, Th. 3.1], there exists a standard representation $(\sigma, D(\sigma), H)$ of $\mathcal{P}(x, y)$ which is an extension of π, where

$$D(\sigma)=\bigcap_n (D(A^n)\cap D(B^n)),$$

$$A=\mathrm{Re}\, N=\frac{1}{2}\overline{(N+N^*)}, \qquad B=\mathrm{Im}\, N=\frac{1}{2i}\overline{(N-N^*)}.$$

But, A being an extension of the selfadjoint operator $\overline{\pi(x)}$, one has $\overline{\pi(x)}=A$; and similarly, $\overline{\pi(y)}=B$. Hence $\overline{\pi(x)}+i\overline{\pi(y)}=N$. Since π is closed,

$$\begin{aligned} D(\pi)=D(\bar{\pi})&=\bigcap \{D(\overline{\pi(P(x, y))})\,|\,P(x, y)\in\mathcal{P}(x, y)\} \\ &=\bigcap_n \{D(\overline{P(\pi(x), \pi(y))})\} \\ &=\bigcap_n (D(\overline{\pi(x)^n})\cap D(\overline{\pi(y)^n})) \\ &=\bigcap_n (D((\overline{\pi(x)})^n)\cap D((\overline{\pi(y)})^n)=D(\sigma). \end{aligned}$$

Thus $\pi=\sigma$; and it follows that $(4)\Rightarrow(1)$. It is immediate that $(4')\Rightarrow(4)$.

For the remaining assertions, let $h=x^2+y^2$. By [9 (I), p. 99], $\mathcal{P}(h)$ dominates $\mathcal{P}(x, y)$ in any $*$-representations, i.e., given a hermitian representation π of $\mathcal{P}(x, y)$, for each $z\in\mathcal{P}(x, y)$, there exists $w\in\mathcal{P}(h)$ such that $\|\pi(z)\xi\|\leqq\|\pi(w)\xi\|$ $(\xi\in D(\pi))$. Hence,

$$\begin{aligned} \|\pi(z)^2\xi\|&=\|\pi(z)\pi(z)\xi\|\leqq\|\pi(w)\pi(z)\xi\|=\|\pi(wz)\xi\| \\ &=\|\pi(zw)\xi\|=\|\pi(z)\pi(w)\xi\|\leqq\|\pi(w)\pi(w)\xi\| \\ &=\|\pi(w)^2\xi\| \quad (\xi\in D(\pi)). \end{aligned}$$

By repeating, $\|\pi(z)^n\xi\|\leqq\|\pi(w)^n\xi\|$ for all $n\in\boldsymbol{N}$, $\xi\in D(\pi)$. Let $(\pi_1, D(\pi_1), H)$ be the hermitian representation of $\mathcal{P}(h)$ defined on $D(\pi_1)=D(\pi)$ as $\pi_1(k)=\pi(k)$ $(k\in\mathcal{P}(h))$. It follows from above that:

(a) on $D(\pi)=D(\pi_1)$, the induced topologies (as defined in the proof of Theorem 2.1) agree; viz. $t_{\pi(\mathcal{P}(x,y))}=t_{\pi(\mathcal{P}(h))}$. Hence π is closed iff π_1 is closed.

(b) every analytic (resp. bounded) vector in $D(\pi)$ for $\pi(w)$ is analytic (resp. bounded) for $\pi(z)$.

Now we use above to show that $(1)\Leftrightarrow(5)\Leftrightarrow(6)\Leftrightarrow(7)$. Clearly $(1)\Rightarrow(5)$. By [5, Th. 2.1] or Corollary 3.2, $(5)\Leftrightarrow(6)\Leftrightarrow(7)\Leftrightarrow\pi_1$ is standard $\Leftrightarrow\pi_1$ is selfadjoint. Assume (5). For any $w\in\mathcal{P}(h)$, $\pi(w)$ is essentially normal with invariant domain $D(\pi)$. Thus $D_\omega(\pi(w))$ is dense in H. It follows from above that, for any $z\in\mathcal{P}(x, y)$, $D_\omega(\pi(z))$ is dense; hence by [8, Lemma 3.2], $\overline{\pi(z)}$ is normal. Thus π is standard. This completes the proof.

Remarks. Unlike $\mathcal{P}(x)$ (see Corollary 3.2), a selfadjoint hermitian representation of $\mathcal{P}(x, y)$ need not a standard. [9(I), p. 102] gives an example of an infinte dimensional irreducible selfadjoint representation π of $\mathcal{P}(x, y)$ such that (i) $\pi(\mathcal{P}(x, y))'_W$ is a von Neumann algebra, (ii) $\pi(x)^n$ and $\pi(y)^n$ are essentially selfadjoint for all n, (iii) π is not standard, (iv) $\pi(x)+i\pi(y)$ is not essentially normal. Infact, in this case, $\pi(x)+i\pi(y)$ does not have a normal extension, even in a possibly larger Hilbert space [10]. This shows that for a representation π of $\mathcal{P}(x, y)$, $\pi(\mathcal{P}(x, y))'_W$ being an algebra does not imply that $\pi(x)+i\pi(y)$ is normal (compare with Proposition 3.3).

Question I. Let $(\pi, D(\pi), H)$ be a hermitian representation of $\mathcal{P}(x, y)$. Find suitable necessary and sufficient conditions for π to be selfadjoint?

A comparision of Corollary 3.2 and Theorem 4.2 suggests the following. For a closed hermitian representation $(\pi, D(\pi), H)$ of $\mathcal{P}(x, y)$, are the following equivalent?

(1) π is selfadjoint.

(2) $\pi(x)^n$ and $\pi(y)^n$ are essentially selfadjoint for all n.

(3) $\pi(x)$ and $\pi(y)$ are selfadjoint and $D(\pi)=\bigcap_n \{D(\overline{(\pi(x))^n})\cap D((\overline{\pi(y)})^n)\}$.

(4) $\pi(\mathcal{P}(x, y))'_W D(\pi)=D(\pi)$.

Question II. What is an analogue of Proposition 3.3 for $\mathcal{P}(x, y)$?

Let π be a hermitian representation of $\mathcal{P}(x, y)$. Are the following equivalent?

(1) π^* is hermitian (equivalently, selfadjoint [9(I)]).

(2) $\pi(\mathcal{P}(x, y))'_W$ is an algebra.

(3) $\pi(x)$ and $\pi(y)$ are essentially selfadjoint.

Further, is it ture that π^* is standard iff $\pi(x)+i\pi(y)$ is essentially normal?

Analogous to Proposition 3.4, it can be shown that given a closed hermitian representation $(\pi, D(\pi), H)$ of $\mathcal{P}(x, y)$, π has a standard extension in H (resp. in a larger Hilbert space K containing H isometrically) iff π has a selfadjoint extension in H (resp. in K) and $\pi(x)+i\pi(y)$ has a normal extension in H (resp. in K). Given a symmetric operator T with dense domain $D(T)$ in a Hilbert space H, there exists a Hilbert space K containing H isometrically, and a selfadjoint operator S in K which is an extension of T [13]. Contrarily, there exist formally normal operators N in a Hilbert space H which fail to admit normal extension in H or in any large Hilbert space K containing H isometrically. In fact, if A and B are selfadjoint operators which commute on a common core D and for which the *spectral* projections do not commute, then $N=(A+iB)|_D$ is a formally normal operator having no normal extension in a possibly large Hilbert space [10]. Using these, one can construct a selfadjoint representation of $\mathcal{P}(x, y)$ that does not admit a standard extension in any larger Hilbert space [11].

Question III. Does Theorem 2.1 hold for noncommutative $*$-algebras?

Question IV. Let $(\pi, D(\pi), H)$ be a standard representation of a commutative $*$-algebra A. Is $D(\pi)=D(\pi_\omega)$? (Here $D(\pi_\omega)$=closure of $D_\omega(\pi)$ in $D(\pi)$ in the induced topology defined by seminorms $\xi \rightarrow \|\pi(x)\xi\|$, $x \in A$.) Is $B(\pi)$ a core for π?

A Final Note: After the preparation of the manuscript, the author came across a recently published (November 1992) paper by I. Ikeda and A. Inoue: Invariant subspaces for closed $*$-representations of $*$-algebras, *Proc. Amer. Math. Soc.,* **116** (1992), 737-745. Ikeda and Inone have also discussed the decompositions given in Corollary 2.4 and Remark 2.5, as well as part (B) of Theorem 4.1.

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