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ALGEBRAIC ELEMENTS IN GROUP RINGS

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ABSTRACT. In this brief note, we study algebraic elements in the complex group algebra $\mathbf{C}[G]$. Specifically, suppose $\xi \in \mathbf{C}[G]$ satisfies $f(\xi) = 0$ for some nonzero polynomial $f(x) \in \mathbf{C}[x]$. Then we show that a certain fairly natural function of the coefficients of ξ is bounded in terms of the complex roots of $f(x)$. For G finite, this is a recent observation of [HLP]. Thus the main thrust here concerns infinite groups, where the inequality generalizes results of [K] and [W] on traces of idempotents.

INTRODUCTION

Let $\alpha = \sum_{g\in\kappa} a_g g$ belong to the group algebra $F[G]$. If κ is a conjugacy class of G, then the κ-trace of α is defined by $\alpha_\kappa = \sum_{g\in\kappa} a_g$. It is clear that the map ${}_\kappa\colon F[G] \to F$ is F-linear. Furthermore, if $g, h \in G$, then $hg = g^{-1}(gh)g$ so $(gh - hg)_\kappa = 0$. Thus, by linearity, $(\alpha\beta)_\kappa = (\beta\alpha)_\kappa$ for all $\alpha, \beta \in F[G]$ and ${}_\kappa$ is indeed a trace map. In this paper we study algebraic elements in the complex group algebra $\mathbf{C}[G]$ and our goal is to prove

Theorem 1. *Let ξ be an element of $\mathbf{C}[G]$ and suppose that $f(\xi) = 0$ for some nonzero polynomial $f(x) \in \mathbf{C}[x]$. If λ denotes the maximum of the absolute values of the complex roots of f, then*

$$\sum_\kappa \xi_\kappa \bar{\xi}_\kappa / |\kappa| \leq \lambda^2$$

where $|\kappa|$ is the size of the class and $^-$ denotes complex conjugation.

For G finite, this is a result of [HLP] which was proved using character theory. So the real content here concerns infinite groups. In this case, if κ is a conjugacy class of infinite size, then the summand $\xi_\kappa \bar{\xi}_\kappa / |\kappa|$ is obviously zero and hence has no effect on the above formula. Thus we need only restrict our attention to the conjugacy classes of G of finite size.

Two special cases of the theorem are worth mentioning. First, if ξ is nilpotent, then $\lambda = 0$ and hence we have $\xi_\kappa = 0$ for all finite classes κ. Second,

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if $\xi = e$ is an idempotent, then $\lambda = 1$ and the formula $e_\kappa \overline{e}_\kappa \leq |\kappa|$ is in fact a result of [W]. As we will see, the proof of the above theorem is in some sense a combination of these two cases. Furthermore, in the course of the proof, we will show that the upper bound λ^2 can be replaced by a suitable weighted average of the squares of the absolute values of all roots of $f(x)$. We will also precisely describe when equality occurs.

The semisimple case

We first consider semisimple elements using an analytic proof. Recall that if $\alpha = \sum_{g\in G} a_g g$ and $\beta = \sum_{g\in G} b_g g$ are in $\mathbf{C}[G]$ then, by definition, $(\alpha, \beta) = \sum_{g\in G} a_g \overline{b}_g$ and $\alpha^* = \sum_{g\in G} \overline{a}_g g^{-1}$. Clearly $(\ ,\ )$ is a Hermitian inner product and we have $(\alpha\beta, \gamma) = (\alpha, \gamma\beta^*) = (\beta, \alpha^*\gamma)$. In addition, we note that $(\alpha, \beta) = \operatorname{tr} \alpha\beta^*$ where $\operatorname{tr} \alpha = \alpha_1 = a_1$.

Following [M], we complete $\mathbf{C}[G]$ to a Hilbert space H. Then, via left multiplication, each $g \in G$ is a unitary operator on H and thus $\mathbf{C}[G]$ embeds in $\mathbf{B}(H)$, the $\mathbf{C}^*$-algebra of bounded operators on H. Furthermore, we see from the above formulas that $*$ on $\mathbf{C}[G]$ extends to the adjoint map $*$ on $\mathbf{B}(H)$. We now let $\mathscr{A}$ denote the uniform closure of $\mathbf{C}[G]$ in $\mathbf{B}(H)$. Then $\mathscr{A}$ is also a $\mathbf{C}^*$-algebra and, as in [M], if e is any idempotent of $\mathscr{A}$ there exists a projection p with $e\mathscr{A} = p\mathscr{A}$. Thus p is a self-adjoint idempotent with $ep = p$ and $pe = e$.

Note that if G is finite, then $\mathscr{A} = \mathbf{C}[G]$.

Lemma 2. *Let $e_1, e_2, \ldots, e_n$ be orthogonal idempotents of $\mathscr{A}$. Then there exist orthogonal projections $p_1, p_2, \ldots, p_n$ such that $T(e_i) = T(p_i)$ for any trace map $T\colon \mathscr{A} \to \mathbf{C}$. Furthermore, if either e_i or p_i is central, then $e_i = p_i$.*

Proof. We show inductively that we can replace $e_1, e_2, \ldots, e_n$ in turn by orthogonal projections without changing their traces. Thus suppose that $e_1, e_2, \ldots, e_{k-1}$ are already projections and that $k \leq n$. The goal here is to replace $e_1, e_2, \ldots, e_n$ by orthogonal idempotents $f_1, f_2, \ldots, f_n$ satisfying $T(e_i) = T(f_i)$ for all i and such that $f_1, f_2, \ldots, f_k$ are projections. To this end, recall that there exists a projection p_k with $p_k e_k = e_k$ and $e_k p_k = p_k$. Set $f_i = (1 - p_k)e_i$ for $i \neq k$ and $f_k = p_k$.

Now $e_i e_k = 0$ for $i \neq k$ and hence $e_i p_k = e_i(e_k p_k) = 0$. Thus for $i, j \neq k$ we have

$$f_i f_j = (1 - p_k)e_i \cdot (1 - p_k)e_j = (1 - p_k)e_i e_j .$$

It follows that $f_i^2 = f_i$ and that $f_i f_j = 0$ for $i \neq j$. Furthermore, $f_k^2 = f_k$ and, since $f_k = p_k$, we have $f_k f_i = f_i f_k = 0$ for $i \neq k$. Thus $f_1, f_2, \ldots, f_n$ are also orthogonal idempotents. Note that $f_k = p_k$ is a projection and for $i < k$ we have $0 = (e_i p_k)^* = p_k e_i$ since $e_i^* = e_i$. Thus $f_i = e_i$ here and $f_1, f_2, \ldots, f_k$ are projections.

Finally notice that for $i \neq k$ we have $f_i e_i = f_i$ and $e_i f_i = e_i(1 - p_k)e_i = e_i$. On the other hand, for $i = k$ we have the reverse formulas $f_k e_k = e_k$ and

$e_k f_k = f_k$. It follows that if T is any trace map, then

$$T(e_i) = T(e_i f_i) = T(f_i e_i) = T(f_i)$$

for $i \neq k$ and similarly $T(e_k) = T(f_k)$. Furthermore, if either e_i or f_i is central, then clearly $e_i = f_i$. With these observations, the result follows by induction on k. ■

Note that as a consequence we have the well-known fact that any central idempotent is a projection.

Now by [M, Lemma 1], tr extends to a trace map $\mathrm{tr}: \mathscr{A} \to \mathbf{C}$ satisfying $\mathrm{tr}(\alpha\alpha^*) > 0$ for all nonzero $\alpha \in \mathscr{A}$. We can therefore extend the Hermitian inner product $(\ , \ )$ to $\mathscr{A}$ by defining $(\alpha, \beta) = \mathrm{tr}\,\alpha\beta^*$ for all $\alpha, \beta \in \mathscr{A}$.

Recall that if $e \neq 0, 1$ is an idempotent of $\mathbf{C}[G]$ then Kaplansky's Theorem (see [K, page 122 or M, Lemma 2]) asserts that $0 < \mathrm{tr}\, e < 1$.

Lemma 3. *Let $\xi = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n \in \mathbf{C}[G]$ with $e_1, e_2, \ldots, e_n$ orthogonal idempotents and $\lambda_i \in \mathbf{C}$. Then*

$$\sum_\kappa \xi_\kappa \bar{\xi}_\kappa / |\kappa| \leq |\lambda_1|^2 \,\mathrm{tr}\, e_1 + |\lambda_2|^2 \,\mathrm{tr}\, e_2 + \cdots + |\lambda_n|^2 \,\mathrm{tr}\, e_n$$

with equality if and only if ξ is central. Furthermore, if $\lambda = \max_i |\lambda_i|$, then

$$\sum_\kappa \xi_\kappa \bar{\xi}_\kappa / |\kappa| \leq \lambda^2$$

with equality if and only if ξ is central, $|\lambda_i| = \lambda$ for all i with $e_i \neq 0$, and $e_1 + e_2 + \cdots + e_n = 1$.

Proof. We can clearly delete the terms with $\lambda_i = 0$ and we can merge those terms with the same λ_i's. Thus we can assume that the λ_i's are distinct and nonzero. This is needed only when we consider the possibility of equality in the formulas.

Now for each finite conjugacy class κ, let $\hat{\kappa}$ denote the central element of $\mathbf{C}[G]$ given by $\hat{\kappa} = \sum_{g \in \kappa} g$. Then for any element $\alpha \in \mathbf{C}[G]$ we have $\alpha_\kappa = \mathrm{tr}\,\alpha\hat{\kappa}^* = (\alpha, \hat{\kappa})$. Define the central element η of $\mathbf{C}[G]$ by

$$\eta = \sum_\kappa \xi_\kappa \hat{\kappa} / |\kappa|$$

where the sum is over the finitely many finite classes κ with $\xi_\kappa \neq 0$. It follows easily from $\alpha_\kappa = (\alpha, \hat{\kappa})$ that

$$(\xi, \eta) = \sum_\kappa \xi_\kappa \bar{\xi}_\kappa / |\kappa| = (\eta, \eta).$$

Now let $p_1, p_2, \ldots, p_n$ be the projections given by Lemma 2 and set $\xi' = \lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_n p_n \in \mathscr{A}$. Since η is central in $\mathbf{C}[G]$, it is clear that both η and η^* are central in the uniform closure $\mathscr{A}$. Thus the map $T: \mathscr{A} \to \mathbf{C}$

defined by $T(\alpha) = \operatorname{tr} \alpha\eta^* = (\alpha, \eta)$ is a trace and we conclude from Lemma 2 that

$$(\xi', \eta) = (\xi, \eta) = (\eta, \eta) \geq 0.$$

By the Cauchy–Schwarz inequality we then have

$$(\eta, \eta)^2 = (\xi', \eta)^2 \leq (\xi', \xi')(\eta, \eta)$$

and hence

$$\sum_{\kappa} \xi_{\kappa}\overline{\xi}_{\kappa}/|\kappa| = (\eta, \eta) \leq (\xi', \xi').$$

Since the p_i's are orthogonal, we have $(p_i, p_j) = (p_i p_j^*, 1) = (p_i p_j, 1) = 0$ for $i \neq j$. Furthermore, $(p_i, p_i) = (p_i, 1) = \operatorname{tr} p_i$ and, since tr is a trace, $\operatorname{tr} p_i = \operatorname{tr} e_i$ by Lemma 2 again. We conclude that

$$(\xi', \xi') = \sum_{i=1}^{n} |\lambda_i|^2 (p_i, p_i) = \sum_{i=1}^{n} |\lambda_i|^2 \operatorname{tr} e_i$$

and the first inequality is proved. We now describe when equality occurs.

Suppose first that ξ is central so that clearly $\xi = \eta$. Since all λ_i are distinct and nonzero, it follows from $\xi^k = \lambda_1^k e_1 + \lambda_2^k e_2 + \cdots + \lambda_n^k e_n$ that each e_i is also central. Hence $e_i = p_i$ by Lemma 2 and $\xi' = \xi = \eta$. Thus $(\eta, \eta) = (\xi', \xi')$ and equality occurs. Conversely, suppose equality occurs. If $\eta \neq 0$, then ξ' must be a scalar multiple of η and hence ξ' is central. As above, this implies that each p_i is central, so $p_i = e_i$ and $\xi = \xi'$ is central. On the other hand if $\eta = 0$, then $\operatorname{tr} e_i = 0$ for all i, so $e_i = 0$ and again ξ is central.

Finally observe that $e = e_1 + e_2 + \cdots + e_n$ is an idempotent so

$$\operatorname{tr} e_1 + \operatorname{tr} e_2 + \cdots + \operatorname{tr} e_n = \operatorname{tr} e \leq 1$$

by Kaplansky's Theorem. Thus the second inequality follows immediately from the first. Moreover, equality clearly occurs here if and only if $|\lambda_i| = \lambda$ for all i with $\operatorname{tr} e_i \neq 0$, $\operatorname{tr} e = 1$, and equality holds in the first formula. Since $\operatorname{tr} e_i = 0$ if and only if $e_i = 0$ and $\operatorname{tr} e = 1$ if and only if $e = 1$, the result follows. ■

Several remarks are now in order. First, if G is a finite group and e is an idempotent in $\mathbf{C}[G]$, then

$$\operatorname{tr} e = \frac{\dim_{\mathbf{C}} e\mathbf{C}[G]}{|G|} = \frac{\dim_{\mathbf{C}} e\mathbf{C}[G]}{\dim_{\mathbf{C}} \mathbf{C}[G]}.$$

Thus the trace of e is a measure of its rank and this is also true in some vague sense for G infinite. It follows that the sum

$$|\lambda_1|^2 \operatorname{tr} e_1 + |\lambda_2|^2 \operatorname{tr} e_2 + \cdots + |\lambda_n|^2 \operatorname{tr} e_n$$

is a weighted average of the various $|\lambda_i|^2$ with $\operatorname{tr} e_i$ being a measure of the multiplicity of λ_i as an eigenvalue of ξ.

Second, if ξ itself is an idempotent, say $\xi = e$, then the inequality of Lemma 3 becomes $\sum_{\kappa} e_{\kappa}\overline{e}_{\kappa}/|\kappa| \leq \operatorname{tr} e$. This clearly extends [W, Theorem 2] where it was shown that $e_{\kappa}\overline{e}_{\kappa} \leq |\kappa| \operatorname{tr} e$ for each finite class κ.

Finally, it is quite possible that the above result can be proved entirely within $\mathbf{C}[G]$ using the ideas of [P, §2.1].

THE NILPOTENT CASE

Nilpotent elements can be handled by a fairly standard technique. Let κ be a conjugacy class of G, let n be an integer and define $\kappa^n = \{g^n | g \in \kappa\}$. From $(g^n)^x = (g^x)^n$ for all $g, x \in G$, we see immediately that κ^n is also a conjugacy class of G.

Lemma 4. *Let κ be a finite conjugacy class and let $g \in G$. If $g^p \in \kappa^p$ for infinitely many distinct primes, then $g \in \kappa$.*

Proof. Since κ is finite and $g^p \in \kappa^p$ for infinitely many primes, there exist $x \in \kappa$ and distinct primes p, q with $g^p = x^p$ and $g^q = x^q$. Since $(p, q) = 1$ we then have $1 = pa + qb$ and hence

$$g = (g^p)^a (g^q)^b = (x^p)^a (x^q)^b = x.$$

Thus $g = x \in \kappa$. ■

With this, we can prove

Lemma 5. *Let $\xi \in \mathbf{C}[G]$ be nilpotent. If κ is a finite conjugacy class of G, then $\xi_\kappa = 0$.*

Proof. Assume by way of contradiction that $\xi_\kappa \neq 0$ and say $\xi^n = 0$. By the Extension Theorem for Places (see [P, Theorem 2.2.4]) there exist infinitely many primes p such that

1. $\xi \in R_p[G]$ where R_p is a valuation subring of $\mathbf{C}$,
2. $\xi_\kappa \neq M_p$ where M_p is the maximal ideal of R_p,
3. $R_p/M_p = F_p$, a field of characteristic p.

Furthermore since $\xi = \sum_{g \in G} a_g g$ has finite support, it follows from the previous lemma that there exists a prime p as above with

4. $p > n$,
5. if $g \in \operatorname{Supp} \xi$, then $g^p \in \kappa^p$ if and only if $g \in \kappa$.

Now let p satisfy (1)–(5) and let $\bar{\ }: R_p[G] \to F_p[G]$ be the natural epimorphism. Since $p > n$, it follows from [P, Lemma 2.3.1] that

$$0 = \bar{\xi}^p = \sum_g (\bar{a}_g)^p g^p + \bar{\eta}$$

where $\bar{\eta}$ is a sum of Lie commutators. In particular, computing the trace with respect to the class κ^p yields $\bar{\eta}_{\kappa^p} = 0$. With this, (5) yields

$$0 = (\bar{\xi}^p)_{\kappa^p} = \sum_{g \in \kappa} (\bar{a}_g)^p = \left(\sum_{g \in \kappa} \bar{a}_g \right)^p = (\bar{\xi}_\kappa)^p$$

and we have a contradiction since $\bar{\xi}_\kappa = \overline{\xi_\kappa} \neq 0$ by (2). Thus $\xi_\kappa = 0$ and the lemma is proved. ■

Conclusion

It is now a simple matter to put all the ingredients together. The following is a more precise version of Theorem 1.

Proposition 6. *Let* ξ *be an element of* $\mathbf{C}[G]$ *and suppose that* $f(\xi) = 0$ *for some nonzero polynomial* $f(x) \in \mathbf{C}[x]$. *If* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are the complex roots of* f, *then*

$$\xi = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n + \xi'$$

where $1 = e_1 + e_2 + \cdots + e_n$ *is a decomposition of* 1 *into orthogonal idempotents and where* ξ' *is nilpotent. Furthermore we have*

$$\sum_\kappa \xi_\kappa \overline{\xi}_\kappa / |\kappa| \le |\lambda_1|^2 \operatorname{tr} e_1 + |\lambda_2|^2 \operatorname{tr} e_2 + \cdots + |\lambda_n|^2 \operatorname{tr} e_n$$

with equality if and only if $\xi - \xi'$, *the semisimple part of* ξ, *is central. Finally, if* $\lambda = \max_i |\lambda_i|$, *then*

$$\sum_\kappa \xi_\kappa \overline{\xi}_\kappa / |\kappa| \le \lambda^2$$

with equality if and only if $\xi - \xi'$ *is central and* $|\lambda_i| = \lambda$ *for all* i *with* $e_i \neq 0$.

Proof. Since $f(\xi) = 0$, $\mathbf{C}[\xi]$ is a finite-dimensional commutative $\mathbf{C}$-algebra. Thus, since $\mathbf{C}$ is algebraically closed, it follows as in [P, Theorem 2.3.8] that

$$\xi = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n + \xi'$$

where $1 = e_1 + e_2 + \cdots + e_n$ is a decomposition of 1 into orthogonal idempotents and where ξ' is nilpotent. Let $\eta = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n$ be the semisimple part of ξ. If κ is a finite conjugacy class of G, then $\xi'_\kappa = 0$ by Lemma 5 so $\xi_\kappa = \eta_\kappa$. Hence $\sum_\kappa \xi_\kappa \overline{\xi}_\kappa / |\kappa| = \sum_\kappa \eta_\kappa \overline{\eta}_\kappa / |\kappa|$ and Lemma 3 yields the result. ■

Notice that if $\xi - \xi'$ is central in the above, then ξ is the sum of a central element and a nilpotent element. Conversely, suppose $\xi = \zeta + \nu$ with ζ central and ν nilpotent. Then ξ, ζ and ν commute so $\mathbf{C}[\xi, \zeta, \nu] = \mathbf{C}[\xi, \nu]$ is again a finite-dimensional commutative $\mathbf{C}$-algebra. Write $\zeta = \eta + \zeta'$ where $\eta \in \mathbf{C}[\zeta]$ is semisimple and ζ' is nilpotent. Then $\xi = \eta + (\zeta' + \nu)$ so it follows that η is also the semisimple part of ξ. Thus $\xi - \xi'$ is central if and only if ξ is the sum of a central element and a nilpotent element.

We close with two corollaries.

Corollary 7. *Let $\xi \in \mathbf{C}[G]$ be algebraic over $\mathbf{C}$ and suppose that its minimal polynomial $f(x)$ has distinct roots, say $\lambda_1, \lambda_2, \dots, \lambda_n$. If λ is the maximum of absolute values of the λ_i, then $\sum_\kappa \xi_\kappa \overline{\xi}_\kappa / |\kappa| \le \lambda^2$ with equality if and only if ξ is central and $|\lambda_i| = \lambda$ for all i.*

Proof. Since $f(x)$ has distinct roots, it follows that ξ is semisimple. Furthermore, since $f(x)$ is the minimal polynomial satisfied by ξ, it follows, in the notation of the preceding proposition, that all e_i are nonzero. Proposition 6 now yields the result. ■

Corollary 8. *Let ξ be a unit of finite order in the group algebra $\mathbf{C}[G]$. Then $\sum_\kappa \xi_\kappa \overline{\xi}_\kappa / |\kappa| \le 1$ with equality if and only if ξ is central. Furthermore, if ξ is in the integral group ring $\mathbf{Z}[G]$, then equality occurs if and only if $\pm\xi$ is a central torsion element of G.*

Proof. Since ξ satisfies $x^m - 1$ for some m, its minimal polynomial has distinct roots all of absolute value 1. With this, the first part follows from the preceding corollary. Now suppose equality occurs and $\xi \in \mathbf{Z}[G]$. Since ξ is central, it follows that for each finite class κ either $\xi_\kappa = 0$ or $\xi_\kappa \overline{\xi}_\kappa \ge |\kappa|^2$. We conclude therefore, from $\sum_\kappa \xi_\kappa \overline{\xi}_\kappa / |\kappa| = 1$, that ξ_κ is zero for all but one class. Furthermore, the remaining class κ has size 1 and $\xi_\kappa = \pm 1$. Thus $\xi = \pm g$ with g a central element of G. ■

Finally we remark that if $\xi \in \mathbf{C}[G]$ is algebraic over the rationals $\mathbf{Q}$, then ξ_κ is also algebraic over $\mathbf{Q}$ for each finite conjugacy class κ. This is an immediate consequence of Lemma 5 and the result [B, Theorem 8.1] on idempotents. It also follows directly from Theorem 1 by noting that ξ_κ and its algebraic conjugates have bounded absolute values. An analogous result of interest is [HLP, Corollary 2.8].

References

[B] H. Bass, *Euler characteristics and characters of discrete groups*, Inventiones Math. **35** (1976), 155–196.

[HLP] A. W. Hales, I. S. Luthar and I. B. S. Passi, *Partial augmentations and Jordan decomposition in group rings* (to appear).

[K] I. Kaplansky, *Fields and Rings,* Chicago Lectures in Mathematics, University of Chicago Press, Chicago.

[M] M. S. Montgomery, *Left and right inverses in group algebras*, Bull. Amer. Math. Soc. **75** (1969), 539–540.

[P] D. S. Passman, *The algebraic structure of group rings*, Wiley-Interscience, New York, 1977.

[W] A. Weiss, *Idempotents in group rings*, J. Pure and Appl. Algebra **16** (1980), 207–213.

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