

MOLECULES AND LATE-TYPE STELLAR MODELS

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ABSTRACT

A difficulty encountered by Osterbrock, when he tried to identify particular red dwarf stars by interpolating between his models of late-type main-sequence stars, is shown to be alleviated by including the effects produced by hydrogen molecules in the convective zone.

For late-type main-sequence stars the early models, which consisted of convective cores and radiative envelopes, have been superseded by models with radiative cores and convective envelopes, topped by an atmosphere in radiative equilibrium (Osterbrock 1953; Limber 1958). But in the available models of the latter type the representation of the convective zone has been rather schematic, insofar as the adiabats used would apply only if the gas were completely ionized. Instead, a general expression for the adiabatic gradient of hydrogen-helium mixtures has been derived from thermodynamics by one of the authors (Vardya 1959, 1960). This derivation takes account of the interlocking ionization of hydrogen and helium and of its coupling with the association of hydrogen into molecules. Though it has been known for some time that hydrogen molecules play a significant role in late-type stellar atmospheres and, in particular, affect the onset of convection (Wildt 1934, 1957), no detailed study incorporating the molecular effects has so far been undertaken. Three sets of adiabats, corresponding to different abundance ratios of hydrogen to helium, each covering a large domain of $\log T$, $\log \rho$ plane, are now available (Vardya 1960).

The purpose of this note is to examine whether these new adiabats will tend to reduce a discrepancy which Osterbrock's as well as Limber's discussions left unresolved. This discrepancy relates to the constant of integration K of the adiabatic pressure-temperature relation, viz.,

$$\ln P = \int^T \frac{\Gamma}{\Gamma - 1} d \ln T + \ln K, \quad (1)$$

where

$$\frac{\Gamma}{\Gamma - 1} = \left(\frac{d \ln P}{d \ln T} \right)_{ad}. \quad (2)$$

This quantity Γ is sometimes called the "effective ratio of the specific heats," which differs from the better-known ratio, $\gamma = C_p/C_v$, by including the effects of dissociation and ionization. If the variability of Γ is disregarded, then, for a monatomic gas, $\Gamma = \gamma = \frac{5}{3}$, and equation (1) assumes the form:

$$P = KT^{5/2}. \quad (3)$$

The integration constant K appears in a dimensionless quantity E , which plays a central role in the theory of red dwarfs; for E is the characteristic parameter of the one-parameter sequence of stellar models constructed by Osterbrock. E is defined by

$$E = 4\pi K \left(\frac{\mu m_0}{k} \right)^{5/2} G^{3/2} M^{1/2} R^{3/2}, \quad (4)$$

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where G is the gravitational constant, M is the mass of the star, R is its radius, and μ is the dimensionless molecular weight in units of m_0 . Note that E depends on the chemical composition through μ .

For any star of known mass, radius, and luminosity (M, R, L) and tentatively assigned chemical composition, E is determinate, according to Osterbrock's theory. Let this be called the "interior value" of E . In order to validate this assumed composition, it is then necessary to compute an atmospheric model in radiative equilibrium, with a surface gravity and effective temperature corresponding to the same set (M, R, L), and the same tentative composition, and to determine at what values of P_1, T_1 the onset of convection occurs. The adiabat passing through P_1, T_1 defines an "exterior value" of K and thereby of E , and, if the chemical composition tentatively adopted was realistic, the exterior and interior value of E ought to coincide, provided, of course, that the underlying physical theory is adequate; otherwise, in order to obtain the identification of this set (M, R, L), with a model in Osterbrock's sequence, the tentative composition has to be varied and the above procedure repeated, until matching is accomplished.

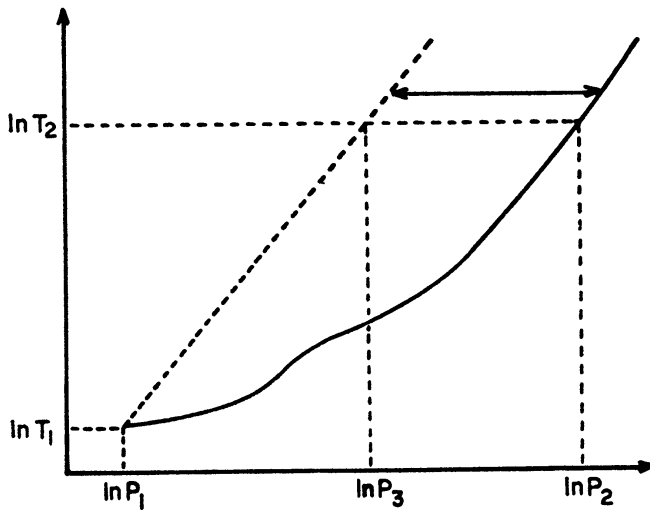


FIG. 1.—Course of hydrogen-helium adiabats in the $\ln T, \ln P$ plane (diagrammatic)

With no reasonable composition was it possible to match the interior and exterior values of E for the stars Castor C av (M0 V) and Krüger 60A (M4 V), the only two red dwarfs tested (Osterbrock 1953; Limber 1958); the exterior values of E turned out to be always smaller than the interior ones. This failure is the discrepancy referred to above. Osterbrock and Limber attributed this mismatch to the inadequacy of the atmospheric models.

We will now show that the inclusion of hydrogen molecules alters the structure of the convective envelope in such a way that the exterior value of E is increased.

Because of $\Gamma \leq \frac{5}{3}$, implying $\Gamma/(\Gamma - 1) \geq \frac{5}{2}$, integration of equation (2) leads to

$$\ln \left(\frac{P_2}{P_1} \right) = \int_{T_1}^{T_2} \frac{\Gamma}{\Gamma - 1} d \ln T \geq \frac{5}{2} \ln \left(\frac{T_2}{T_1} \right) = \ln \left(\frac{P_3}{P_1} \right). \tag{5}$$

As illustrated in Figure 1, this inequality asserts that in progressing from (P_1, T_1) along the true adiabat (with variable Γ), the pressure increment $\ln (P_2/P_1)$, for a fixed temperature increment $\ln (T_2/T_1)$, is greater than the pressure increment $\ln (P_3/P_1)$, pre-

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dicted from the fictitious adiabat starting likewise at (P_1, T_1) but having $\Gamma = \gamma = \frac{5}{3}$ throughout. It follows from equation (5) that

$$\ln \left(\frac{P_2}{P_3} \right) = \int_{T_1}^{T_2} \left(\frac{\Gamma}{\Gamma - 1} - \frac{5}{2} \right) d \ln T \geq 0. \quad (6)$$

Here the sign of equality would apply only if, at the starting point (P_1, T_1) , the gas were already completely ionized. In the present context, the sign $>$ in equation (6) applies. Therefore, $\ln (P_2/P_3)$ (indicated on Fig. 1 by double-headed arrow) represents the positive increment

$$\Delta \ln K = \ln K - \ln K^*, \quad (7)$$

where the true and the fictitious adiabats are labeled K and K^* , respectively. This implies that the K -values of successive adiabats increase toward right on the $\ln P, \ln T$ diagram as oriented in Figure 1. A numerical example follows.

TABLE 1
POSSIBLE INTERIOR VALUES OF E FOR
CASTOR C AV (OSTERBROCK 1953)

X	Y	Z	E
0 90	0 091	0 009	18 7
80	184	016	19 4
0 70	0 271	0 029	19 9

According to Osterbrock's (1953) model for Castor C av, which ignores the existence of hydrogen molecules, the atmosphere becomes unstable against convection at

$$\log_{10} T = 3.6, \quad \log_{10} \rho = -5.960, \quad \text{and} \quad \log_{10} P = 5.560. \quad (8)$$

Integrating inward from these boundary values and taking account of the varying ionization, Osterbrock finds that this particular adiabat is characterized by $\log_{10} K = -1.15$. Since his model atmosphere consisted of pure hydrogen, it follows that $E = 13.6$ (exterior value). Several tentative interior values of E obtained for Castor C av for a reasonable range in chemical composition are given in Table 1; $X, Y,$ and Z denote the fraction by mass of hydrogen, helium, and heavy elements, respectively.

We proceed to examine what happens if the presence of hydrogen molecules is taken into account, using the recently computed adiabats (Vardya 1960). In order to determine the exterior value of K and the corresponding E for the adiabat passing through the point defined by equation (8), an interpolation can be made either in the $(\log T, \log \rho)$ plane or in the $(\log T, \log P)$ plane. Reflection will show that the values obtained from these two planes will not be the same, for the simple reason that $\log_{10} T, \log_{10} P,$ and $\log_{10} \rho,$ as given in equation (8), are consistent only if the molecules are neglected; needless to say, they will also be inconsistent if the composition is not of pure hydrogen. Keeping this in mind, $\log_{10} K$ has been interpolated linearly in both these planes, for each of the three chemical compositions for which adiabats are available. These interpolated values of $\log_{10} K$ are given in Table 2, together with the quantity E appropriate to Castor C av.

According to Table 2, the exterior value of E for pure hydrogen is very close to 19,

the value Osterbrock obtained from the interior (Table 1). But for any reasonable hydrogen-helium mixture, the predicted atmospheric values of E in Table 2 are much higher than the E 's derived from the core solution (Table 1). Although the effect produced by hydrogen molecules operates in the direction predicted by our qualitative argument (see eq. [7]), the exterior values of E now exceed the interior ones. This overcompensation of Osterbrock's discrepancy, though surprising, need not occasion dismay, because the determination of the E -values listed in Table 2 from the boundary values (8) was not a self-consistent procedure; for the former data (Table 2) include the manifestations of hydrogen molecules, which were explicitly ignored in arriving at the latter (eq.

TABLE 2
POSSIBLE EXTERIOR VALUES OF E FOR CASTOR C AV

X	Y	Z	(LOG T, LOG ρ) PLANE		(LOG T, LOG P) PLANE	
			log ₁₀ K	E	log ₁₀ K	E
1 00	0 00	0 00	-0 987	19 9	-0 950	21 7
80	20	00	- 979	28 3	- 896	34 3
0 67	0 33	0 00	-0 965	37 6	-0 862	47 6

[8]). As shown by Wildt (1957), the boundary of the convective zone extends farther out, owing to the formation of hydrogen molecules, than was implied in Osterbrock's atmospheric model (eq. [8]). Since in the outer layers the pressure gradient is much steeper than the temperature gradient, the true adiabats will have smaller values of log₁₀K, than those listed in Table 2, and hence they would better accord with Table 1. Evidently, a definitive appraisal of the worth of the accurate adiabats now available must await completion of a correspondingly elaborate theory of the atmospheres of late-type stars.

REFERENCES

Limber, D. N. 1958, *A p. J.*, 127, 387.
 Osterbrock, D. E. 1953, *A p. J.*, 118, 529.
 Vardya, M. S. 1959, Thesis, Yale University
 ———. 1960, *A p. J. Suppl.*, 4, 281.
 Wildt, R. 1934, *Zs. f. A p.*, 9, 176.
 ——— 1957, *Mém. Soc. R. Sci. Liège*, 4th ser., p 319.

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