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ON AUTOMATIC CONTINUITY OF HOMOMORPHISMS

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ABSTRACT. Introducing a weaker notion of regularity in a topological algebra, we examine and improve an automatic continuity theorem given by the second author. Examples and applications are given.

All topological algebras considered are commutative and Hausdorff, having a unit element. A topological algebra A is a *Q-algebra* if the set $G(A)$ of invertible elements is open. Let B be a subalgebra of an algebra A. Then B is *inverse closed in A* if $G(B) = B \cap G(A)$. A is *strongly semisimple* if for every $x \in A$, $x \neq 0$, there exists a nonzero continuous multiplicative linear functional χ such that $\chi(x) \neq 0$. A is *advertibly complete* if a Cauchy net x_α in A converges in A whenever for some y in A, $x_\alpha + y - x_\alpha y$ converges to 0. A Q-algebra is advertibly complete [Ma, p. 45]. A *uniform seminorm* on an algebra A is a seminorm p such that $p(x^2) = p(x)^2$ for all x in A. Such a p is submultiplicative [BK]. A *uniform topological algebra A* is a topological algebra whose topology is defined by a family of uniform seminorms. Such an A is semisimple. The abbreviation lmca will stand for locally m-convex algebra.

In [B], the following is given.

Theorem ([B, Theorem 2.2])**.** *Let A be a spectrally bounded, regular, complete, uniform topological algebra. If B is an lmca and $\phi : A \to B$ is a one-to-one homomorphism such that $(\operatorname{Im}\phi)^-$ (the closure of $\operatorname{Im}\phi$) is a semisimple Q-algebra, then $\phi^{-1}/\operatorname{Im}\phi$ is continuous.*

In the proof, the author considers the map $\phi^* : \sigma(C) \to \sigma(A)$, with $\phi^*(f) = f \circ \phi$, $\sigma(C)$ and $\sigma(A)$ denoting respectively the spaces of nonzero continuous multiplicative functionals on $C = (\operatorname{Im}\phi)^-$ and A. In Math. Reviews, the reviewer R. J. Loy [L] asserted that the continuity of ϕ has been implicitly used in [B]. Indeed, ϕ^* is not always well defined when ϕ is not continuous as the following example shows.

Example 1. Let Ω denote the first uncountable ordinal and $[0, \Omega)$ the set of all ordinals smaller than Ω. Consider the algebra $C[0, \Omega)$ of complex continuous functions on $[0, \Omega)$ with compact open topology τ. Every $f \in C[0, \Omega)$ is bounded. It is a regular uniform lmca. The identity map $\phi : (C[0, \Omega), \tau) \to (C[0, \Omega), \| \|_\infty), \phi(f) = f$, satisfies the hypotheses of the above statement. It is well known that $(C[0, \Omega), \tau)$ has discontinuous multiplicative linear functionals (see, for example, [Mi], [Z]); let χ

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be such a functional. It is in $\sigma(C[0,\Omega), \| \ \|_\infty)$, but $\phi^*(\chi) = \chi$ is not in $\sigma([0,\Omega), \tau)$. One may give another example in a more general situation. Let X be a non-compact, locally compact pseudocompact space. Take $A = C(X)$ the algebra of all continuous functions on X with compact open topology, B the algebra $C(X)$ endowed with the sup norm $\| \ \|_\infty$ and $\phi : A \to B$ be $\phi(f) = f$. Since X is pseudo-compact, non-compact, it is not realcompact [S, p. 44]. Let y be an element of the real compactification of X with y not in X. The evaluation δ_y at y is in $\sigma(B)$ but not in $\sigma(A)$. Hence ϕ^* is not defined on δ_y.

On the other hand, ϕ^* can be well defined even when ϕ is not continuous; in this case, the proof given in [B] works. Here is an example of such a situation.

Example 2. Let X be a compact Hausdorff space. Consider the algebra $C(X)$ of continuous complex functions on X. Take $A = (C(X), \tau_d)$, τ_d being the topology of uniform convergence on all countable compact subsets of X, B the algebra $(C(X), \| \ \|_\infty)$ and ϕ the identity map from A to B. It is of course discontinuous, only if X is uncountable. But in this case, $\sigma(A)$ and $\sigma(B)$ are both homeomorphic to X. So ϕ^* is well defined.

The following theorem repairs the above result. On one hand, it provides a positive result in a context more general than above; on the other hand, it shows that if one assumes the continuity of ϕ in the above result, then the algebra A is necessarily a Banach algebra.

Theorem 1. *Let A be weakly regular, advertibly complete, uniform topological algebra, let B be an lmca, and let $\phi : A \to B$ be a one-to-one homomorphism such that $(\operatorname{Im}\phi)^-$ is a semisimple Q-algebra.*

(1) *If A is functionally continuous, then $\phi^{-1}/\operatorname{Im}\phi$ is continuous.*
(2) *If ϕ is continuous, then the topology of A is normable.*

Following [Mi, p. 51], A is *functionally continuous* (FC) if every multiplicative functional on A is continuous. Note that $(C(X), \tau_d)$ in Example 2 is FC, but not Q. A major unsolved problem in topological algebras is the Michael problem: Is every multiplicative linear functional on a Frechet lmc algebra continuous? This has led to several sufficient conditions for A to be FC. This makes FC a reasonable assumption.

Let A be a commutative topological algebra. A is *weakly regular* if given a closed set $F \subset \sigma(A)$, $F \neq \sigma(A)$, there exists $x \neq 0$ in A such that $f(x) = 0$ for all $f \in F$. In the context of Banach algebras, weak regularity arises naturally in the study of uniqueness of the uniform norm [BD]; and it is weaker than regularity. This is exhibited in an example due to Barnes [Me, Example 1]. Let $D = \{z \in C : |z| < 1\}$, $X = \bar{D} \times [0,1]$. Let $A = \{f \in C(X) : f \text{ is holomorphic on } D \times \{0\}\}$, a uniform Banach algebra. Then A is weakly regular, but not regular. Since regularity in a uniform algebra is a stringent property, the validity of Theorem 1 under weak regularity is interesting.

Proof of Theorem 1. (1) Let $C = (\operatorname{Im}\phi)^-$. Since A is FC, $\phi^* : \sigma(C) \to \sigma(A)$, $\phi^*(f) = f \circ \phi$ is well defined. It is also continuous with respective Gelfand topologies. Since the algebra C is a locally convex Q-algebra, $\sigma(C)$ is compact [Ma, p. 187]. Then $\phi^*(\sigma(C))$ is a compact subset of $\sigma(A)$. We have $\phi^*(\sigma(C)) = \sigma(A)$. Indeed, if $\phi^*(\sigma(C)) \neq \sigma(A)$, there exists $x \neq 0$ in A such that $\chi(\phi(x)) = 0$ for all χ in $\sigma(C)$.

Since C is commutative semisimple and lmc, $\phi(x) = 0$; and then $x = 0$ for ϕ is one-to-one. Thus $\phi^*(\sigma(C)) = \sigma(A)$. Now let $P = (p_\alpha)$ be a family of uniform seminorms defining the topology τ of A. Since A is advertibly complete, the spectrum $\mathrm{Sp}_A(x) = \{\chi(x) : \chi \in \sigma(A)\}$ and the spectral radius $\rho_A(x) = \sup_\alpha\{\lim_{n\to\infty}(p_\alpha(x^n))^{1/n}\}$ for all x in A [Ma, p. 104, p. 99]. Since $\sigma(A)$ is compact, $\mathrm{Sp}_A(x)$ is bounded. Since $p_\alpha(x^2) = p_\alpha(x)^2$ for all x and α, $\rho_A(x) = \sup_\alpha p_\alpha(x)$ for all $x \in A$. Also, $\sigma(A) = \phi^*(\sigma(C))$ gives $\mathrm{Sp}_A(x) = \{f(\phi(x)) : f \in \sigma(C)\} \subset \mathrm{Sp}_C(\phi(x))$. Further, as C is a Q-algebra, $s(C) = \{x \in C : \rho_C(x) \leq 1\}$ is a neighbourhood of 0 by [Mi, Prop. 13.5, p. 58]; and there exists a convex balanced open set W such that $0 \in W \subseteq s(C)$. The Minkowski functional q of W in C is a continuous seminorm satisfying $\rho_C(y) \leq q(y)$ for all $y \in C$. Hence for each α, for each $x \in A$, $p_\alpha(x) \leq \rho_A(x) \leq \rho_C(\phi(x)) \leq q(\phi(x))$. This proves that $\phi^{-1}/\mathrm{Im}\,\phi$ is continuous.

(2) Suppose ϕ is continuous. Then $\phi^* : \sigma(C) \to \sigma(A)$ is well defined even if A is not FC. Then $\phi^{-1}/\mathrm{Im}\,\phi$ is continuous as above making ϕ a topological isomorphism. Thus $\phi(A)$ is advertibly complete; and hence inverse closed in its completion. Whence it is inverse closed in the Q-algebra C, for the completion of $\phi(A)$ is contained in C. Therefore $\phi(A)$, and so A, is a Q-algebra. Hence the topology on A given by the algebra norm ρ_A is finer than τ. Now since A is a Q-algebra, $s(A) = \{x \in A : \rho_A(x) \leq 1\}$ is a neighbourhood of 0 on (A, τ). Thus ρ determines τ. □

Remark. Once ϕ^* is defined, the full strength of weak regularity has not been used. In fact, one has only to find a nonzero element vanishing on a given compact set.

We now give a result in the absence of FC. We consider the space $\sigma^*(A)$ consisting of all nonzero multiplicative functionals on A endowed with the weak topology $\sigma(A^*, A)$. We then introduce the following notion of weak σ^*-compact-regular weakened in the sense of the previous remark.

Definition. A commutative topological algebra A is called *weakly σ^*-compact-regular* if for a compact subset K of $\sigma^*(A)$, $K \neq \sigma^*(A)$, there exists a nonzero $x \in A$ such that $\chi(x) = 0$ for all $\chi \in K$.

Theorem 2. *Let A be a weakly σ^*-compact-regular advertibly complete uniform algebra, B a locally convex algebra and $\phi : A \to B$ a one-to-one homomorphism such that $C = (\mathrm{Im}\,\phi)^-$ is a strongly semisimple Q-algebra. Then $\phi^{-1}/\mathrm{Im}\,\phi$ is continuous. If ϕ is continuous, then the topology of A is normable.*

For the proof, consider the map $\phi^{**} : \sigma(C) \to \sigma^*(A)$, $\phi^{**}(\chi) = \chi \circ \phi$. If $\chi \circ \phi$ is identically zero, then by the continuity of χ, one obtains that χ is also identically zero. This contradicts $\chi \in \sigma(C)$. Thus ϕ^{**} is defined; and then it is continuous. Now one obtains $\phi^{**}(\sigma(C)) = \sigma^*(A)$; and the proof can be completed as in Theorem 1.

We conjecture that the semisimplicity of $(\mathrm{Im}\,\phi)^-$ in Theorem 1 (and strong semisimplicity in Theorem 2) can be omitted. The following supports this.

Theorem 3. *Let A be a uniform lmca, B a locally convex algebra, and $\phi : A \to B$ a one-to-one homomorphism such that $(\mathrm{Im}\,\phi)^-$ is a Q-algebra. Assume that at least one of the following holds.*

(a) *A is advertibly complete and $\mathrm{Im}\,\phi$ is FC with continuous product.*
(b) *A is FC, Ptak (as a l.c. space), regular, having locally equicontinuous spectrum $\sigma(A)$ (in particular, A is FC, Frechet, regular, having locally compact spectrum*

$\sigma(A)$), and B is lmca. Then $\phi^{-1}/\operatorname{Im}\phi$ is continuous. If ϕ is continuous, then the topology of A is normable.

Proof. (1) Assume (a). Then $\sigma^*((\operatorname{Im}\phi)^-) = \sigma((\operatorname{Im}\phi)^-)$ (since a Q-algebra is FC) $= \sigma(\operatorname{Im}\phi)$ (by the joint continuity of multiplication in $\operatorname{Im}\phi$) $= \sigma^*(\operatorname{Im}\phi)$ and $\phi^*(\sigma^*(\operatorname{Im}\phi)) = \sigma^*(A)$ as ϕ is one-to-one. Then, for all $x \in A$, $\operatorname{Sp}_A(x) = \{\chi(x) : \chi \in \sigma(A)\} = \{\chi(x) : \chi \in \sigma^*(A)\} = \{f(\phi(x)) : f \in \sigma^*(\operatorname{Im}\phi)\} = \{f(\phi(x)) : f \in \sigma^*(C)\}$. Hence for some continuous seminorm q, $\rho_A(x) = \rho_C(x) \leq q(\phi(x))$ $(x \in A)$.

(2) Assume (b). By [Ma, Coro. 1.3, p. 184], local equicontinuity of $\sigma(A)$ implies continuity of the Gelfand map $x \to \hat{x}$ and local compactness of $\sigma(A)$. We show that $\sigma(A) = \phi^*(\sigma(C))$. Note that $\phi^*(\sigma(C)) \subset \sigma(A)$. Suppose $\chi \in \sigma(A)\backslash\phi^*(\sigma(C))$. By the local compactness, there exists a compact set $K \subseteq \sigma(A)$ and disjoint open sets U, V in $\sigma(A)$ such that $\chi \in K \subset U$, $\phi^*(\sigma(C)) \subset V$. As A is Ptak, regular and having continuous Gelfand map, [Ma, Coro. 4.4, p. 344] implies that there exist $x, y \in A$ such that $g(x) = 1$ $(g \in \phi^*(\sigma(C)))$, $g(x) = 0$ $(g \in \sigma(A)\backslash V)$; $g(y) = 1$ $(g \in K)$, $g(y) = 0$ $(g \in \sigma(A)\backslash U)$. Then $g(x)g(y) = 0$ for all $g \in \sigma(A)$. By the semisimplicity of A, $xy = 0 = \phi(x)\phi(y)$. On the other hand, for all $f \in \sigma(C)$, $f(\phi(x)) = 1$. Thus $0 \notin \{f(\phi(x)) : f \in \sigma(C)\} = \operatorname{Sp}_C(\phi(x))$, C being lmc and a Q-algebra. Thus $\phi(x)$ is invertible in C. Hence $\phi(y) = \phi(x)^{-1}\phi(x)\phi(y) = 0$, so that $y = 0$, a contradiction. It follows that $\phi^*(\sigma(C)) = \sigma(A)$. Now the proof can be completed as in Theorem 1. Note that if A is Frechet, then every compact subset of $\sigma(A)$ is equicontinuous [Mi, Prop. 4.2, p. 17]. Hence by [Ma, Th. 1.1, p. 182], the Gelfand map is continuous. Further if $\sigma(A)$ is locally compact, then it is locally equicontinuous [Ma, Cor. 1.3, p. 184]. □

Remarks. (1) If B is lmca and $C = (\operatorname{Im}\phi)^-$ is a semisimple Q-algebra, then C is strongly semisimple.

(2) Actually in the above theorems, $\phi^{-1} : \phi(A) \to (A, \|\ \|)$ is continuous, where $\|\ \|$ is the uniform norm given by $\|x\| = \sup\{p(x) : p$ is a continuous uniform seminorm$\}$. The existence of this norm implies that A is spectrally bounded.

(3) Theorem 2 also applies to Example 1. Indeed, the algebra $(C[0,\Omega), \tau)$ is a complete uniform algebra. It is weakly σ^*-compact-regular, for $\sigma^*(C[0,\Omega))$ is homeomorphic to the Stone-Cech compactification $\beta[0,\Omega)$ of $[0,\Omega)$ and $C[0,\Omega)$ is isomorphic to the algebra $C(\beta[0,\Omega))$ [GJ].

(4) The hypothesis $(\operatorname{Im}\phi)^-$ is a Q-algebra cannot be omitted. Let $A = C_b(\mathbb{R})$ be the algebra of all continuous bounded functions on the real line. Endowed with the sup norm, it is a uniform Banach algebra. By the same arguments as in (3), one shows that A is weakly σ^*-compact-regular. Consider $B = C(\mathbb{R})$ to be the algebra of all continuous functions with the compact open topology. Consider $\phi : A \to B$, $\phi(f) = f$. Then $(\operatorname{Im}\phi)^- = C(\mathbb{R})$. It is well known that it is not a Q-algebra. Clearly ϕ^{-1} is discontinuous.

(5) The referee has asked: (In above theorems) does the automatic continuity of ϕ^{-1} on $\operatorname{Im}\phi$ necessitate $(\operatorname{Im}\phi)^-$ a Q-algebra? The following answers this.

Proposition 4. *Let A be a complete Q-lmca, B an lmca, and $\phi : A \to B$ a one-to-one homomorphism such that $\phi^{-1}/\operatorname{Im}\phi$ is continuous. Then $(\operatorname{Im}\phi)^-$ is a Q-algebra.*

Proof. We may assume that B is complete, hence $C = (\operatorname{Im}\phi)^-$ is complete. Then $\phi^{-1}/\operatorname{Im}\phi$ extends as a continuous homomorphism $\psi : C \to A$. By assumption, there exists a continuous seminorm p on A such that for all x in A, $r_C(\phi(x)) \leq r_{\operatorname{Im}\phi}(\phi(x)) \leq r_A(x) \leq p(x) \leq p(\psi(\phi(x)))$. Since ψ is continuous, there exists a continuous seminorm q on C such that $p(\psi(y)) \leq q(y)$ $(y \in C)$. Hence in the above, $r_C(\phi(x)) \leq p(\psi(\phi(x))) \leq q(\phi(x))$ for all x in A. Now let $y \in C$, $y = \lim \phi(x_\alpha)$ for some net (x_α) in A. For any continuous multiplicative functional f on C, $|f(y)| \leq |f(y - \phi(x_\alpha))| + |f(\phi(x_\alpha))| \leq |f(y - \phi(x_\alpha))| + q(\phi(x_\alpha)) \to q(y)$. Hence $r_C(y) = \sup |f(y)| \leq q(y)$ $(y \in C)$ showing that C is a Q-algebra.

If A is not a Q-algebra, then this does not hold. For the open unit disc U in the complex plane, let $A = H(U)$ be the uniform Frechet algebra consisting of holomorphic functions on U with the compact-open topology, $B = C(U)$ with the compact-open topology, and $\phi : A \to B$ be $\phi(f) = f$. Clearly ϕ is a homeomorphism and $(\operatorname{Im} \phi)^- = B$ fails to be a Q-algebra. □

Applications

(1) Proposition. *Let A be an advertibly complete lmca. Let $\| \ \|$ be any continuous norm on A. Then A cannot be simultaneously weakly regular and uniform unless the topology of A is normable.*

Proof. Let τ denote the lmc topology on A. Let $P = (p_\alpha)$ be a family of submultiplicative seminorms on A defining τ. Then $P_0 = P \cup \{\| \ \|\}$ also determines τ. Suppose (A, τ) is uniform. Then τ is defined by a family $S = (q_i)$ of uniform seminorms. By closing S with maxima of finite subfamilies and applying the continuity of $\| \ \|$, there exists a q in S which is a norm. Let A_q be the uniform Banach algebra obtained by completing (A, q). Let $\phi : A \to A_q$ be $\phi(x) = x$. Now if A is weakly regular, then Theorem 1 applied to ϕ implies that τ is normable.

It follows that an advertibly complete non-normed weakly regular uniform algebra cannot support a continuous norm. Let X be a compact Hausdorff space. By a well known result of Kaplansky, if $| \ |$ is any norm on $C(X)$ making it a normed algebra, then the supnorm $\| \ \| \leq | \ |$. The following has a bearing with this. A norm on an algebra A is *semisimple* if the completion of $(A, | \ |)$ is semisimple [BD]. □

(2) Corollary. *Let $\| \ \|$ be a uniform norm on an algebra A such that $(A, \| \ \|)$ is a weakly regular Q-normed algebra. Let $| \ |$ be any submultiplicative norm on A.*

(i) *If $| \ |$ is semisimple, then $\| \ \| \leq | \ |$. Further if $| \ |$ is continuous, then $| \ |$ is equivalent to $\| \ \|$.*
(ii) *Let $(A, \| \ \|)$ be complete and regular. Then $\| \ \| \leq | \ |$ for any submultiplicative norm $| \ |$.*

Indeed let $\phi : (A, \| \ \|) \to (\widetilde{A}, | \ |)$ ($\widetilde{A} =$ completion of $(A, | \ |)$), $\phi(x) = x$. Theorem 1 implies that there exists $k > 0$ such that $\| \ \| \leq k | \ |$. Since $(A, \| \ \|)$ is Q, $\rho_A(x) = \inf \|x^n\|^{1/n} = \lim \|x^n\|^{1/n} = \|x\| \leq \lim |x^n|^{1/n} \leq |x|$ for all $x \in A$. (ii) follows by Theorem 3(b).

(3) Let $A = \mathbb{C} \times C_c^\infty(\mathbb{R})$ (resp. $B = \mathbb{C} \times C_c(\mathbb{R})$) be the algebra of all complex C^∞-functions (resp. continuous functions) on $\mathbb{R}$ which are constant outside some compact set (depending on the function). We endow A (resp. B) with the inductive limit topology τ_D (resp. τ_K). The algebra (A, τ_D) is a complete regular lmca, (B, τ_K) is a lmc Q-algebra [Ma, p. 128] and A is dense in B [K, p. 148]. Consider $\phi : A \to B$, $\phi(f) = f$. Since the topology τ_D is finer than τ_K on A, ϕ is continuous. It is classical that A is not normable. Hence by Theorem 1, (A, τ_D) cannot be uniform.

Let A be the algebra $\mathbb{C} \times C_c^\infty(\mathbb{R})$ endowed with the compact open topology τ. It is a weakly σ^*-compact-regular uniform lmca. Since it is inverse closed in $C(\mathbb{R})$, it is advertibly complete. Take (B, τ_K) as above. Let $\phi : (A, \tau) \to (B, \tau_K)$, $\phi(f) = f$. By Theorem 1, ϕ^{-1} is continuous.

(4) Let $U \subset \mathbb{C}^d$ be open. Let $H(U)$ be the uniform Frechet algebra of all holomorphic functions on U with the compact open topology. Let $H^\infty(U) = \{f \in H(U) : f \text{ is bounded}\}$, a uniform Banach algebra. Let $X \subset \mathbb{C}^d$ be compact. Let $H(K)$ be the algebra of holomorphic germs on X. Choose a decreasing sequence (U_n) of open neighbourhoods of X such that $\bar{U}_{n+1} \subset U_n$ and $\bar{U}_{n+1}$ is compact. In view of the continuous embeddings

$$\cdots \to H^\infty(U_n) \to H(U_n) \to H^\infty(U_{n+1}) \to H(U_{n+1}) \to \cdots,$$

$H(X)$ can be realized as inductive limits $H(X) = \varinjlim H(U_n) = \varinjlim H^\infty(U_n)$, its topology τ being the finest locally convex topology making all $\phi_n : H(U_n) \to H(X)$, $\phi_n(f) = f/X$ and similarly making all $\psi_n : H^\infty(U_n) \to H(X)$, $\psi_n(f) = f/X$ continuous.

None of $H^\infty(U)$ and $H(U)$ is weakly regular. Note that $(H(X), \tau)$ is a complete semisimple Q-algebra [Ma, p. 134]. If $H(U_n)$ is weakly regular, then by Theorem 1, it becomes a Banach algebra and ϕ_n becomes a homeomorphism. If $H^\infty(U_n)$ is weakly regular, then ψ_n becomes a homeomorphism. Either of these forces $H(X)$ to be a uniform Banach algebra. Being a complete, non-normed Q-algebra, $H(X)$ is not a uniform algebra [BD].

(5) Let $D = \{z \in \mathbb{C} : |z| < 1\}$, $Y = D \times [0,1]$, $Z = \bar{D} \times [0,1)$. Let $A = \{f \in C(Y) : f \text{ is holomorphic on } D \times \{0\}\}$, $B = \{f \in C(Z) : f \text{ is holomorphic on } D \times \{0\}\}$. Let $0 < r < 1$. Let $\|f\|_r = \sup\{|f(x)| : x \in \bar{D}_r \times [0,1]\}$ $(f \in A)$; $|f|_r = \sup\{|f(x)| : x \in \bar{D} \times [0,r]\}$ $(f \in B)$. Each of A and B with the topology defined respectively by $\{\| \ \|_r : 0 < r < 1\}$ and $\{| \ |_r : 0 < r < 1\}$ is a uniform Frechet algebra. Any $f \in C(Y)$ (resp. $f \in C(Z)$) vanishing on $D \times \{0\}$ is in A (resp. in B). This implies that both A and B are weakly regular, not regular. Thus each of A and B fails to support a continuous norm.

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