# Nonadiabatic Particle Motion in Magnetic Mirror Traps

H. Irié, S. Otsuka,† R. K. Varma,* T. Watanabe** and Kyoji Nishikawa†

College of Science and Technology, Nihon University
† Faculty of Science, Hiroshima University
* Physical Research Laboratory, Ahmedabad, India
** Institute for Fusion Theory, Hiroshima University

Abstract: By numerical integration of the equation of single particle motion, the basic features of the actual nonadiabatic escape of particles are studied. The results are compared with the predictions of two existing theoretical models: "diffusion" model derived by B. V. Chirikov and "tunneling" model introduced by R. K. Varma.

In a strong magnetic field, a charged particle gyrates around a line of magnetic field, and simultaneously drifts across and over the magnetic surface. We restrict ourselves to the case of axially-symmetric and constant magnetic field, then the drift motion occurs around the axis of symmetry and can be removed. When a magnetic field slowly (adiabatically) varies during the gyration motion of a particle, the magnetic moment, defined by $\mu = v_\perp^2/2\Omega$, is approximately constant, where $v_\perp$ is the velocity component perpendicular to the magnetic field $\vec{B}$ , $\Omega = |\vec{B}|$ is the gyration frequency of the particle and we set the mass and the charge as unity for simplicity. Because the magnetic moment vector is directed opposite to $\vec{B}$ and its strength $\mu$ is an adiabatic invariant, a particle in a non-uniform magnetic field is pushed along a magnetic line into the area of weak field. A magnetic mirror whose field strength has minimum in the mirror is a plasma confinement scheme which utilizes the nature of this adiabatic invariance of particle magnetic moment.

Let us consider a single charged-particle motion in an axi-symmetric magnetic mirror without electric field. The Hamiltonian is written in the cylindrical coordinates $(r, \theta, z)$ as follows:

$$H(p_z, p_r, z, r) = \frac{1}{2}\{p_z{}^2 + p_r{}^2 + [\frac{p_\theta}{r} - A_\theta(z, r)]^2\}, \quad (1)$$

where $p_\theta$ is the angular momentum which is constant, and $A_\theta$ is the vector potential which produces a magnetic mirror whose field strength has minimum, $\Omega_{min}$, at $z = 0$ and maximum, $\Omega_{max}$, at $z = \pm L/2$ along the field line.

If the adiabatic approximation is valid, the particle that has initially a magnetic moment $\mu_o$ larger than the critical value $\mu_c \equiv E/\Omega_{max}$ can be trapped and bounces with the bounce frequency, $\omega_b$, in the mirror, where E is the particle energy which is a constant of motion. This feature can be seen from the expression: $H = v_{\parallel}^2/2 + \mu\Omega = E$. The condition of $\mu_o > \mu_c$ is equivalent to that of $\psi_o < \psi_c$ where $\psi$ is the pitch angle in the velocity space being defined by $\psi \equiv \sin^{-1}(v_{\perp}/v)$, and $\psi_o$ and $\psi_c$ are the initial and the critical values, respectively.

In actual situations, however, the magnetic moment $\mu$ is not a strictly constant during the particle motion, because the ratio of the gyration period to the bouncing period is finite, that is, the magnetic field strength varies during the gyration period. Then the particle that adiabatically trapped in the mirror, i.e. $\mu_o > \mu_c$ can escape from the mirror when the moment $\mu$ becomes less than the critical value $\mu_c$ due to its nonadiabatic change.

There exist two theoretical models which describe the nonadiabatic escape. One is a diffusion model in $\mu$-space derived by B. V. Chirikov.[1] This theory makes use of the fact that the magnetic moment rapidly changes by an amount $\Delta\mu = \xi \sin\phi$ everytime the particle crosses the median plane of the mirror, where $\phi$ is the gyrophase of the particle at the median plane and $\xi$ is a function of $\mu$. From this fact Chirikov uses a standard mapping in $(\mu, \phi)$ at median plane of the mirror. This mapping is linearized with respect to $\mu$ and is characterized by only one parameter, the so-called stability parameter, S. When S is less than unity, there is only bounded oscillation of $\mu$ around each resonance value which is evaluated by the resonance between the gyration oscillation and the bouncing oscillation of the particle. If S is much larger than unity, there occur resonance overlaps and stochastic changes of $\mu$. The diffusion in $\mu$-space can be observed by use of the standard mapping in the case of $S \gg 1$. When $\mu$ decreases to a value less than the critical value $\mu_c$, the particle escapes from the mirror.

The other theoretical model is based on the ensemble view point introduced by R. K. Varma.[2] According to this theory, the probability density, F(x, t), at the coordinate x parallel to the magnetic field and at the time t is expressed as a sum of the contribution of a set of wave functions $\Psi_n(x, t)$ which are distinguished by the ensemble mode number n, $F(x, t) = \sum_{n=1} |\Psi_n(x, t)|^2$.

Each of the wavefunctions obeys a one-dimensional (along x) Schrödinger-like equation where the role of $\hbar$ is replaced by $\mu_o/n$ with initial magnetic moment $\mu_o$. This model describes the non-adiabatic escape as the escape due to "tunneling-like effect", and predicts that the lifetime of adiabatically trapped particles in a magnetic mirror with given energy and given magnetic moment $\mu_o$ is approximately determined by the inverse of the product of the number of hit on the both walls per unit time, $\omega_b/\pi$, and the transmission rate. If the WKB approximation can be used for the calculation of the transmission rate, the lifetime of the n-th ensemble mode becomes

$$\tau_n = \frac{\pi}{\omega_b} \exp\{n \frac{2}{\mu_o} \int_a^b \sqrt{2[\mu_o \Omega(x) - E]}dx\} , \qquad (2)$$

where the integral takes place over the region of the potential barrier, $\mu_o\Omega(x) > E$, from one zero of the integrand(at x = a) to the other (at x = b). The argument of the exponential is proportional to $n/\varepsilon$ where $\varepsilon^{-1} \equiv L\Omega_o/v$, $\Omega_o$ is the typical gyrofrequency, so there exist multiple lifetimes.

Numerical Calculation The purpose of the present work is to examine the basic features of the actual nonadiabatic escape of the particles by a numerical calculation, and to compare the results with the predictions of these two existing theories.

The basic equations are the single particle equation of motion,

$$\frac{d\vec{r}}{dt} = \vec{v} , \quad \frac{d\vec{v}}{dt} = \vec{v} \times \vec{\Omega} \quad (\vec{\Omega} = \vec{B}) . \qquad (3)$$

We numerically integrated these equations by the 6-th order Runge-Kutta method. Two cases are studied.

Case (A) In this case, we consider particles at various values of stability parameter S by using a simple magnetic field configuration which is divergence free but not rotation free. It is expressed as follows:

$$\Omega_r = -\Omega_o \frac{\pi r}{L} a \sin(2\pi z/L), \quad \Omega_\theta = 0 ,$$
$$\Omega_z = \Omega_o[1-a \cos(2\pi z/L)]. \qquad (4)$$

We take the parameter a as 0.2, the corresponding mirror ratio being R = (1+a)/(1-a) = 1.5. We chose the initial conditions such that $\cos \psi_o = 0.550 < \cos \psi_c = 0.577$ ($\mu_o > \mu_c$) and 500 particles with the same energy and the same magnetic moment $\mu_o$ (or the same pitch angle $\psi_o$) are uniformly distributed on a circle around the guiding center at $r_{go} = 0.05L$. These are distinguished each other by their initial gyrophases $\phi_o$. Initial z-coordinates are the same, i.e. $z_o = 0$. The energy is varied as $\varepsilon^{-1} \equiv L\Omega_o/v = 4$ through 20, the corresponding stability parameter being varied as S = 1 through 14. The stability parameter has the form

$$S = \frac{\alpha}{\varepsilon^2} e^{-\kappa/\varepsilon} , \tag{5}$$

where, in the present case, $\alpha \cong 2.84$ and $\kappa \cong 0.332$. The particle that reaches at $z = \pm L$ is considered as escaped. In Fig. 1 the stability parameter is ploted by the solid curve. The crosses and the dashed curve denote the survival particles in the mirror after 200 times the gyration period. The crosses are the numerical values and the dashed curve is the theoretical one which is derived by the consideration of diffusion in μ-space with a constant diffusion coefficient $D = \omega_b \xi^2 (1-1/S)^2/4\pi$ and with the absorbing boundary at $\mu = \mu_c$. The number of survival particles is calculated by this diffusion model with absorbing boundary as[3])

$$N(t) = \frac{N_o}{\sqrt{\pi}} \gamma(\frac{1}{2}, \frac{(\mu_o-\mu_c)^2}{4Dt}) , \tag{6}$$

where $\gamma(\nu, x)$ $(\equiv \int_0^x t^{\nu-1} e^{-t} dt)$ is the incomplete gamma function.

We can clearly see a threshold of the stability parameter in order to have a substantial particle loss.

When $\varepsilon^{-1}$ is equal to 4, the Störmer radius is greater than the radial coordinate of the guiding center, therefore no particles escape.[4])

Case (B) We found multiple lifetimes in the case of particles having stability parameter below threshold. The magnetic field, which is taken to be both divergence free and rotation free, is of the following form

$$\begin{aligned} \Omega_r &= -\Omega_o a \sin(2\pi z/L) I_1(2\pi r/L), \quad \Omega_\theta = 0, \\ \Omega_z &= \Omega_o[1 - a\cos(2\pi z/L) I_o(2\pi r/L)] , \end{aligned} \tag{7}$$

where $I_o$ and $I_1$ are the modified Bessel functions and we take the parameter a as 0.2. Cosine of initial pitch angle is larger than before and is $\cos\psi_o = 0.570$ which is still smaller than the critical one, $\cos\psi_c = 0.577$. The initial guiding center coordinate divided by the mirror length is $r_{go}/L = 0.177$. In this case, all the particles have nearly the same guiding center and nearly the same magnetic moment $\mu_o$ which is measured at their guiding center. The parameter $\varepsilon^{-1}$ is taken from 14 to 16.3. The corresponding stability parameter is from 0.4 to 0.8. Figure 2 shows the number of survival particles N(t) on logarithmic scale versus time. We can identify two lifetimes which are defined by the inverse of the slope. In Fig. 3, the escaping time on the initial gyrophases are ploted. Figure 3 shows that in some regions of the gyrophase the dispersion of the escaping time is large, while in other regions it is small. Figure 4 shows the logarithm of two lifetimes for each value of $\varepsilon^{-1}$. One can find that the two slopes are nearly proportional to $\varepsilon^{-1}$ in qualitative agreement with the WKB approximation for the tunneling model of R. K. Varma. The stability parameter, in this case, is depicted by the solid line in Fig. 4, so that Chirikov's model predicts only bounded oscillations of the magnetic moment.

Summary of the Results

1) For particle of pitch angles not very close to the critical value, substantial nonadiabatic loss occurs when stability parameter exceeds a critical value nearly equal to 3.

2) Dependence of the number of remaining particles at given time on stability parameter is in good agreement with the diffusion model with absorbing boundary.

3) Even if Chirikov's stability parameter is below unity, particles of pitch angle very close to the critical value can escape from the magnetic mirror by nonadiabatic effects.

4) In this case, multiple lifetimes (at least two) are observed in accord with Varma's ensemble theory.

References

1) B. V. Chirikov: Sov. J. Plasma Phys. 4 (1978) 289, and references cited therein.

2) R. K. Varma: Phys. Rev. Lett. 26 (1971) 417.

3) T. Tange: Private communication.

4) O. V. Serdyuk, Sov. Phys. Tech. Phys. 21 (1976) 1252.

Figure Captions

Fig. 1 Stability parameter (solid curve) and the number of survival particles $N(t = 200\ 2\pi/\Omega_o)$ in the mirror (crosses: numerical results, dashed curve: diffusion model with absorbing boundary) versus $\varepsilon^{-1}$.

Fig. 2 The number of survival particles in the mirror in logarithmic scale versus time for $\varepsilon^{-1} = 14.4$.

Fig. 3 Dependence of the escaping time of each particle on the initial gyrophase $\phi_0$.

Fig. 4 Lifetimes $\tau$ for each $\varepsilon^{-1}$.

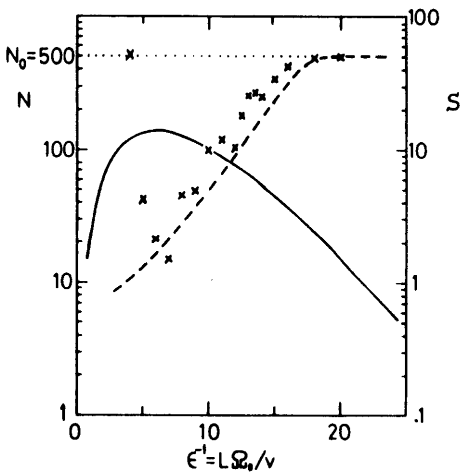


Fig. 1

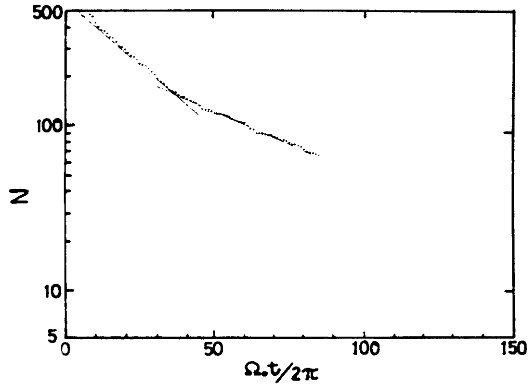


Fig. 2

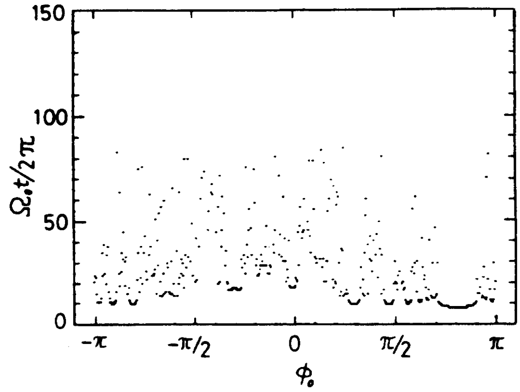


Fig. 3

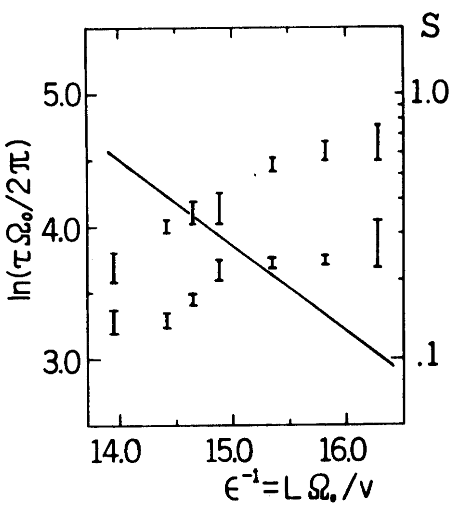


Fig. 4