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Gravitational fields with space-times of Bianchi type IX

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Abstract. Spatially homogeneous space-times of Bianchi type IX are considered. A general scheme for the derivation of exact solutions of Einstein's equations corresponding to perfect fluid plus pure radiation fields is outlined. Some simple rotating Bianchi type IX cosmological models are presented. The details of these solutions are also discussed.

Keywords. Bianchi type IX space-times; cosmology; gravitational field.

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1. Introduction

Metric of space-times which satisfy Weyl postulate can be expressed in the form

$$ds^2 = dt^2 - g_{\alpha\beta}\, dx^\alpha\, dx^\beta, \tag{1}$$

α, β running over 1, 2, 3 corresponding to three space-like coordinates x^1, x^2, x^3. Bianchi (1897) was the first to give classification of such space-times, with homogeneous 3-spaces $t = \text{constant}$, into nine distinct types. The homogeneous and isotropic Robertson-Walker space-times which are used to describe standard cosmological models are particular cases of Bianchi type I, V or IX according as the constant curvature of the physical 3-space $t = \text{constant}$ is zero, negative or positive.

However an impression is gaining ground among cosmologists that perhaps the standard cosmological models are too restrictive because of their insistance on the isotropy of the physical 3-space and several attempts have been made to study what have come to be known as non-standard cosmological models (see e.g. MacCallum 1979; Narlikar and Kembhavi 1980; Narlikar 1983).

It would therefore be fruitful to carry out detailed studies of gravitational fields which can be described by space-times of various Bianchi types. Now the classical Einstein universe and the deSitter universe have physical 3-space with positive curvature. Therefore they are particular cases of Bianchi type IX. To express the metric of such a Bianchi type IX space-time, one can choose convenient angular co-ordinates $x^1 = \psi$, $x^2 = \theta$, $x^3 = \phi$ in the homogeneous 3-space $x^4 = t = \text{constant}$ and write this metric in the form

$$\begin{aligned} ds^2 = dt^2 - l^2(d\psi + \cos\theta\, d\phi)^2 - m^2(\sin\psi\, d\theta - \cos\psi\, \sin\theta\, d\phi)^2 \\ - n^2(\cos\psi\, d\theta + \sin\psi\, \sin\theta\, d\phi)^2 \end{aligned} \tag{2}$$

$l = l(t)$, $m = m(t)$, $n = n(t)$ (Michalin aand Melvin 1980).

The authors felicitate Prof. D S Kothari on his eightieth birthday and dedicate this paper to him on this occasion.

We shall first draw out the relation between this metric and the Robertson-Walker metric, describing space-time with topology $R^1 \times S^3$ (i.e. having space sections t = constant with positive constant curvature) viz

$$ds^2 = dt^2 - S^2(t)\, d\sigma^2,$$

where $d\sigma^2$ is the metric on the unit 3-sphere S^3. Thus

$$d\sigma^2 = dx^2 + dy^2 + dz^2 + \frac{(x\,dx + y\,dy + z\,dz)^2}{1 - (x^2 + y^2 + z^2)}.$$

Using a set of transformations first used by Schrödinger (1956),

$$x = \sin\alpha \cos\beta, \quad y = \sin\alpha \sin\beta, \quad z = \cos\alpha \cos\gamma,$$

we rewrite $d\sigma^2$ as

$$d\sigma^2 = d\alpha^2 + \sin^2\alpha\, d\beta^2 + \cos^2\alpha\, d\gamma^2.$$

One can now go over to angular co-ordinates ψ, θ, ϕ of (2) by the substitutions

$$\beta = \tfrac{1}{2}(\phi - \psi), \quad \gamma = \tfrac{1}{2}(\phi + \psi), \quad \alpha = \theta/2.$$

The Robertson-Walker metric then takes the form

$$ds^2 = dt^2 - \frac{S^2(t)}{4}[(d\psi + \cos\theta\, d\phi)^2 + d\theta^2 + \sin^2\theta\, d\phi^2], \tag{3}$$

which is of Bianchi type IX with

$$l^2 = m^2 = n^2 = \tfrac{1}{4} S^2(t).$$

Several physically interesting gravitational situations have been studied earlier using the metric form (3) above e.g. (i) Einstein-Gödel universe (Vaidya 1978) (ii) de Sitter and Taub—NUT space times (Vaidya 1985) and (iii) A rotating homogeneous universe with an electromagnetic field (Vaidya and Patel 1984). It is our aim here to study the gravitational situations described by the general Bianchi type IX metric (2).

In the next section we shall use the formalism of differential forms to describe the geometric properties of the space-time described by the metric (2). In the following section we use Einstein's field equations to relate these geometrical properties with the physical properties of the gravitational fields. In the last two sections we present some details of a few simple particular cases of homogeneous rotating world models.

2. Ricci tensor

For the metric (2) we choose real tetrads $\xi^{(a)}$ as follows:

$$\xi^{(1)} = l(d\psi + \cos\theta\, d\phi), \quad \xi^{(2)} = m(\sin\psi\, d\theta - \cos\psi \sin\theta\, d\phi)$$

$$\xi^{(3)} = n(\cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi), \quad \xi^{(4)} = dt \tag{4}$$

so that the metric (2) becomes

$$ds^2 = (\xi^{(4)})^2 - (\xi^{(1)})^2 - (\xi^{(2)})^2 - (\xi^{(3)})^2$$

and that

$$dt = \xi^{(4)}, \quad d\psi = \frac{\xi^{(1)}}{l} + \cot\theta\left(\frac{\xi^{(2)}}{m}\cos\psi - \frac{\xi^{(3)}}{n}\sin\psi\right)$$

$$d\theta = \frac{\xi^{(2)}}{m}\sin\psi + \frac{\xi^{(3)}}{n}\cos\psi, \quad \sin\theta\, d\phi = -\frac{\xi^{(2)}}{m}\cos\psi + \frac{\xi^{(3)}}{n}\sin\psi.$$

From Cartan's first equation of structure

$$d\xi^{(a)} = -w^a_b \Lambda \xi^{(b)}$$

We find the connection 1-forms w_{ab} as

$$w_{12} = \frac{l^2 + m^2 - n^2}{2lmn}\xi^{(3)}, \qquad w_{14} = -\frac{l_t}{l}\xi^{(1)},$$

$$w_{23} = \frac{m^2 + n^2 - l^2}{2lmn}\xi^{(1)}, \qquad w_{24} = -\frac{m_t}{m}\xi^{(2)},$$

$$w_{31} = \frac{n^2 + l^2 - m^2}{2lmn}\xi^{(2)}, \qquad w_{34} = -\frac{n_t}{n}\xi^{(3)}.$$

The subscript t indicates differentiation with respect to t. From Cartan's second equation of structure

$$\Omega^a_b = dw^a_b + w^a_c \Lambda w^c_b$$

one can find the components of the curvature 2-forms Ω^a_b which in their turn lead to the tetrad components R^a_{bcd} of the curvature tensor through the relations

$$\Omega^a_b = \tfrac{1}{2} R^a_{bcd}\xi^{(c)}\Lambda\xi^{(d)}.$$

And from these tetrad components of the curvature tensor it is easy to work out the tetrad components of the Ricci tensor. All this is straightforward calculation and is not described here in detail. The final expressions of the tetrad components $R_{(ab)}$ of the Ricci tensor turn out to be

$$R_{(11)} = -\frac{l_{tt}}{l} - \frac{l_t}{l}\left(\frac{m_t}{m} + \frac{n_t}{n}\right) - \frac{(l^2 + m^2 - n^2)(l^2 + n^2 - m^2)}{2l^2m^2n^2},$$

$$R_{(22)} = -\frac{m_{tt}}{m} - \frac{m_t}{m}\left(\frac{n_t}{n} + \frac{l_t}{l}\right) - \frac{(m^2 + n^2 - l^2)(m^2 + l^2 - n^2)}{2l^2m^2n^2},$$

$$R_{(33)} = -\frac{n_{tt}}{n} - \frac{n_t}{n}\left(\frac{l_t}{l} + \frac{m_t}{m}\right) - \frac{(n^2 + l^2 - m^2)(n^2 + m^2 - l^2)}{2l^2m^2n^2},$$

$$R_{(44)} = \frac{l_{tt}}{l} + \frac{m_{tt}}{m} + \frac{n_{tt}}{n}, \tag{5}$$

$$R_{(ab)} = 0 \text{ if } a = b.$$

3. The field equations

The Einstein's field equations are

$$R_{ik}-\tfrac{1}{2}g_{ik}R = -8\pi T_{ik}-\Lambda g_{ik}, \tag{6}$$

where Λ is the cosmological constant. We assume that the source of the gravitational field consists of a perfect fluid distribution and a pure radiation field. Hence

$$T_{ik} = (p+\rho)v_i v_k - p g_{ik} + \sigma w_i w_k, \tag{7}$$

where $\sigma w_i w_k$ is the tensor arising from the flowing null radiation and the velocities satisfy the relations

$$v_i v^i = 1, \quad w_i w^i = 0, \quad v_i w^i = 1. \tag{8}$$

The last relation in (8) is the normalizing condition. It is easy to see that the field equations (6) with (7) can be expressed in the form

$$R_{ik} = -8\pi\left[(p+\rho)v_i v_k - \tfrac{1}{2}(\rho - p)g_{ik} + \sigma w_i w_k\right] + \Lambda g_{ik}. \tag{9}$$

The geometry of the metric (2) shows that the 3-space t = constant is homogeneous but not isotropic. As a matter of fact, with distinct functions l, m, n of t, no particular direction can be singled out. However, for considering the fluid distribution which can be described by this geometry, we shall limit ourselves to the cases of the fluids having a unidirectional flow. For such fluid distributions, at every point we can take the fluid flow to be directed along one of the three tetrad directions $\xi^{(1)}$, $\xi^{(2)}$ or $\xi^{(3)}$, say along $\xi^{(1)}$. We therefore choose the flow vector to have tetrad components

$$v_{(a)} = (\sinh\lambda, 0, 0, \cosh\lambda), \tag{10}$$

λ being a function of t to be determined by the field equations. With this form of $v_{(a)}$ the first of the three relations (8) is obviously satisfied. The second and the third relations of (8) will be satisfied if we take

$$w_{(a)} = (-e^{-\lambda}, 0, 0, e^{-\lambda}) \quad \text{or} \quad w_{(a)} = (e^{\lambda}, 0, 0, e^{\lambda}).$$

It will be clear from the field equation corresponding to $R_{(14)} = 0$ that the choice of $w_{(a)}$ between the two possible forms given above essentially depends on the sign of λ. For definiteness we take $\lambda \geqslant 0$ so that

$$w_{(a)} = (-e^{-\lambda}, 0, 0, e^{-\lambda}). \tag{11}$$

Using (10) and (11) in (9) we obtain the following relations

$$R_{(22)} = R_{(33)}, \tag{12}$$

$$8\pi p = \Lambda + \tfrac{1}{2}(R_{(11)} - R_{(44)}), \tag{13}$$

$$8\pi\rho = -\Lambda + \tfrac{1}{2}(R_{(11)} - R_{(44)}) - 2R_{(22)}, \tag{14}$$

$$8\pi\sigma = \sinh\lambda\cosh\lambda(-R_{(11)} - R_{(44)}), \tag{15}$$

$$e^{2\lambda} = \frac{R_{(11)} + R_{(44)}}{2R_{(22)} + R_{(44)} - R_{(11)}}. \tag{16}$$

The last relation (16) can also be written as

$$\sinh^2 \lambda = \frac{(R_{(11)} - R_{(22)})^2}{(R_{(11)} + R_{(44)})\,(R_{(44)} - R_{(11)} + 2R_{(22)})}. \tag{17}$$

Having found $R_{(ab)}$ from the geometry of the metric (2) in the previous section, we [illegible] see that given the metric coefficients of (2), (13) to (16) determine the ph[illegible] parameters p, ρ, σ, e^{λ} for the source of the gravitational field, while (12) remains [illegible] equation to be satisfied by the three metric potentials l, m, n. It may be recalled th[illegible] is a consequence of the assumption of what we have termed unidirectional [illegible] Equation (12) is one equation to determine l, m, n. Two other equations ne[illegible] specify the three metric functions $l(t)$, $m(t)$, $n(t)$ are supplied by (i) an equation of [illegible] the fluid and (ii) by an equation further to specify the fluid motions. In order [illegible] idea of this further specification of fluid motion, we work out the rotation-vec[illegible] stream-lines.

Using (10) and the relations

$$v_i = e_i^{(a)} v_{(a)}, \quad \xi^{(a)} = e_i^{(a)}\,\mathrm{d}x^i,$$

we can easily obtain the tensor components v_i as

$$v_i = (l \sinh \lambda,\ 0,\ l \sinh \lambda \cos \theta,\ \cosh \lambda). \tag{18}$$

The angular velocity of this flow vector is given by

$$\Omega^i = (\varepsilon^{ijkl}/\sqrt{-g})\, w_{jkl},$$

where ε^{ijkl} is the usual Levi-Civita symbol and

$$w_{jkl} = \frac{1}{3!}\left[v_j(v_{k,l} - v_{l,k}) + v_k(v_{l,j} - v_{j,l}) + v_l(v_{j,k} - v_{k,j})\right].$$

A lengthy but straightforward calculation gives

$$\Omega^i = \left(\frac{1}{mn} \sinh \lambda \cosh \lambda,\ 0,\ 0,\ \frac{l}{mn} \sinh^2 \lambda\right). \tag{19}$$

The vector Ω^i is, of course, space-like and the magnitude ω of this vector is [illegible]

$$\omega^2 = -g_{ik}\Omega^i\Omega^k = \frac{l^2}{m^2 n^2} \sinh^2 \lambda.$$

Thus the source of this gravitational field is a rotating fluid distribution, the ma[illegible] of the twist being $(l/mn)\sinh \lambda$. A specification of this twist may give us the [illegible] equation to determine l, m and n.

The following important conclusion, can be drawn at this stage.

The necessary and sufficient condition that the source of the gravitational [illegible] represented by the metric (2) be a perfect fluid (and not a mixture of perfect flu[illegible] null fluid) is that the twist of the stream-lines is zero and that the stream-[illegible] orthogonal to the surfaces ρ = constant.

This result is clear from (15). If the source is a perfect fluid we must take $\sigma = 0$. Equation (15) then implies $\sinh\lambda = 0$ which in its turn leads to the twist $\omega = 0$. Again $\sinh\lambda = 0$ implies that the co-ordinates used in metric (2) are co-moving and so the streamlines are normal to the surfaces t = constant which are the same as the surfaces ρ = constant.

Though several perfect fluid distributions can be worked out which give rise to gravitational fields described by a Bianchi type IX metric (2), in what follows we would be interested in fluid distributions with rotating stream-lines and so with $\sigma \neq 0$.

Let us end this section with a short discussion of the first of these three field equations viz (12). Using the forms of $R_{(22)}$ and $R_{(33)}$ for the metric (2) as given by (5) we find that $R_{(22)} = R_{(33)}$ implies

$$[l(nm_t - mn_t)]_t + \frac{(m^2 - n^2)(m^2 + n^2 - l^2)}{lmn} = 0.$$

It is clear that this relation is identically satisfied if $m = n$. In the next section we work out this case in some detail. In the following section we work out a couple of simple examples when $m \neq n$, the detailed discussion of this case being left to a later paper.

4. Solutions with $m = n$

In this case we shall work with the equations of state of the type $p = \gamma\rho$ where γ is a constant. Substitution of the values of p and ρ from (13) and (14) in $p = \gamma\rho$ yields the differential equation

$$2(\gamma+1)\left[\frac{m_{tt}}{m} + \frac{l_t m_t}{lm} - \Lambda\right] + 2\left[2\gamma\frac{m_t^2}{m^2} - (\gamma-1)\frac{l_{tt}}{l}\right] + \frac{1}{2m^4}\{8m^2\gamma - l^2(5\gamma-1)\} = 0. \tag{20}$$

Case (i): $l^2 = 2km^2$, $p = \gamma\rho$, k = constant, $\Lambda = 0$.

The relation $l^2 = 2km^2$ reduces (20) to the form

$$4\frac{m_{tt}}{m} + 2(3\gamma+1)\frac{m_t^2}{m^2} + \frac{1}{m^2}\{4\gamma - k(5\gamma-1)\} = 0.$$

The first integral of the above equation can be easily found to be

$$m_t^2 = Am^{-(3\gamma+1)} + \frac{k(5\gamma-1) - 4\gamma}{2(3\gamma+1)} \tag{21}$$

where A is a constant of integration. From the relations (14), (15) and (16) we obtain the physical parameters ρ, σ and $e^{2\lambda}$ as

$$8\pi\rho = 3Am^{-3(\gamma+1)} - \frac{2(2k-1)}{3\gamma+1}m^{-2}, \tag{22}$$

$$8\pi\sigma = \frac{(2k-1)[3A(\gamma+1)(3\gamma+1)+(2k-1)(\gamma-1)m^{3\gamma+1}]}{m^2(\gamma+1)[3A(3\gamma+1)-2(2k-1)m^{3\gamma+1}]}, \tag{23}$$

$$e^{2\lambda} = \frac{3A(\gamma+1)(3\gamma+1)-4\gamma(2k-1)m^{3\gamma+1}}{3A(\gamma+1)(3\gamma+1)-2(\gamma+1)(2k-1)m^{3\gamma+1}}. \tag{24}$$

From (23) it is clear that when $2k = 1$ the radiation density σ vanishes and we recover a special case of the Robertson-Walker universe. This is a remarkable feature of the above solution.

Here the constants k, A and γ must be so adjusted that the physical requirements $\rho > 0$, $p \geqslant 0$, $\sigma \geqslant 0$, $e^{2\lambda} > 0$ are satisfied. We have verified that the rotation of the flow vector v^i is non-zero in this case. Note that for the matter dominated epochs we have $\gamma = 0$ and for the radiation-dominated epochs we have $\gamma = 1/3$.

A more general method of obtaining solutions of this type is outlined in the appendix.

Case (ii) l = constant, $p = 0$.

Substituting $\gamma = 0$ and l = constant in (20) we obtain

$$2(m_{tt}/m) + (l^2/2m^4) = 2\Lambda. \tag{25}$$

The first integral of the above differential equation can be readily found to be

$$m_t^2 = B + \Lambda m^2 + (l^2/4m^2), \tag{26}$$

where B is an arbitrary constant of integration. The solution of (26) can be, in general, expressed in terms of elliptic functions. We shall not give the explicit solution of (26). The physical parameters ρ, σ and $\exp(2\lambda)$ are determined from (14), (15) and (16) as

$$8\pi\rho = 2\Lambda + (2B/m^2) + (1/m^4)(2m^2 - l^2),$$

$$8\pi\sigma = \frac{(B+1)(m^2 - l^2 + 2\Lambda m^4 + Bm^2)}{m^2(l^2 - 2m^2 - 2Bm^2 - 2\Lambda m^4)},$$

$$e^{2\lambda} = (2\Lambda m^4 - l^2)(l^2 - 2m^2 - 2Bm^2 - 2\Lambda m^4)^{-1}.$$

When we put m = constant = l, we must have $B = -\frac{1}{2}$ and $\Lambda = (1/4m^2)$. In this case σ vanishes and we recover the usual Einstein universe. We have seen that the twist of the stream-lines of the dust distribution filling the model is non-vanishing. For the Einstein universe it becomes zero. Thus the solution described here represents a non-static rotating generalization of Einstein static universe.

Case (iii) m = constant, $p = 0$.

Since this case is similar to the case (ii), we shall be brief and simply state the results. The differential equation to be satisfied by the function $l(t)$ is

$$l_t^2 = C + \Lambda l^2 - (l^4/8m^4), \tag{28}$$

which can be solved in terms of elliptic functions. Here C is a constant of integration.

The parameters ρ, σ and $e^{2\lambda}$ are given by

$$8\pi\rho = -2\Lambda + [(2m^2 - l^2)/m^4],$$

$$8\pi\sigma = \frac{(4m^2 - l^2 - 4\Lambda m^4)(3l^2 - 4m^2 + 4\Lambda m^4)}{16m^4(2m^2 - l^2 - 2\Lambda m^4)}, \tag{29}$$

$$e^{2\lambda} = \frac{l^2}{2(2m^2 - l^2 - 2\Lambda m^4)}.$$

Here also it can be seen that the solution describes a non-static rotating generalization of Einstein static universe.

In the cases (ii) and (iii) we have considered the matter dominated epochs (i.e. $p = 0$) because of our interest in the generalizations of Einstein universe. The solutions for other equations of state can also be found on similar lines.

5. Solutions with $m \neq n \neq l$

In the early epochs of a big-bang cosmology one may expect radiation in thermal equilibrium i.e. $\rho = 3p$ and in the late stages of evolution, where galaxies move freely, one may expect dust i.e. $p = 0$. Therefore in this section we consider these two equations of state along with $m \neq n \neq l$.

Case (*i*) $p = 0$, $\Lambda = 0$

Putting $p = 0$, $\Lambda = 0$, $m = l\cos q$, $n = l\sin q$, $q =$ constant in (12), we get the equation

$$(2l_{tt}/l) + (l_t^2/l^2) + (1/l^2) = 0. \tag{30}$$

It is easy to obtain the first integral of the above equation in the form

$$l_t^2 = (D/l) - 1, \tag{31}$$

where D is a constant of integration. The solution of (31) may be parametrized by an angle $H(t)$ with

$$l = \frac{D}{2}(1 - \cos H), \quad t = \frac{D}{2}(H - \sin H).$$

The parameters ρ, σ and $\exp(2\lambda)$ can be easily determined. They are given by

$$8\pi\rho = \frac{1}{l^3}(3D - 4l), \quad \exp(2\lambda) = 3D/(3D - 4l),$$

$$8\pi\sigma = \frac{2}{l^2}\frac{(3D - 2l)}{(3D - 4l)}. \tag{32}$$

For $\rho > 0$, $\exp(2\lambda) > 0$ and $\sigma > 0$ we must have $3D - 4l > 0$ i.e. $l < 3D/4$. The model is valid for those values of l which satisfy this restriction. Here it should be noted that the constant D is always positive.

Case (ii) $\rho - 3p = 0, \quad \Lambda = 0.$

In this case we obtain the following differential equation for the function l:

$$3(l_{tt}/l) + 3(l_t^2/l) + 1/l^2 = 0.$$

Here also we have taken $m = l\cos q, n = l\sin q, q =$ constant. The first integral for the above equation is found to be

$$l_t^2 = (N/l^2) - \tfrac{1}{3}, \tag{34}$$

where N is a constant of integration. The parameters ρ, σ and $\exp(2\lambda)$ in this case are determined as

$$8\pi\rho = (3N - 2l^2)/l^3, \quad 8\pi\sigma = \frac{1}{l^2}\frac{(6N - l^2)}{(3N - 2l^2)},$$

$$e^{2\lambda} = (3N + l^2)/(3N - 2l^2). \tag{35}$$

Here also the constant N is always positive.

It can be easily seen from (19) that the rotation vector Ω^i of the stream-lines is non-vanishing in both the cases. Therefore the solutions of the above two cases represent rotating cosmological models.

Appendix

In the axially symmetric case $m = n$ it is convenient to replace differentiation with regard to t by one with regard to m. As a matter of fact we use the substitutions $l^2 = 2\eta m^2, m_t^2 = 4\mu\eta$ and replace d/dt by d/dz where $e^z = m$. Using these substitutions in (5) and also noting that we are now dealing with the axially symmetric case $m = n$, we shall find,

$$R_{(11)} = -\exp(-2z)[2\mu\eta_{zz} + (\mu_z + 8\mu)\eta_z + (2\mu_z + 8\mu + 1)\eta]$$

$$R_{(22)} = R_{(33)} = -\exp(-2z)[4\mu\eta_z + (2\mu_z + 8\mu - 1)\eta + 1]$$

$$R_{(44)} \equiv \exp(-2z)[2\mu\eta_{zz} + (\mu_z + 8\mu)\eta_z + 4(\mu_z + \mu + \tfrac{1}{8})\eta],$$

where the subscript z denotes differentiation with respect to z. The advantage of these substitutions is that for simple cosmologically relevant equations of state like $p = 0$, $\rho = 3p$ or $p =$ (constant)ρ we get a tractable relation between η and μ. For example, for the equation of state $p = 0$, taking the cosmological constant $\Lambda = 0$, we find the equation

$$2\mu\eta_{zz} + (\mu_z + 8\mu)\eta_z + 4(\mu_z + \mu + \tfrac{1}{8})\eta = 0. \tag{A1}$$

We are still left with the freedom of choosing a third relation between l, m, n. We can choose it in a way which can help us in working with (A.1). For example choosing $\mu_z = az + b$, a, b, constants, one can make (A.1) amenable to a power series solution for η in terms of z.

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