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On degrees of maps between Grassmannians

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Abstract. Let $\tilde{G}_{n,k}$ denote the oriented grassmann manifold of oriented k-planes in $\mathbb{R}^n$. It is shown that for any continuous map $f: \tilde{G}_{n,k} \to \tilde{G}_{m,l}$, $\dim\tilde{G}_{n,k} = \dim\tilde{G}_{m,l} = l(m-l)$, the Brouwer's degree is zero, provided $l > 1$, $n \neq m$. Similar results for continuous maps $g: \mathbb{C}G_{m,l} \to \mathbb{C}G_{n,k}$, $h: \mathbb{H}G_{m,l} \to \mathbb{H}G_{n,k}$, $1 \leqslant l < k \leqslant n/2$, $k(n-k) = l(m-l)$ are also obtained.

Keywords. Grassmann manifolds; Brouwer degree; characteristic classes.

1. Introduction

Let $\tilde{G}_{n,k}$ denote the oriented grassmann manifold of *oriented* k-dimensional vector subspaces of $\mathbb{R}^n$. For $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$, let $\mathbb{F}G_{n,k}$ denote the $\mathbb{F}$-grassmannian of k-dimensional (left) $\mathbb{F}$-vector subspaces of $\mathbb{F}^n$. In their work on self maps of homogeneous varieties, Paranjape and Srinivas [7] prove, among other things, that if $f: \mathbb{C}G_{n,k} \to \mathbb{C}G_{m,l}$, $l \geqslant 2$, $\dim\mathbb{C}G_{m,l} = l(m-l) = k(n-k) = \dim\mathbb{C}G_{n,k}$ is a finite morphism of projective varieties, then $(n,k) = (m,l)$ and f is an *isomorphism*. It is obvious that if $f: \mathbb{C}G_{n,k} \to \mathbb{C}G_{m,l}$ is any morphism of projective varieties, then f is complex analytic when the varieties involved are regarded as complex analytic manifolds. It follows that f is orientation preserving and that its Brouwer degree can be calculated as $\#f^{-1}(x)$ for most points $x \in \mathbb{C}G_{m,l}$. In particular its (Brouwer) degree must be positive. It is a well-known fact that if $f: N \to M$ is any continuous map of non-zero degree between compact, connected, oriented manifolds, then the induced map in the rational cohomology $f^*: H^*(M; \mathbb{Q}) \to H^*(N; \mathbb{Q})$ is a monomorphism. Using this observation we are able to show the following.

Theorem 1. *Let $h: \tilde{G}_{n,k} \to \tilde{G}_{m,l}$, be any map between oriented grassmann manifolds of the same dimension, where $(n,k) \neq (m,l)$, $2 \leqslant l \leqslant m/2$, $1 \leqslant k \leqslant n/2$. Then* $\deg(h) = 0$.

Theorem 2. *Let $f: \mathbb{C}G_{m,l} \to \mathbb{C}G_{n,k}$, $g: \mathbb{H}G_{m,l} \to \mathbb{H}G_{n,k}$, $1 \leqslant l < k \leqslant [n/2]$ be any map between the complex (respectively quaternionic) grassmannians of the same dimension. Then* $\deg(f) = 0 = \deg(g)$.

In the case, when $h: \tilde{G}_{n,k} \to \tilde{G}_{m,1} = S^d$, $k(n-k) = d = m-1$, is a continuous map, one knows by Hopf–Whitney theorem (Theorem 5, ch.1, [6]), that there are maps of every degree $m \in \mathbb{Z}$, from $\tilde{G}_{n,k}$ to S^d and that any two maps of the same degree are homotopic.

The restriction $1 \leqslant l < k \leqslant [n/2]$ is needed in our proof in the case of complex and quaternionic grassmannians. However it seems plausible that the theorem will continue to be true, even for $2 \leqslant k < l \leqslant [m/2]$. We were able to verify this only for small values of l, k, m and n.

Note that Theorem 2 implies part of the result of Paranjape and Srinivas quoted earlier, namely, if $f: \mathbb{C}G_{m,l} \to \mathbb{C}G_{n,k}$, $2 \leqslant l \leqslant k \leqslant n/2$, is a finite surjective morphism, then we deduce from Theorem 2 that $(n, k) = (m, l)$. On the other hand, our theorem applies to any *continuous* map f. It is not true in general that any given continuous map can be homotoped to a complex analytic map. Hence, Theorem 2 does not follow from the work of Paranjape and Srinivas [7].

Our proofs are in fact quite elementary, and for the most part follow from a purely algebraic lemma (see Lemma 4 (v)). The cases when $f: \mathbb{C}G_{n,k} \to \mathbb{C}P^d$, $g: \mathbb{H}G_{n,k} \to \mathbb{H}P^d$, $k(n-k) = d$, are any continuous maps between the complex (respectively quaternionic) grassmannians will be discussed at the end of §2. We give applications to K-theory in some cases in §3.

For the sake of completeness, we give a proof of the following well-known lemma, quoted earlier.

Lemma 3. Let $f: N \to M$ be any continuous map between two compact connected oriented manifolds of the same dimension d such that $\deg(f) \neq 0$. *Then $f^*: H^*(M; \mathbb{Q}) \to H^*(N; \mathbb{Q})$ is a monomorphism.*

Proof. Let $[M]$ denote the orientation class in $H_d(M; \mathbb{Z}) \subsetneq H_d(M; \mathbb{Q}) \cong \mathbb{Q}$. One has a non-degenerate pairing

$$H^p(M; \mathbb{Q}) \times H^{d-p}(M; \mathbb{Q}) \to \mathbb{Q}$$
$$(\alpha, \beta) \mapsto \langle \alpha \cup \beta, [M] \rangle$$
$$= \langle \alpha, \beta \cap [M] \rangle.$$

Let $\deg(f) = \lambda \neq 0$, $\lambda \in \mathbb{Z}$. Thus $f_*([N]) = \lambda[M]$. Now if $0 \neq \alpha \in H^p(M; \mathbb{Q})$, choose $\beta \in H^{d-p}(M; \mathbb{Q})$ such that $(\alpha, \beta) = 1$. Then $\langle f^*(\alpha) \cup f^*(\beta), [N] \rangle = \langle \alpha, \cup \beta, f_*[N] \rangle = \langle \alpha \cup \beta, \lambda[M] \rangle = \lambda \neq 0$. Therefore f^* is a monomorphism. □

2. Proofs of main results

Let $1 \leqslant k \leqslant n/2$, $n, k \in \mathbb{Z}$. Let $H_{n,k}$ denote the graded $\mathbb{Q}$-algebra with generators $x_1, x_2, \ldots, x_k$, $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-k}$; $\deg(x_i) = i = \deg(\bar{x}_i)$ with relations given by the inhomogeneous relation

$$(1 + x_1 + x_2 + \cdots + x_k)(1 + \bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_{n-k}) = 1 \tag{1}$$

(i.e. the relations are all generated by the basic relations $\Sigma_{0 \leqslant i \leqslant r} x_i \bar{x}_{r-i} = 0$, $1 \leqslant r \leqslant n$). Note that one can regard the basic relations $\Sigma_{i+j=r} x_i \bar{x}_j = 0$ as defining the $\bar{x}_r$ as a polynomial in x_i, inductively for $1 \leqslant r \leqslant n-k$. Therefore $H_{n,k} = \mathbb{Q}[x_1, x_2, \ldots, x_k]/\sim$. $H_{n,k}$ is readily recognized as being isomorphic to the cohomology algebra $H^*(\mathbb{C}G_{n,k}; \mathbb{Q})$ by an isomorphism that doubles the gradation, sending x_i to c_i, the ith Chern class of the canonical k-plane bundle over $\mathbb{C}G_{n,k}$. We denote the vector space of homogeneous elements of $H_{n,k}$ of degree r by $H^r_{n,k}$.

Lemma 4. With the above notation

(i) $H^r_{n,k} = 0$ *if* $r > k(n-k)$,

(ii) $x_1^{k(n-k)}$ *is a generator of* $H^{k(n-k)}_{n,k}$, *which is a 1-dimensional $\mathbb{Q}$-vector space,*

(iii) $\dim_{\mathbb{Q}} H_{n,k} = \binom{n}{k}$,

(iv) *there are no algebraic relations among* $x_1, x_2, \dots, x_k$ *up to degree* $n-k$.
(v) *If* $1 \leqslant l < k \leqslant n/2$, *and* $m \in \mathbb{Z}$ *is such that* $k(n-k) \leqslant l(m-l)$, *then for any homomorphism* $\phi: H_{n,k} \to H_{m,l}$ *of graded* $\mathbb{Q}$*-algebras,* $\ker(\phi) \neq 0$.

Proof. Parts (i), (ii), (iii) and (iv) follow from the isomorphism of graded algebras $H_{n,k} \cong H^*(\mathbb{C}G_{n,k}; \mathbb{Q})$, and well-known facts about the cohomology of $\mathbb{C}G_{n,k}$.

Proof of (v): Let $z_1, z_2, \dots, z_l$ denote the defining algebra generator of $H_{m,l}$, $\deg(z_j) = j$, $1 \leqslant j \leqslant l$. Let, if possible, $\phi: H_{n,k} \to H_{m,l}$ be a monomorphism of $\mathbb{Q}$-algebras. Since $k > l$, there exists an $i \leqslant l+1$ such that $\phi(x_i)$ is in the subalgebra generated by $z_1, z_2, \dots, z_{i-1}$. We may assume that $1 \leqslant i$ is the smallest such integer. If $i = 1$, then $\phi(x_1) = 0$. Therefore $i \geqslant 2$ and

$$\phi(x_1) = \lambda_1 z_1, \quad \lambda_1 \neq 0, \tag{2}$$

$$\phi(x_j) = \lambda_j z_j + P_j(z_1, z_2, \dots, z_{j-1}), \quad \lambda_j \neq 0 \tag{3}$$

and

$$\phi(x_i) = P_i(z_1, z_2, \dots, z_{i-1}), \tag{4}$$

for suitable polynomials P_j, $1 \leqslant j \leqslant i$. In view of (2) and (3), one can express the z_j as a polynomial in $\phi(x_1), \phi(x_2), \dots, \phi(x_j)$ for $1 \leqslant j < i$. Thus

$$\mathbb{Q}[z_1, z_2, \dots, z_{i-1}] = \mathbb{Q}[\phi(x_1), \phi(x_2), \dots, \phi(x_{i-1})]. \tag{5}$$

In particular, for a suitable polynomial Q, one has

$$P_i(z_1, z_2, \dots, z_{i-1}) = Q(\phi(x_1), \phi(x_2), \dots, \phi(x_{i-1})) \tag{6}$$

and hence

$$\phi(x_i) = Q(\phi(x_1), \phi(x_2), \dots, \phi(x_{i-1})) = \phi(Q(x_1, x_2, \dots, x_{i-1})). \tag{7}$$

Therefore, $x_i - Q(x_1, x_2, \dots, x_{i-1}) \in \ker(\phi) = 0$. But this contradicts (iv). Therefore, we must have $\ker(\phi) \neq 0$. □

Let $2 \leqslant k \leqslant n/2$. We recall the structure of the cohomology algebra $H^*(\tilde{G}_{n,k}; \mathbb{Q})$. Let $\gamma_{n,k}$ (resectively $\beta_{n,k}$) denote the canonical real k-plane (respectively $(n-k)$-plane) bundle over $\tilde{G}_{n,k}$. Denote by p_i the Pontrjagin class $p_i(\gamma_{n,k}) \in H^{4i}(\tilde{G}_{n,k}; \mathbb{Q})$, $1 \leqslant i \leqslant [k/2]$ and denote by $\bar{p}_j$ the class $p_j(\beta_{n,k})$, $1 \leqslant j \leqslant [(n-k)/2]$. Note that since $\gamma_{n,k} \oplus \beta_{n,k} \approx n\varepsilon$, the trivial bundle of rank n, one has

$$p(\gamma_{n,k}) \cdot p(\beta_{n,k}) = 1, \tag{8}$$

where p denotes the total Pontrjagin class. The subalgebra generated by $p_1, p_2, p_3, \dots, p_s$, $\bar{p}_1, \bar{p}_2, \dots, \bar{p}_t$ is isomorphic to the graded algebra $H_{s+t,s}$ where $k = 2s + \varepsilon$, $n - k = 2t + \eta, \varepsilon, \eta \in \{0, 1\}$ under the isomorphism $p_i \mapsto x_i$; $1 \leqslant i \leqslant s$. Note that since $\gamma_{n,k}$ and $\beta_{n,k}$ are oriented in a natural way, one has the rational Euler classes $e_k := e(\gamma_{n,k}) \in H^k(\tilde{G}_{n,k}; \mathbb{Q})$, $e_{n-k} := e(\beta_{n,k}) \in H^{n-k}(\tilde{G}_{n,k}; \mathbb{Q})$. (Our notation is ambiguous when $2k = n$, but as it is unlikely to cause confusion, we retain this notation throughout.) Since the (integral) Euler class of an oriented bundle of odd rank is of order two, e_k (respectively e_{n-k}) is zero if k (respectively $n-k$) is odd. Again using the relation $\gamma_{n,k} \oplus \beta_{n,k} \approx n\varepsilon$, we get $e(\gamma_{n,k}).e(\beta_{n,k}) = 0$. That is, $e_k . e_{n-k} = 0$. Furthermore, $e_k^2 = p_s$, (respectively $e_{n-k}^2 = \bar{p}_t$), when $k = 2s$ (respectively $n - k = 2t$).

In case n is even and k is odd there is a cohomology class $\sigma = \sigma_{n,k} \in H^{n-1}(\tilde{G}_{n,k}; \mathbb{Q})$, which transgresses to $p_{[n/2]}(\gamma_{\infty,n}) \in H^{2n}(BSO(n); \mathbb{Q})$ in the fibration

$$\tilde{G}_{n,k} \hookrightarrow BSO(k) \times BSO(n-k) \to BSO(n)$$

as can be seen using a spectral sequence argument and known facts about the rational cohomology of the classifying spaces $BSO(n)$ [5]. Here $\gamma_{\infty,n}$ denotes the universal oriented n-plane bundle over $BSO(n)$. We are ready to describe the cohomology algebra $H^*(\tilde{G}_{n,k}; \mathbb{Q})$:

PROPOSITION 5

With the above notation

(i) $H^*(\tilde{G}_{2s+2t,2s}; \mathbb{Q}) \cong H_{s+t,s}[e_{2s}, e_{2t}]/\sim$ where $e_{2s}^2 = p_s$, $e_{2t}^2 = \bar{p}_t$, $e_{2s} \cup e_{2t} = 0$.

(ii) $H^*(\tilde{G}_{2s+2t+1,2s}; \mathbb{Q}) \cong H_{s+t,s}[e_{2s}]/\sim$ where $e_{2s}^2 = p_s$.

(iii) $H^*(\tilde{G}_{2s+2t+2,2s+1}; \mathbb{Q}) \cong H_{s+t,s}[\sigma]$, $\sigma^2 = 0$, $\sigma \in H^{2s+2t+1}(\tilde{G}_{2s+2t+2,2s+1}; \mathbb{Q})$.

Here $H_{s+t,s} \cong \mathbb{Q}[p_1, p_2, \ldots, p_s]/\sim$ *is as defined earlier.*

We omit the proof of this proposition. For (i) and (ii) see Théorème 26.1 [2]. □

Note. In $H^*(\tilde{G}_{2s+2t,2s}; \mathbb{Q})$ one has $e_{2s}\bar{p}_t = e_k \cup e_{n-k} \cup e_{n-k} = 0$, and similarly, $e_{2t}\,p_s = 0$. However, there are no linear relations over $H_{s+t,s}$ satisfied by e_{2s} in dimensions less than $2s + 4t = 2n - k$. □

Proof of Theorem 1. Let $1 < l < k \leqslant n/2 < m/2$, $k(n-k) = l(m-l) = d$. Let $f: \tilde{G}_{n,k} \to \tilde{G}_{m,l}$, $g: \tilde{G}_{m,l} \to \tilde{G}_{n,k}$ be any continuous map. We must show that $\deg(f) = 0 = \deg(g)$.

In view of Lemma 3, it suffices to show that neither f^* nor g^* is a monomorphism in the rational cohomology.

Write $k = 2s + \varepsilon$, $n - k = 2t + \eta$ with $\varepsilon, \eta \in \{0, 1\}$, $l = 2a + \varepsilon'$, $m - l = 2b + \eta'$, $\varepsilon', \eta' \in \{0, 1\}$. Let q_i, $1 \leqslant i \leqslant a$ denote the ith Pontrjagin class of $\gamma_{m,l}$ and let p_j, $1 \leqslant j \leqslant s$ denote the jth Pontrjagin class of $\gamma_{n,k}$.

First consider the case when $d = \dim \tilde{G}_{n,k}$ is odd. In this case n, m are both even and k, l are both odd. Thus $\varepsilon = \varepsilon' = \eta = \eta' = 1$, $n = 2s + 2t + 2$ and $m = 2a + 2b + 2$. Clearly, $g^*(p_1) = \lambda q_1$ for some $\lambda \in \mathbb{Z}$ and $f^*(q_1) = \mu p_1$ for some $\mu \in \mathbb{Z}$. Recall, for a nilpotent element x in any algebra, the height of x is defined to be the smallest integer n such that $x^n \neq 0$ and $x^{n+1} = 0$. Using Lemma 4(ii) and the above Proposition, note that the height of p_1 is st, whereas the height of q_1 is ab. Since $st = (d-(n-1))/4 > (d-(m-1))/4 = ab$ it follows that $g^*(p_1^{ab+1}) = \lambda^{ab+1} q_1^{ab+1} = 0$ and so g^* is not a monomorphism. On the other hand $f^*(q_1) = 0$ because f^* is a ring homomorphism and $st > ab$.

Now assume $d = \dim \tilde{G}_{n,k}$ is even. Again, one can see easily that the height of p_1 equals st and the height of q_1 equals ab. But st equals ab only when $\varepsilon = \eta = \varepsilon' = \eta' = 0$. Therefore, only the case $n = 2s + 2t$, $k = 2s$, $m = 2a + 2b$, $l = 2a$ remains to be considered. Consider any continuous maps $f: \tilde{G}_{n,k} \to \tilde{G}_{m,l}$ and $g: \tilde{G}_{m,l} \to \tilde{G}_{n,k}$.

First we show that $g^*(e_k) = 0$. Observe that using the relations in $H^*(\tilde{G}_{m,l}; \mathbb{Q})$ one can write

$$g^*(e_k) = e_l P + Q \tag{9}$$

for suitable homogeneous polynomials $P = P(q_1, q_2, \ldots, q_a)$ and $Q = Q(q_1, q_2, \ldots, q_a)$ of degrees $k - l$ and l respectively. Similarly,

$$g^*(e_{n-k}) = e_l P' + Q', \tag{10}$$

where P' and Q' are homogeneous polynomials in $q_1, q_2, \ldots, q_a$ of degrees $n-k-l$ and $n-k$ respectively. Since $e_k \cup e_{n-k} = 0$, one has

$$0 = g^*(e_k \cup e_{n-k}) = (e_l P + Q)(e_l P' + Q') \tag{11}$$

$$= q_a PP' + QQ' + e_l(PQ' + P'Q) \tag{12}$$

in $H^n(\tilde{G}_{m,l}; \mathbb{Q})$. Note that as there are no *linear* relations satisfied by the Euler class e_l over $H_{a+b,a}$ up to dimension $2m-l-1$, and since $n < 2m-l$, we obtain

$$q_a PP' + QQ' = 0, \tag{13}$$

$$PQ' + P'Q = 0. \tag{14}$$

Since by Lemma 4 (iv), there are no algebraic relations satisfied by q_j in degrees up to $4b$, the above relations actually hold in the *polynomial* algebra $A = \mathbb{Q}[q_1, q_2, \ldots, q_a]$. Therefore, multiplying (13) by Q and substituting for $P'Q$ from (14), we obtain

$$Q'Q^2 = q_a P^2 Q' \tag{15}$$

in A. This implies $Q^2 = q_a P^2$ in A. Since A is a UFD this is clearly a contradiction, unless $P = Q = 0$. Hence $g^*(e_k) = 0$. Now by Lemma 4 (v), g^* cannot be a monomorphism and so $\deg(g) = 0$.

To show $\deg(f) = 0$ note that since $a < s, f^*(e_{2a})$ can be expressed as a polynomial $P(p_1, p_2, \ldots, p_{[a/2]})$ in $p_1, p_2, \ldots, p_{[a/2]}$. To obtain a contradiction assume that f^* is a monomorphism. Then proceeding as in the proof of Lemma 4 (v) one can express p_i, $1 \leqslant i \leqslant [a/2]$ as a polynomial in $f^*(q_1), f^*(q_2), \ldots, f^*(q_i)$. Hence

$$f^*(e_{2a}) = P(p_1, \ldots, p_{[a/2]}) = P'(f^*q_1), f^*(q_2), \ldots, f^*(q_{[a/2]})) \tag{16}$$

for some suitable polynomial P'. In particular $e_{2a} - P'(q_1, q_2, \ldots, q_{[a/2]})$ is in $\ker(f^*) = 0$. That is, $e_{2a} = P'(q_1, \ldots, q_{[a/2]})$. But this contradicts Proposition 5. Hence $\deg(f)$ has to be zero. This completes the proof of the theorem. □

Proof of Theorem 2. Let $1 \leqslant l < k \leqslant [n/2]$, $k(n-k) = l(m-l)$. Let $f: \mathbb{C}G_{m,l} \to \mathbb{C}G_{n,k}$ be any continuous map. Then f induces an algebra homomorphism $f^*: H^*(\mathbb{C}G_{n,k}; \mathbb{Q}) \to H^*(\mathbb{C}G_{m,l}; \mathbb{Q})$. As $H^*(\mathbb{C}G_{n,k}; \mathbb{Q}) \cong H_{n,k}$, it is immediate from Lemma 4(v) that $\ker(f^*) \neq 0$. Hence $\deg(f) = 0$. Proof for $g: \mathbb{H}G_{m,l} \to \mathbb{H}G_{n,k}$ is similar. □

Remark 6. Let $f: \mathbb{C}G_{n,k} \to \mathbb{C}P^d$, $d = k(n-k)$ be any continuous map. Then $f^*(c_1(\gamma_{d+1,1})) = \lambda_f c_1(\gamma_{n,k})$ for some integer λ_f. Using the fact that $\mathbb{C}P^d$ is the $2d+1$-skeleton of $\mathbb{C}P^\infty \cong K(\mathbb{Z}, 2)$, one sees readily that if $g: \mathbb{C}G_{n,k} \to \mathbb{C}P^d$ is another continous map, then f is homotopic to g if and only if $\lambda_f = \lambda_g$. Moreover, there exists a map $f: \mathbb{C}G_{n,k} \to \mathbb{C}P^d$ with λ_f as any pre-assigned integer. Note that with respect to the orientation obtained from the complex structure on $\mathbb{C}P^d$, the positive generator of $H^{2d}(\mathbb{C}P^d, \mathbb{Z})$ is $(-c_1(\gamma_{d+1,1}))^d$. The degree of f can be determined to be $(\lambda_f)^d \cdot \langle(-c_1(\gamma_{n,k}))^d, [\mathbb{C}G_{n,k}]\rangle$ which equals $(\lambda_f)^d[(1!2!\ldots(k-1)!d!)/((n-k)!\ldots(n-1)!)]$, using (Eg. 14.7.11, [3]).

By an entirely analogous argument one shows that the set $[\mathbb{H}G_{n,k}; \mathbb{H}P^d] \cong [\mathbb{H}G_{n,k}; \mathbb{H}P^\infty] \cong [\mathbb{H}G_{n,k}; BSp(1)]$. But $Sp(1) \cong SU(2) \cong S^3$. So, $[\mathbb{H}G_{n,k}; \mathbb{H}P^d]$ is in bijective correspondence with the set of isomorphism cases of $SU(2)$-bundles over

$\mathbb{H}G_{n,k}$. If $f: \mathbb{H}G_{n,k} \to \mathbb{H}P^d$, then the degree of f is given by the same formula as in the case of complex grassmannians, where λ_f is defined by

$$c_2(f^*(\gamma_{d+1,1})) = \lambda_f c_2(\gamma_{n,k}).$$

We do not know if there exists a continuous map f with λ_f as any pre-assigned integer. □

3. Application to K-theory

Let $f: X \to Y$ be any continuous map between two finite CW complexes. One has a commutative diagram [1]

$$\begin{array}{ccc} \tilde{K}(Y) & \xrightarrow{f^*} & \tilde{K}(X) \\ \downarrow \mathrm{ch}(Y) & & \downarrow \mathrm{ch}(X) \\ \tilde{H}^{ev}(Y;\mathbb{Q}) & \xrightarrow{f^*} & \tilde{H}^{ev}(X;\mathbb{Q}) \end{array}$$

where ch(−) denotes the Chern character. In case X has cells only in even dimensions then Chern character is well-known to be a monomorphism. In any case $\mathrm{ch}_{\mathbb{Q}}(X) := \mathrm{ch}(X) \otimes \mathbb{Q} : \tilde{K}(X) \otimes \mathbb{Q} \to \tilde{H}^{ev}(X;\mathbb{Q})$ is a monomorphism ([4], p. 238).

Lemma 7. Suppose that $f^*: \tilde{H}^{ev}(Y;\mathbb{Q}) \to \tilde{H}^{ev}(X;\mathbb{Q})$ is zero and that $K(X)$ and $K(Y)$ are free abelian groups, then $f^*: \tilde{K}(Y) \to \tilde{K}(X)$ is zero.

Proof. It suffices to show that $f^*: \tilde{K}(Y) \otimes \mathbb{Q} \to \tilde{K}(X) \otimes \mathbb{Q}$ is zero. This follows from the fact that $\mathrm{ch}_{\mathbb{Q}}(X)$ is a monomorphism and the hypothesis that $f^*: \tilde{H}^{ev}(Y) \to \tilde{H}^{ev}(X)$ is zero. □

Lemma 8. Let $1 \leqslant l < k \leqslant n/2 < m/2$, $k(n-k) \leqslant l(m-l)$. Assume that $n \geqslant k^2/(k-l)$. Then any graded $\mathbb{Q}$-algebra homomorphism $\phi: H_{n,k} \to H_{m,l}$, has image in $\mathbb{Q}$, the elements of degree zero in $H_{m,l}$.

Proof. Let us write the canonical generators of $H_{n,k}$ (respectively $H_{m,l}$) as $x_1, x_2, \ldots, x_k$ (respectively $y_1, y_2, \ldots, y_l$). Write $u_i = \phi(x_i) \in H_{m,l}$, $1 \leqslant i \leqslant k$. We must show that $u_i = 0$ for each i. To obtain a contradiction, assume that $u_i \neq 0$ for some $i \geqslant 1$. Then $\bar{u}_j := \phi(\bar{x}_j) \neq 0$ for some j, $1 \leqslant j \leqslant n-k$. Let p and q be the largest integers so that $u_p \neq 0$, $\bar{u}_q \neq 0$. Applying ϕ to both sides of the relation

$$(1 + x_1 + x_2 + \ldots + x_k)(1 + \bar{x}_1 + \bar{x}_2 + \ldots + \bar{x}_{n-k}) = 1 \tag{17}$$

and collecting the terms of degree $p+q$ we get $u_p \cdot \bar{u}_q = 0$ in $H_{m,l}$. Since $p \leqslant k$, $q \leqslant n-k$, we get $p+q \leqslant n$. But, $n \geqslant k^2/(k-l) \Rightarrow n(k-l) \geqslant k^2 \Rightarrow k(n-k) \geqslant nl \Rightarrow l(m-l) \geqslant nl \Rightarrow (m-l) \geqslant n$. Hence $(p+q) \leqslant m-l$. This contradicts Lemma 4 (iv). □

As in the proof of Theorem 1, we write $k = 2s + \varepsilon$, $n - k = 2t + \eta$, $l = 2a + \varepsilon'$, $m - l = 2b + \varepsilon'$, with $\varepsilon, \eta, \varepsilon', \eta' \in \{0, 1\}$.

Theorem 9. (i) *Let $1 \leqslant l < k \leqslant n/2 < m/2$, and let $k(n-k) \leqslant l(m-l)$. Assume that $n \geqslant k^2/(k-l)$. Then for any continuous maps, $g: \mathbb{C}G_{m,l} \to \mathbb{C}G_{n,k}$, $h: \mathbb{H}G_{m,l} \to \mathbb{H}G_{n,k}$, one has $g^*: \tilde{K}(\mathbb{C}G_{n,k}) \to \tilde{K}(\mathbb{C}G_{m,l})$ and $h^*: \tilde{K}(\mathbb{H}G_{n,k}) \to K(\mathbb{H}G_{m,l})$ are zero.* (ii) *Let $1 \leqslant l \leqslant m/2$, $2 \leqslant k \leqslant n/2$, and let $1 \leqslant a < s \leqslant t$, $st \leqslant ab$. Assume that $(s+t) \geqslant s^2/(s-a)$. If $l \not\equiv 0 \bmod 4$ and $(m-l) \not\equiv 0 \bmod 4$, then for any continuous map $f: \tilde{G}_{m,l} \to \tilde{G}_{n,k}$, one has $f^*: \tilde{K}(\mathbb{C}G_{n,k}) \to \tilde{K}(\mathbb{C}G_{m,l})$ is zero.*

Proof. By Theorem 3.6 [1] one knows that $K(X)$ is a free abelian when $X = G/K$, where G is any compact connected Lie group and K is a connected subgroup of maximal rank. In particular, this shows that $K(X)$ is a free abelian for $X = \mathbb{C}G_{n,k}$, $\mathbb{H}G_{n,k}$, $\tilde{G}_{p,q}$ when p is odd or q is even. $K^*(\tilde{G}_{p,q})$ has been calculated in [8] for any p and q, and in particular, it follows from Theorem 3.6, [8] that $K(\tilde{G}_{p,q})$ is a free abelian for p is even and q is odd. (i) The above result shows $g^* = 0 = h^*$ by applying Lemmas 7 and 8. (ii) Now let $l \not\equiv 0 \bmod 4$, $(m-l) \not\equiv 0 \bmod 4$. Then by a straightforward dimension argument, $f^*: H^{ev}(\tilde{G}_{n,k}; \mathbb{Q}) \to H^{ev}(\tilde{G}_{m,l}; \mathbb{Q})$ must map the subalgebra $H_{s+t,s} \subset H^{ev}(\tilde{G}_{n,k}; \mathbb{Q})$ into the subalgebra $H_{a+b,a} \subset H^{ev}(\tilde{G}_{m,l}; \mathbb{Q})$. By our hypotheses on s, t, a, b it follows from Lemma 8 that $f^*|H_{s+t,s}$ is zero.

In case $\eta = 0$, so that $n - k = 2t$, one has $f^*(e_{n-k}) \in H_{a+b,a}$ if $n - k \equiv 0 \bmod 4$, and $f^*(e_{n-k}) \in e_l H_{a+b,a} \subset H^{ev}(\tilde{G}_{m,l}; \mathbb{Q})$ if $n - k \equiv 2 \bmod 4$. (Here $e_l = 0$ if l is odd). This is because l is *not* divisible by 4. Suppose $f^*(e_{n-k}) = P \in H_{a+b,a}$. Then

$$0 = f^*(\bar{p}_t) = f^*(e_{n-k}^2) = P^2. \tag{18}$$

Since $st \leqslant ab$ and $a < s$, one has $t < b$. This implies $2(n-k) < 4b$. Hence, $P^2 = 0$ implies $P = 0$ by Lemma 4(iv) and hence $f^*(e_{2t}) = 0$. Similarly, when $n - k \equiv 2$ mod 4, we show that $f^*(e_{n-k}) = 0$. By an analogous argument, when $\varepsilon = 0$, so that $k = 2s$, we show that $f^*(e_{2s}) = 0$. Hence, from Lemma 7, $f^*: \tilde{K}(\tilde{G}_{n,k}) \to \tilde{K}(\tilde{G}_{m,l})$ is zero. □

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