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The quasi-linear evolution of the density field in models of gravitational instability

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ABSTRACT

Two quasi-linear approximations, the frozen-flow approximation (FFA) and the frozen-potential approximation (FPA), have been proposed recently for studying the evolution of a collisionless, self-gravitating fluid. In FFA it is assumed that the velocity field remains unchanged from its value obtained from the linear theory, whereas in FPA the same approximation is made for the gravitational potential. In this paper, we compare these and the older Zel'dovich approximation by calculating the evolution of the density in perturbation theory. In particular, we compute the skewness, including the smoothing effects, and the kurtosis for FFA, FPA and the Zel'dovich approximation, and compare their relative accuracy.

Key words: galaxies: clustering – cosmology: theory – large-scale structure of Universe.

1 INTRODUCTION

Large-scale structures observed today are believed to have developed from small density fluctuations generated in the early Universe. The growth of these fluctuations has been studied by regarding the system to be a collisionless, self-gravitating fluid. When the amplitudes of the fluctuations are small, perturbation theory can be used to study the evolution of the system. In particular, the growth rate of the rms fluctuations can be described by the linearized equations of fluid motion. When the fluctuations eventually become non-linear, however, the perturbation theory is no longer accurate, and *N*-body simulations have been widely used to overcome this difficulty. The understanding of the physical processes that take place in such a self-gravitating fluid necessitates, however, the use of analytical models and approximations that can be more easily studied.

The best-known model for describing the mildly non-linear evolution is that of Zel'dovich (1970) (see also Shandarin & Zel'dovich 1989). In this approximation the motion of each particle is determined by its initial Lagrangian displacement. Presentation of this approximation has been made by Moutarde et al. (1991), Bouchet et al. (1992) and Buchert (1992) in the frame of a Lagrangian description. The evolution of the density field can be studied in this approximation until the formation of caustics, when this approach breaks down. Recently, two new approximations have been proposed, with the aim of improving upon the Zel'dovich approximation – (1) the frozen-flow approximation (FFA) (Matarrese et al. 1992), and (2) the frozen-potential approximation (FPA) (Brainerd, Scherrer & Villumsen 1993; Bagla & Padmanabhan 1994). In FFA the velocity flows are 'frozen' to their local initial linear values, and at any time the velocity of each particle is the one associated to the point at which it lies. The evolution of the density is then treated exactly. In the second approximation, FPA, the gravitational potential is 'frozen' at its linear value. That is, the Eulerian potential is kept constant, and the particles obey the standard Eulerian equations of motion in this potential.

All the three approximations mentioned above are naturally consistent with the linear theory. However, beyond the linear order the density evolution is different in each case. To compare them we calculate, assuming Gaussian initial conditions, the third- and fourth-order moments of density by means of perturbation theory. We then compare the results obtained for each of these approximations with those obtained from perturbation theory using the exact dynamics.

In Section 2, we recall the basic equations of motion and the calculations of the third and fourth moments of density (Peebles 1980; Fry 1984; Grinstein & Wise 1987; Bouchet et al. 1992; Bernardeau 1993) for the exact dynamics, as well as for the Zel'dovich approximation. These computations are repeated for FFA and FPA in Sections 3 and 4, respectively. Throughout we assume an $\Omega=1$, spatially flat universe, so that the scale factor evolves as $a(t)=a_0(t/t_0)^{2/3}$.

2 BASIC EQUATIONS

The evolution of a collisionless, self-gravitating fluid in an expanding Robertson–Walker universe is described by the

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following equations in the Newtonian limit (Peebles 1980):

$$\dot{\delta}+\frac{1}{a}\nabla\cdot[(1+\delta)\,\boldsymbol{v}]=0, \tag{1}$$

$$\dot{\boldsymbol{v}}+\frac{\dot{a}}{a}\boldsymbol{v}+\frac{1}{a}[\boldsymbol{v}\cdot\nabla]\,\boldsymbol{v}=-\frac{1}{a}\nabla\phi, \tag{2}$$

$$\nabla^2\phi=4\pi G\rho_{\rm b}a^2\delta. \tag{3}$$

Here $\delta\equiv[\rho(\boldsymbol{x},t)-\rho_{\rm b}(t)]/\rho_{\rm b}(t)$ is the enhancement of the true density $\rho(\boldsymbol{x},t)$ over the mean density $\rho_{\rm b}(t)$, $\boldsymbol{v}$ is the proper peculiar velocity, relative to the Hubble flow, and $\phi(\boldsymbol{x},t)$ is the peculiar gravitational potential.

From these equations it follows that the density contrast $\delta(\boldsymbol{x},t)$ evolves according to the equation

$$\ddot{\delta}+\frac{2\dot{a}}{a}\dot{\delta}-4\pi G\rho_{\rm b}\delta$$
$$=4\pi G\rho_{\rm b}\delta^2+\frac{1}{a^2}\nabla_i\delta\cdot\nabla_i\phi+\frac{1}{a^2}\nabla_i\nabla_j[(1+\delta)\,v^iv^j]. \tag{4}$$

In the linear theory, all the terms on the right-hand side of (4) are dropped, and $\delta\equiv\delta^{(1)}(\boldsymbol{x},t)$ has the solution $\delta^{(1)}(\boldsymbol{x},t)=A(\boldsymbol{x})\,D(t)$, with $D(t)\propto a(t)\propto t^{2/3}$ (growing mode) for $\Omega=1$.

It is convenient to define the potential $\Delta(\boldsymbol{x},t)$ through the relation $\phi=4\pi G\rho_{\rm b}a^2\Delta$, so that $\nabla^2\Delta=\delta$. The peculiar velocity $\boldsymbol{v}^{(1)}$ in the linear theory is

$$\boldsymbol{v}^{(1)}=-\frac{a\dot{D}}{D}\nabla\Delta^{(1)}, \tag{5}$$

where

$$\Delta^{(1)}(\boldsymbol{x})=-\frac{1}{4\pi}\int{\rm d}^3x'\,\frac{\delta(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}=\frac{\phi^{(1)}}{4\pi G\rho_{\rm b}a^2}. \tag{6}$$

The solution for δ in perturbation theory is obtained via the perturbation expansion,

$$\delta=\sum_{n=1}^{\infty}\delta^{(n)},\qquad \boldsymbol{v}=\sum_{n=1}^{\infty}\boldsymbol{v}^{(n)},\qquad \Delta=\sum_{n=1}^{\infty}\Delta^{(n)}, \tag{7}$$

with respect to the initial Gaussian fluctuations, so that $\delta^{(n)}$ satisfies the equation (Fry 1984)

$$\ddot{\delta}^{(n)}+\frac{2\dot{a}}{a}\dot{\delta}^{(n)}-4\pi G\rho_{\rm b}\delta^{(n)}$$
$$=\sum_{k=1}^{n-1}\Bigg[4\pi G\rho_{\rm b}\delta^{(k)}\delta^{(n-k)}+4\pi G\rho_{\rm b}\nabla_i\delta^{(k)}\nabla_i\Delta^{(n-k)}$$
$$+\frac{1}{a^2}\nabla_i\nabla_jv^{(k)i}v^{(n-k)j}+\frac{1}{a^2}\sum_{m=1}^{k-1}\nabla_i\nabla_j\delta^{(m)}v^{(k-m)i}v^{(n-k)j}\Bigg]. \tag{8}$$

The derivation of the behaviour of the first cumulants of the density distribution at large scale can be done assuming Gaussian initial conditions. It can be shown that, in general (Goroff et al. 1986; Bernardeau 1992),

$$\langle\delta^p\rangle_{\rm c}=S_p\langle\delta^2\rangle^{p-1}, \tag{9}$$

where S_p is a coefficient that depends weakly on the cosmological parameters. Moreover, when the smoothing effects are neglected, these coefficients are independent of the shape of the power spectrum. Bernardeau (1992) gives a method to derive the whole series of the coefficients for the exact dynamics, but we consider here only the first two coefficients, S_3 and S_4.

Let us first recall the principle of the calculation of S_3 and S_4 for the exact dynamics. The derivation of the skewness of the distribution function requires the calculation of the density contrast at second order, $\delta^{(2)}$. It is obtained by setting $n=2$ in equation (8):

$$\ddot{\delta}^{(2)}+\frac{2\dot{a}}{a}\dot{\delta}^{(2)}-4\pi G\rho_{\rm b}\delta^{(2)}$$
$$=4\pi G\rho_{\rm b}[\delta^{(1)2}+\nabla_i\delta^{(1)}\nabla_i\Delta^{(1)}]+\frac{1}{a^2}\nabla_i\nabla_j[v^{(1)i}v^{(1)j}].$$

The solution for the growing mode is

$$\delta^{(2)}=\tfrac{5}{7}\delta^{(1)2}+\delta^{(1)}_{,i}\Delta^{(1)}_{,i}+\tfrac{2}{7}\Delta^{(1)2}_{,ij}. \tag{10}$$

Using this solution, it is straightforward to calculate the skewness $S_3\equiv\langle\delta^3\rangle/\langle\delta^2\rangle^2$, which to the lowest order is $3\langle\delta^{(1)2}\delta^{(2)}\rangle/\langle\delta^{(1)2}\rangle^2=34/7$ (Peebles 1980).

The calculation of the fourth cumulant involves the knowledge of the density field at the third order in perturbative calculation. It can be shown that the coefficient S_4 can be written (Fry 1984; Bernardeau 1992) as $S_4=12R_{\rm a}+4R_{\rm b}$, where

$$R_{\rm a}=4\langle\delta^{(1)2}\delta^{(2)2}\rangle/\langle\delta^{(1)2}\rangle^3, \tag{11}$$

and

$$R_{\rm b}=\langle\delta^{(1)3}\delta^{(3)}\rangle/\langle\delta^{(1)2}\rangle^3. \tag{12}$$

The value of $R_{\rm a}$ can be easily obtained from equation (10); thus $R_{\rm a}=(34/21)^2=2.62$. The third-order term of the density contrast, $\delta^{(3)}$, can be calculated more easily in Fourier space, by first defining

$$\tilde{\delta}(\boldsymbol{k})=\frac{1}{V}\int{\rm d}^3\boldsymbol{x}\,\delta(\boldsymbol{x})\,{\rm e}^{{\rm i}\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{13}$$

and similar transforms for $\boldsymbol{v}$ and Δ. Equation (8) for $n=3$ then gives a solution for $\tilde{\delta}^{(3)}$, which can be used to show that $R_{\rm b}=682/189=3.61$, and that $S_4=45.88$ (see Fry 1984; Bernardeau 1992).

These results, however, concern the behaviour of the cumulants at a given point. When the field is filtered at a given scale, the values of S_3 and S_4 have to be changed. We recall here the results obtained for a top-hat window function (Bernardeau 1993; Juszkiewicz, Bouchet & Colombi 1993). They read:

$$S_3=\frac{34}{7}+\gamma_1 \tag{14}$$

and

$$S_4=\frac{60712}{1323}+\frac{62}{3}\gamma_1+\frac{7}{3}\gamma_1^2+\frac{2}{3}\gamma_2, \tag{15}$$

where γ_1 and γ_2 are the first two logarithmic derivatives of the variance with scale,

$$\gamma_1 = \frac{\mathrm{d}\log\langle\delta^2\rangle}{\mathrm{d}\log R}, \tag{16}$$

$$\gamma_2 = \frac{\mathrm{d}^2\log\langle\delta^2\rangle}{\mathrm{d}\log^2 R}. \tag{17}$$

The derivation of these smoothing effects is based on geometrical properties of the top-hat window function given by Bernardeau (1993).

For the Zel'dovich approximation, the behaviour of the cumulants at large scale is similar to the one encountered in the real dynamics, but the coefficients S_p are slightly changed due to the approximation that is made (Grinstein & Wise 1987). Recently, Bernardeau (1993) has derived the expression of the coefficients S_3 and S_4 when a top-hat filter is applied to the density field:

$$S_3^{\mathrm{Zel}} = 4 + \gamma_1, \tag{18}$$

$$S_4^{\mathrm{Zel}} = \frac{272}{9} + \frac{50}{3}\gamma_1 + \frac{7}{3}\gamma_1^2 + \frac{2}{3}\gamma_2. \tag{19}$$

This quantitative change of the large-scale cumulants is general for any approximative dynamics starting with Gaussian initial conditions. The coefficients S_p then turn out to be a good tool to test the various approximations with each other by comparing the values of these coefficients. In the next sections, we will generalize the results obtained for the Zel'dovich approximation to the frozen-flow and the frozen-potential approximations.

3 THE FROZEN-FLOW APPROXIMATION

The frozen-flow approximation (FFA), which was proposed by Matarrese et al. (1992), is best defined using slightly different variables from those in equations (1)–(3). Using the scale factor a as the time variable, define the comoving peculiar velocity $\boldsymbol{u} \equiv \mathrm{d}\boldsymbol{x}/\mathrm{d}a = \boldsymbol{v}/a\dot{a}$, $\eta = 1+\delta$, $\psi \equiv (3t_0^2/2a_0^3)\,\phi$, where $a(t) = a_0(t/t_0)^{2/3}$. Equations (1)–(3) then reduce to

$$\frac{\mathrm{d}\eta}{\mathrm{d}a} + \eta\nabla\cdot\boldsymbol{u} = 0, \tag{20}$$

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}a} + \frac{3}{2a}\boldsymbol{u} = -\frac{3}{2a}\nabla\psi, \tag{21}$$

$$\nabla^2\psi = \delta/a, \tag{22}$$

where $\mathrm{d}/\mathrm{d}a = \partial/\partial a + \boldsymbol{u}\cdot\nabla$.

FFA is defined by assuming that the velocity field $\boldsymbol{u}$ is steady: $\partial\boldsymbol{u}/\partial a = 0$; that is, stream lines are frozen to their initial shape. The frozen value of $\boldsymbol{u}$ would then be the constant value it has in the linear theory:

$$\boldsymbol{u}_{\mathrm{FFA}}(x) = -\nabla\psi_{\mathrm{LIN}}(\boldsymbol{x}). \tag{23}$$

[This, of course, implies that $\boldsymbol{v} = \boldsymbol{v}_{\mathrm{LIN}} = \boldsymbol{v}^{(1)}$, as given by equation (5).] The $\boldsymbol{u}_{\mathrm{FFA}}$ of equation (23) is a solution of the Euler equation (21), provided ψ is approximated to be

$$\psi_{\mathrm{FFA}}(\boldsymbol{x}, t) = \psi_{\mathrm{LIN}} - \frac{a}{3}(\nabla\psi_{\mathrm{LIN}})^2. \tag{24}$$

In the notation of equations (1)–(3), FFA corresponds to

$$\boldsymbol{v}_{\mathrm{FFA}} = \boldsymbol{v}^{(1)}, \qquad \phi_{\mathrm{FFA}} = \phi^{(1)} - \frac{a}{3}\left(\frac{3t_0^2}{2a_0^3}\right)[\nabla\phi^{(1)}]^2$$
$$= \phi^{(1)} - \frac{4\pi}{3}G\rho_{\mathrm{b}}a^2[\nabla\Delta^{(1)}]^2$$
$$\equiv \phi^{(1)} + \phi^{(2)}. \tag{25}$$

The form of $\phi^{(2)}$ implies that it is second-order in perturbation, but it is not the same as the second-order potential in true non-linear evolution. To study the density evolution in FFA, we first note that equations (1)–(3) give

$$\ddot{\delta} + \frac{2\dot{a}}{a}\dot{\delta} = \frac{1}{a^2}(1+\delta)\nabla^2\phi + \frac{1}{a^2}\nabla\delta\cdot\nabla\phi + \frac{1}{a^2}\nabla_i\nabla_j[(1+\delta)v^iv^j]. \tag{26}$$

Here we substitute for $\boldsymbol{v}$ as $\boldsymbol{v}_{\mathrm{FFA}}$ and for ϕ as ϕ_{FFA}. Next, we implement the perturbation expansion of equation (7) to get the following equation for $\delta^{(2)}$:

$$\ddot{\delta}^{(2)} + \frac{2\dot{a}}{a}\dot{\delta}^{(2)} - \frac{1}{a^2}\nabla^2\phi^{(2)}$$
$$= 4\pi G\rho_{\mathrm{b}}\delta^{(1)2} + \frac{1}{a^2}\nabla\delta^{(1)}\cdot\nabla\phi^{(1)} + \frac{1}{a^2}\nabla_i\nabla_j[v^{(1)i}v^{(1)j}]. \tag{27}$$

This equation has the solution

$$\delta_{\mathrm{FFA}}^{(2)} = \tfrac{1}{2}\delta^{(1)2} + \tfrac{1}{2}\delta_{,i}^{(1)}\Delta_{,i}^{(1)}, \tag{28}$$

which should be compared with the $\delta^{(2)}$ for the true evolution, in equation (10). From here, it is straightforward to carry through the calculation of skewness as in Peebles (1980, section 18), since the only change in the solution $\delta^{(2)}$ is that in the coefficients. The result is $S_3^{\mathrm{FFA}} = 3$, as compared to the true value of 34/7. The smoothing effects on S_3 can be easily calculated, and the final result for a top-hat window function reads:

$$S_3^{\mathrm{FFA}} = 3 + \frac{\gamma_1}{2}. \tag{29}$$

To obtain S_4 in FFA, we first find R_{a} from equation (11) by substituting the solution $\delta_{\mathrm{FFA}}^{(2)}$. This gives, upon angle averaging, as in Fry (1984), $R_{\mathrm{a}} = 1.0$. From (26), the equation for $\delta^{(3)}$ in FFA reads:

$$\ddot{\delta}^{(3)} + \frac{2\dot{a}}{a}\dot{\delta}^{(3)} = \frac{1}{a^2}\{\delta^{(2)}\nabla^2\phi^{(1)} + \delta^{(1)}\nabla^2\phi^{(2)}\}$$
$$+ \frac{1}{a^2}\{\nabla\delta^{(2)}\cdot\nabla\phi^{(1)} + \nabla\delta^{(1)}\cdot\nabla\phi^{(2)}\}$$
$$+ \frac{1}{a^2}\nabla_i\nabla_j[\delta^{(1)}v^{(1)i}v^{(1)j}]. \tag{30}$$

Before doing the Fourier transform, we note that

$$\nabla\phi^{(2)} = -\frac{8\pi}{3}G\rho_{\mathrm{b}}a^2(\nabla\Delta^{(1)}\cdot\nabla)\,\nabla\Delta^{(1)}, \tag{31}$$

$$\nabla^2\phi^{(2)} = -\frac{8\pi}{3} G\rho_b a^2 \{\Delta^{(1)}_{,i}\delta^{(1)}_{,i} + \Delta^{(1)}_{,ij}\Delta^{(1)}_{,ij}\}. \quad (32)$$

Using the transform (13) and its inverse,

$$\delta(\boldsymbol{x}) = \frac{V}{(2\pi)^3}\int d^3k\,\tilde{\delta}(\boldsymbol{k})\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \quad (33)$$

the Fourier transform of a product can be written as a convolution (Fry 1984):

$$\mathrm{FT}\{F_1(x)\cdots F_N(x)\}$$

$$= \frac{1}{V^N}\int \frac{d^3k_1}{(2\pi)^3}\cdots\frac{d^3k_N}{(2\pi)^3}\left[V\delta_D\left(\sum \boldsymbol{k}_i - \boldsymbol{k}\right)\right]\tilde{F}_1(\boldsymbol{k}_1)\cdots\tilde{F}_N(\boldsymbol{k}_N)$$

$$= \tilde{F}_1 * \cdots * \tilde{F}_N. \quad (34)$$

When applied to equation (30) for $\delta^{(3)}_{\mathrm{FFA}}$, this gives

$$\ddot{\tilde{\delta}}^{(3)} + \frac{2\dot{a}}{a}\dot{\tilde{\delta}}^{(3)} = 4\pi G\rho_b\left\{\tilde{\delta}^{(2)} * \tilde{\delta}^{(1)} - \frac{2}{3}\tilde{\delta}^{(1)} * \left[\frac{k_i\tilde{\delta}^{(1)}}{k^2}\right] * \left[k_i\tilde{\delta}^{(1)}\right]\right\}$$

$$-\frac{2}{3}\tilde{\delta}^{(1)} * \left[\frac{k_ik_j}{k^2}\tilde{\delta}^{(1)}\right] * \left[\frac{k_ik_j}{k^2}\tilde{\delta}^{(1)}\right]$$

$$+\left[k_i\tilde{\delta}^{(2)}\right] * \left[\frac{k_i}{k^2}\tilde{\delta}^{(1)}\right] - \frac{2}{3}\left[k_i\tilde{\delta}^{(1)}\right] * \left[\frac{k_j\tilde{\delta}^{(1)}}{k^2}\right]$$

$$* \left[\frac{k_ik_j\tilde{\delta}^{(1)}}{k^2}\right] + \frac{1}{a^2}(ik_i)(ik_j)\,\tilde{\delta}^{(1)} * \tilde{v}^{(1)i} * \tilde{v}^{(2)j}. \quad (35)$$

The solution can be written down in analogy with equation (46) of Fry (1984), so we do not put it down explicitly, except to note that now $\delta^{(2)}_{\mathrm{FFA}}$ of equation (28) should be used. Using this $\tilde{\delta}^{(3)}$ and carrying out angular averaging as in Fry's paper, we obtain $R_b = 1$, and hence

$$S_4^{\mathrm{FFA}} = 16. \quad (36)$$

The derivation of the smoothing effects for S_4 is slightly more complicated and will not be given.

4 THE FROZEN-POTENTIAL APPROXIMATION

The frozen-potential approximation (FPA) was proposed by Brainerd et al. (1993) and by Bagla & Padmanabhan (1994). FPA is defined by keeping the potential ϕ constant at its value $\phi^{(1)}(\boldsymbol{x})$ in the linear theory, so that $\phi^{(2)}$ and higher terms in the perturbation expansion of ϕ are set to zero. However, unlike in FFA, $\boldsymbol{v}$ is not approximated, but is to be obtained from the Euler equation (2) with $\phi(\boldsymbol{x},t) \equiv \phi^{(1)}(\boldsymbol{x})$. Thus equation (26) for the evolution of density, when written for the case of FPA, becomes

$$\ddot{\delta} + \frac{2\dot{a}}{a}\dot{\delta} = \frac{1}{a^2}(1+\delta)\nabla^2\phi^{(1)} + \frac{1}{a^2}\nabla\delta\cdot\nabla\phi^{(1)}$$

$$+\frac{1}{a^2}\nabla_i\nabla_j[(1+\delta)v^iv^j]. \quad (37)$$

Using the perturbation expansion (7), the equation for $\delta^{(2)}$ is found to be

$$\ddot{\delta}^{(2)} + \frac{2\dot{a}}{a}\dot{\delta}^{(2)} = 4\pi G\rho_b\delta^{(1)2} + \frac{1}{a^2}\nabla\delta^{(1)}\cdot\nabla\phi^{(1)}$$

$$+\frac{1}{a^2}\nabla_i\nabla_j[v^{(1)i}v^{(1)j}]. \quad (38)$$

The solution is

$$\delta^{(2)} = \frac{1}{2}\delta^{(1)2} + \frac{7}{10}\delta^{(1)}_{,i}\Delta^{(1)}_{,i} + \frac{1}{5}\Delta^{(1)2}_{,ij}, \quad (39)$$

as contrasted to the true $\delta^{(2)}$ in equation (10), and $\delta^{(2)}_{\mathrm{FFA}}$ in equation (28). Once again, the skewness is easy to work out, following Peebles (1980), and the result is $S_3^{\mathrm{FPA}} = 17/5$. The derivation of the smoothing effects gives

$$S_3^{\mathrm{FPA}} = \frac{17}{5} + \frac{7}{10}\gamma_1. \quad (40)$$

The calculation of the kurtosis of the density field can be done as usual. Substitution of this expression for $\delta^{(2)}$ in (11) and application of angle averaging as before gives $R_a = (17/15)^2 = 1.28$. The equation for $\delta^{(3)}$ in FPA is

$$\ddot{\delta}^{(3)} + \frac{2\dot{a}}{a}\dot{\delta}^{(3)} = 4\pi G\rho_b\delta^{(1)}\delta^{(2)} + \frac{1}{a^2}\nabla\delta^{(2)}\cdot\nabla\phi^{(1)} + \frac{1}{a^2}\nabla_i\nabla_j$$

$$\times[\delta^{(1)}v^{(1)i}v^{(1)j} + v^{(1)i}v^{(2)j} + v^{(2)i}v^{(1)j}], \quad (41)$$

as contrasted to the equation (30) for $\delta^{(3)}_{\mathrm{FFA}}$. Here $\delta^{(2)}$ is the FPA solution, equation (39), and the solution for $\boldsymbol{v}^{(2)}$ can be found from the continuity equation to be

$$\boldsymbol{v}^{(2)}_{\mathrm{FPA}} = -\frac{4a}{3t}\nabla\Delta^{(2)}_{\mathrm{FPA}} - \delta^{(1)}\boldsymbol{v}^{(1)}. \quad (42)$$

Equation (41) is a special case of the equation for the true $\delta^{(3)}$, and has been obtained simply by setting $\phi^{(2)} = 0$. It is thus easier to handle than the equation for $\delta^{(2)}_{\mathrm{FFA}}$. Carrying out the Fourier transform and angular averaging precisely as in Fry (1984), we obtain $R_b = 457/315 = 1.45$, and hence

$$S_4^{\mathrm{FPA}} = 21.22. \quad (43)$$

5 CONCLUSIONS

In Table 1, we display the leading-order mean values of the third and fourth moments for the true and approximate evolution. For the third moment, the smoothing corrections for a top-hat window function are included ($n+3 = -\mathrm{d}\log\langle\delta^2\rangle/\mathrm{d}\log R$).

Table 1. Third and fourth moments of density.

	Third moment, S_3	Fourth moment, S_4
True evolution	$4.86-(n+3)$	45.88
Zel'dovich	$4.00-(n+3)$	30.22
Frozen potential	$3.40-0.7\,(n+3)$	21.22
Frozen flow	$3.00-0.5\,(n+3)$	16.00

At this stage it is useful to compare the second-order solution for the density field in various approximations. For the Zel'dovich case, the second-order solution is given by (Bouchet et al. 1991)

$$\delta^{(2)}=\frac{1}{2}\,\delta^{(1)2}+\delta^{(1)}_{,i}\Delta^{(1)}_{,i}+\frac{1}{2}\,\Delta^{(1)2}_{,ij}, \tag{44}$$

while the second-order solutions for FFA and FPA are given by equations (28) and (39), respectively. It should be noted that the same terms appear in the three approximations, but the coefficients are, of course, different. The coefficients in the Zel'dovich case are the closest to the true density evolution, followed by FPA and then FFA, and this feature is clearly reflected in the respective values for the skewness. It is instructive to enquire why these approximations underestimate such a quantity. The skewness may be seen as a measure of the ability of the system to create more rare overdense spots compared to underdense spots. Early nonlinearities tend to increase the growth rate of the positive fluctuations and decrease that of the negative fluctuations. The skewness is sensitive to this asymmetry. On the other hand, the kurtosis measures the ability of the system to create rare spots of any kind. This view finds support in the relation between the spherical collapse dynamics and the values of the skewness and the kurtosis (see Bernardeau 1992, 1993). As can be seen from the work of Matarrese et al. (1992), the FFA is less accurate than the Zel'dovich approximation (and the real dynamics) for the spherical collapse: the acceleration is too weak to concentrate the matter efficiently. That leads directly to a smaller skewness, which is again borne out by the result on the kurtosis.

When the smoothing effects are taken into account, however, results for the skewness seem to attenuate these effects. Taking the results at face value, we could summarize them by saying that, in the regime in which perturbation theory is valid, the unsmoothed Zel'dovich approximation performs better than FFA and FPA. However, smoothing improves both FFA and FPA; in the circumstance when the spectral index n is greater than -1, both FFA and FPA perform better than the Zel'dovich approximation. For the particular case $n=-1$, the Zel'dovich approximation, FFA and FPA all give the same result, $S_3=2$ (instead of the exact value $S_3=2.86$). Actually, the smoothing corrections are sensitive to the tidal effects in the density field, as they arise from the term $\delta^{(1)}_{,i}\Delta^{(1)}_{,i}$, and only the Zel'dovich approximation gives the right coefficient for this term. The other two approximations underestimate the tidal effects, with FFA even failing to give a term containing a quadrupole contribution (in $\Delta^{(1)}_{,ij}\Delta^{(1)}_{,ij}$). This is a major consideration for a practical use of these approximations. The disruption of objects, for instance, is expected to be less accurate in FFA or FPA than in the real dynamics.

How do our results compare with those from simulations using N-body, and those based on FFA and FPA? (For simulations, see Matarrese et al. 1992; Brainerd et al. 1993; Melott et al. 1994; Bagla & Padmanabhan 1994). FFA and FPA were developed to improve upon the Zel'dovich approximation at about the time of shell-crossing and thereafter. FFA was proposed to improve the Zel'dovich approximation by avoiding shell-crossing, while FPA attempts to improve upon FFA and the Zel'dovich approximation, by keeping only the potential linear, but evolving both density and velocity exactly (with the allowance for shell-crossing). A visual comparison of N-body numerical results with simulations based on the Zel'dovich approximation, FFA and FPA suggests that FPA does better than FFA, and they both do better than the Zel'dovich approximation, by preventing the thickening of pancakes. This is supported by some, though not all, statistical tests. In particular, Brainerd et al. (1993) report that, in a cross-correlation test, the Zel'dovich approximation, in fact, does better than FPA, while Melott et al. (1994) report that FFA does poorly in cross-correlation with N-body. The smoothing, however, improves FFA and FPA in comparison with the Zel'dovich approximation. Our analysis of the accuracy of FFA and FPA is naturally incomplete, since we use perturbation theory, which is evidently valid only when the density fluctuations are small. In particular, the S_3 and S_4 parameters are given in the low σ limit. For the exact dynamics, numerical simulations indicate that these parameters seem to be nearly σ-independent in the quasi-linear regime (Bernardeau & Kofman 1994). This gives a strong motivation for such calculations. However, for the Zel'dovich approximation the S_3 and S_4 parameters are shown to have a strong σ-dependence even in the quasi-linear regime (Bernardeau & Kofman). This result can be derived from the density probability distribution function given by Kofman et al. (1994). Our theoretical results for the approximate dynamics then do not necessarily contradict the numerical measurements of the skewness or the kurtosis, since they are made for quite high values of σ. In any case, our results suggest that FFA and FPA are only partially successful in their aim, and the quasi-linear regime is rather poorly described by these approximations. Thus, while the simulations and our analytical results are two different ways of testing various approximations, both the means indicate the need for a more careful comparison, before FFA and FPA can be usefully adopted as improvements over the Zel'dovich approximation. These approximations might, of course, be interesting for analytical studies, but one should generally exercise caution in their use.

Recently, Munshi & Starobinsky (1994) have also carried out a comparison of Zel'dovich approximation with FFA and FPA in perturbation theory, and have arrived at results similar to ours.

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