ON MULTIPLE FOURIER SERIES

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§ 1. The object of this note is to announce some of the results obtained by the author on the *Spherical* Summation of Multiple Fourier Series, with an indication of the method of proof. A complete account of these results will appear elsewhere, in due course.

§ 2. *Notations and Definitons.* Let $f(x) = f(x_1 \ldots x_k)$ be a function of the Lebesgue class L, which is periodic in each of the k-variables, with period 2π.

Let

$$a_{n_1 \ldots n_k} = \frac{1}{(2\pi)^k} \int_{-\pi}^{+\pi} \cdots \int_{-\pi}^{+\pi} f(x)\, e^{-i(n_1 x_1 + \cdots + n_k x_k)}\, dx_1 \ldots dx_k$$

The Series

$$\Sigma\, a_{n_1 \ldots n_k}\, e^{i(n_1 x_1 + \cdots + n_k x_k)} \tag{2·1}$$

is called the Multiple Fourier Series of the function $f(x)$.

Let

$$S_R(x) = \sum_{\nu \leqslant R} a_{n_1 \ldots n_k}\, e^{i(n_1 x_1 + \cdots + n_k x_k)}, \quad \nu^2 = n_1^2 + \ldots + n_k^2 \tag{2·2}$$

denote the 'spherical' partial sum of the Series (2·1); that is, we shall consider (2·1) as a simple series

$$\sum_{j=0}^{\infty} \sum_{\nu = R_j} a_{n_1 \ldots n_k}\, e^{i(n_1 x_1 + \cdots + n_k x_k)}, \tag{2·3}$$

where R_j^2 is the sequence of all integers that can be represented as sums of k-squares.

Let

$$S_R^\delta(x) = \sum_{\nu \leqslant R} \left(1 - \frac{\nu^2}{R^2}\right)^\delta a_{n_1 \ldots n_k}\, e^{i(n_1 x_1 + \cdots + n_k x_k)}, \tag{2·4}$$

so that S_R^δ is the Riesz mean of the Series (2·1), of type ν^2 and order δ.

If $\underset{R \to \infty}{lt}\; S_R^\delta$ exists and is finite, then the Series (2·1) will be summable (ν^2, δ). If $S_R^\delta(x)$ is of bounded variation in $0 < R < \infty$, the Series (2·1) will be *absolutely* summable (ν^2, δ), or summable $|\,\nu^2, \delta\,|$.

Let

$$f_p(x, t) = f_p(t) = \frac{c}{t^k} \int f(y) \left(1 - \frac{s^2}{t^2}\right)^{p-1} dy,$$

where dy is the k-dimensional volume element, $\sum_1^k (y_i - x_i)^2 = s^2 \leqslant t^2$, and c is a suitable constant. If $p = 0$, we write $f_0(x, t) = f_x(t)$. $f_p(t)$ may be called the 'spherical mean' of order p of the function $f(x)$.

We prove theorems connecting the behaviour of the spherical mean of a function at a point, with the summability of the corresponding Fourier Series at the point.

§ 3. *Theorems on Summability.*

Theorem 1. If $f_p(t) \to l$ as $t \to 0$, then $\lim_{R \to \infty} S_R^\delta(x) = L$, for $\delta > p + \frac{k-1}{2}$ and $L = 2^{\frac{k-2}{2}} \Gamma\left(\frac{k}{2}\right)$.

Theorem 2. If (i) $f_{p+1}(t) - l = o(1)$, $t \to 0$,

(ii) $t^{-k-2p} \int_0^t s^{k+2p-1} |f_p(t)|\, dt = O(1)$ as $t \to 0$,

or if

$$t^{-k-2p} \int_0^t s^{k+2p-1} |f_p(t)|\, dt = o(1),$$

then, $\lim_{R \to \infty} S_R^\delta(x) = L$, for $\delta > p + \frac{k-1}{2}$.

Theorem 3. If $f_p(t) - l = O(t^\alpha)$ as $t \to 0$, $0 < \alpha < 1$, then, for $\delta = p + \frac{k-1}{2} + \beta$, $0 < \beta$,

$$S_R^\delta(x) - L = \begin{cases} O(R^{-\alpha}), \text{ if } \beta > \alpha, \\ O(R^{-\alpha} \log R), \text{ if } \beta = \alpha, \\ O(R^{-\beta}), \text{ if } \beta < \alpha. \end{cases}$$

Theorem 4. If $\frac{f_p(t) - l}{t^\alpha} \to s_\alpha$,

then,

$$R^\alpha (S_R^\delta - L) \to l_\alpha, \text{ for } \delta > p + \frac{k-1}{2} + \alpha,$$

where

$$l_\alpha = s_\alpha \cdot \frac{2^{\frac{k}{2} - 1 + \alpha}\, \Gamma(\delta + 1)\, \Gamma\left(p + \frac{\alpha}{2} + \frac{k}{2}\right) \Gamma\left(\frac{k}{2}\right)}{\Gamma\left(p + \frac{k}{2}\right) \Gamma\left(\delta + 1 - \frac{\alpha}{2}\right)}.$$

Theorem 5. If $S_R^\gamma(x) \to s$ as $R \to \infty$, then $f_p(y) \to s/2^{K/2-1}\,\Gamma\left(\frac{k}{2}\right)$, as $y \to 0$, provided $p > \max\left(1, \gamma - \frac{k-3}{2}\right)$.

Theorem 6. If $S_R^\gamma(x) - s = O(R^{-\alpha})$ as $R \to \infty$, $0 < \alpha < 2$, then,

$$f_p(y) - s/2^{\frac{k}{2}-1}\,\Gamma\left(\frac{k}{2}\right) = O(y^\alpha),$$

for

$$p > \max\left[1, \gamma - \frac{k-3}{2} + \alpha\right].$$

Combining Theorems 1 and 5, we can state the following.

Theorem 7. A necessary and sufficient condition that the Multiple Fourier Series of a function $f(x)$ should be summable (spherically) at a point is that the meanlimit, of some order, of the function exists at that point.

Combining Theorems 2 and 6, we can state the following.

Theorem 8. If $t^{-K-2p}\int_0^t s^{K+2p-1}\,|f_p(s)|\,ds = O(1)$ as $t \to 0$, or in particular, if $f_p(s) = O(1)$, then the Multiple Fourier Series of $f(x)$ is either summable (ν^2, δ) for every $\delta > p + \frac{k-1}{2}$ or for no δ; a necessary and sufficient condition for it to be summable is that $f_q(t) \to l$ as $t \to 0$, for $q > p + 1$.

Theorem 9. If $f_p(t)$ is of bounded variation in $0 < t < \infty$, then the Series (2·1) is summable $|\nu^2, \delta|$, for $\delta > p + \frac{k-1}{2}$.

Theorem 10. If $S_R^\delta(x)$ is of bounded variation in $0 < R < \infty$, then $f_p(t)$ is of bounded variation in $0 < t < \infty$, for $p > \max\left(1, \delta - \frac{k-3}{2}\right)$.

Theorem 11. If $f_p(t)$ is of bounded variation in $0 < t < \infty$, *and* $p \geqslant 1$, then the Series (2·1) is summable (ν^2, δ) for $\delta > p - 1 + \frac{k-1}{2}$.

Theorem 12. Summability $|\nu^2, \delta|$, for $\delta > \frac{k+1}{2}$, of the Multiple Fourier Series of $f(x)$ at any point depends only on the behaviour of the function in the neighbourhood of that point.

The proof of the above Theorems is essentially based on the following fundamental formula of *Bochner:

* S. Bochner : " Summation of Multiple Fourier Series by Spherical Means," *Trans. American Math. Soc.*, 40 (1936), 175–207.

$$S_R^\delta(x) = 2^\delta \Gamma(\delta+1) R^k \int_0^\infty t^{k-1} f_x(t) V_{\delta+\frac{k}{2}}(tR)\, dt, \tag{3·1}$$

for $\delta > \frac{k-1}{2}$, where $V_l(x) = J_l(x)/x^l$ and J_l denotes the Bessel function of order l.

By partial integration, we can generalize the formula (3·1) and prove *its* reciprocal. We accordingly obtain, on the one hand,

$$S_R^\delta(x) = \frac{2^{\delta-p}\,\Gamma(\delta+1)\,\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(p+\frac{k}{2}\right)} R^{K+2p} \int_0^\infty t^{K+2p-1} f_p(t)\, V_{\delta+p+\frac{k}{2}}(tR)\, dt \tag{3·2}$$

if $\delta > h + \frac{k-1}{2}$, where h is the greatest integer less than p; and on the other,

$$f_p(y) = \frac{\Gamma\left(p+\frac{k}{2}\right)}{2^{\delta-p}\,\Gamma(\delta+1)\,\Gamma\left(\frac{k}{2}\right)} y^{2\delta+2} \int_0^\infty S_R^\delta \cdot R^{2\delta+1} V_{p+\delta+\frac{k}{2}}(yR)\, dR \tag{3·3}$$

if $p > 1$ and $\delta > \frac{k-1}{2}$.

Formulæ (3·2) and (3·3) enable us, on the application of appropriate arguments, to connect the behaviour of $\lim_{R\to\infty} S_R^\delta$ with that of $\lim_{t\to 0} f_p(t)$ and deduce all the results cited above.