A complete set of numerical invariants for a subfactor

VIJAY KODIYALAM and V.S. SUNDER
The Institute of Mathematical Sciences, INDIA

Abstract

We show that certain numerical invariants associated naturally to a *subfactor planar algebra* constitute a complete family in the sense of determining the isomorphism class of the subfactor planar algebra.

In the course of the proof, we show also that planar algebra isomorphisms of subfactor planar algebras can always be chosen to be $*$-preserving. This latter statement generalises the fact that 'Hopf algebra isomorphisms of finite-dimensional Kac algebras can be chosen to be $*$-preserving'.

Keywords: Planar algebra, picture invariant, invariant theory.

1 Introduction

In [J1], Jones associated what he called a *planar algebra* to every extremal subfactor of finite index, and used earlier work of Popa ([Po2]) to show that all 'subfactor planar algebras' (in the terminology we use here) arise in this fashion. A subfactor is uniquely determined by its planar algebra only in the presence of additional hypotheses such as strong amenability (by [Po1]); so our numerical invariants will also determine only such subfactors.

We quickly recall the essential facts about a subfactor planar algebra P. (All these facts are from [J1]; these facts are also covered in a more elaborate fashion in [KLS].)

Facts:

- (1) There is a countable[1] family $\{P_k\}_{k \in \ Col}$ of finite-dimensional Hilbert spaces, where the elements of the indexing set Col will be thought of as *colours*.
- (2) A *planar tangle* T consists of an exterior disc of colour k_0 and an ordered collection of b (≥ 0) internal discs of colours $k_1, \cdots, k_b$ respectively together with some additional data. We will also refer to (and draw) the discs of a tangle as boxes.
- (3) To each planar tangle T is associated a linear map

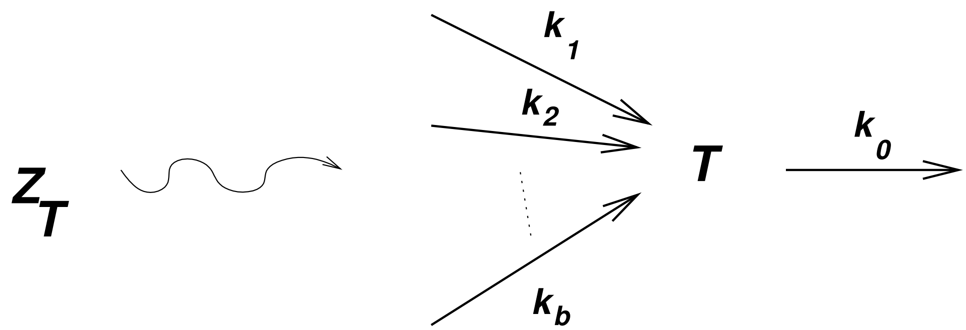
  $$Z_T : \otimes_{i=1}^{b} P_{k_i} \rightarrow P_{k_0}$$

  (When we wish to draw attention to the planar algebra P, we shall write Z_T^P for what we called Z_T above.)
- (4) The assignment $T \mapsto Z_T$ satisfies various 'natural' properties (such as being well-behaved with respect to 'composition of tangles' or 're-numbering of the internal discs).
- (5) Each P_k is a C^*-algebra, with the adjoint and multiplication related to the planar algebra structure in a definite manner.
- (6) There are C^*-algebra inclusions $P_k \subset P_{k+1}$ which are induced by certain 'inclusion tangles'.

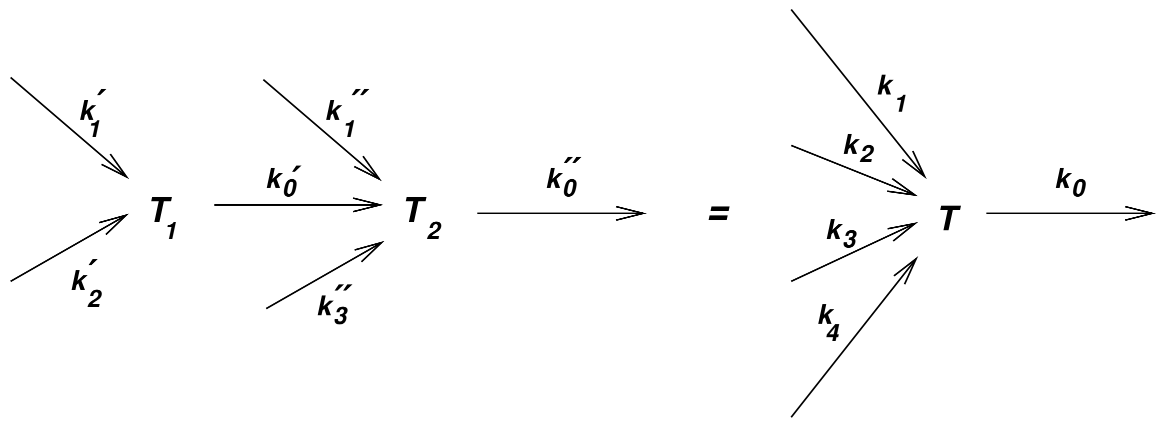
[1] Actually, this set is taken as $\{0_+, 0_-, 1, 2, \cdots\}$ in [J1] and [KLS].

- (7) $P_{0_\pm}$ is a unital one-dimensional algebra over $\mathbb{C}$ and is hence canonically identified with $\mathbb{C}$. Further, there exists a tangle tr_k (which is also denoted, at least when $k \neq 0_-$, by E_k^{0+} in §4 of this paper) whose external disc has colour 0_+ if $k > 0$ or $k = 0_+$, and 0_- if $k = 0_-$, with the property that if we let $\tau_k : P_k \to \mathbb{C}$ be the map obtained by composing $\delta^{-k} Z_{tr_k}$ and the isomorphism of $P_{0_\pm}$ with $\mathbb{C}$ (where δ^2 is the Jones index of the given subfactor), then the family $\{\tau_k : k \in Col\}$ defines a consistent faithful trace τ on the union $\cup P_k$, and the inner product on P_k is given by $\langle x, y \rangle = \tau(y^* x)$.

It will be convenient for us to think pictorially of the operator Z_T associated to the tangle T as follows:



It is clear that we may 'glue' several such pictures together (provided the colours of the arrows match) to obtain more complicated pictures. For instance, suppose we have tangles T_1 and T_2 where T_1 has two internal discs of colours k'_1, k'_2 and external disc of colour k'_0, while T_2 has three internal discs of colours k''_1, k''_2, k''_3 and external disc of colour k''_0; suppose further that $k'_0 = k''_2$. Then we can form a new tangle T (which would be denoted by $T_2 \circ_{D_2} T_1$ in the notation of [J1] and [KLS]) with four internal discs with colours $k_1 = k''_1, k_2 = k'_1, k_3 = k'_2, k_4 = k''_3$ respectively, and external disc of colour $k_0 = k''_0$. This is understood most easily via the picture:



In more explicit detail, we find that

$$Z_T : P_{k_1''} \otimes P_{k_1'} \otimes P_{k_2'} \otimes P_{k_3''} \to P_{k_0''}$$

and is defined by the equation

$$Z_T(x_{k_1''} \otimes x_{k_1'} \otimes x_{k_2'} \otimes x_{k_3''}) = Z_{T_2}(x_{k_1''} \otimes Z_{T_1}(x_{k_1'} \otimes x_{k_2'}) \otimes x_{k_3''}) \ .$$

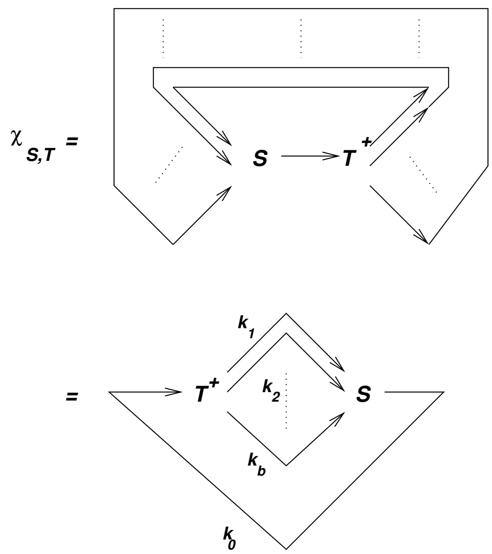
One convention that we will follow, and which is illustrated above, is the following: when we draw such pictures, it will always be assumed that the order of the inputs (resp., outputs) for a tangle (resp., the dagger of a tangle - see §3) will be counter-clockwise (resp., clockwise) starting from just after its unique output (resp., input).

We now come to the most important 'picture' from the point of view of this paper.

DEFINITION 1 *If S, T are two tangles of the same type - i.e., S and T have the same number, say b, of internal discs, the i-th internal disc of each is of the same colour, say k_i, and both have their external discs of the same colour, say k_0 - then the operator $Z_S^P(Z_T^P)^*$ is an endomorphism of P_{k_0}, and so, the right sides of the following equation makes sense:*

$$\chi_{S,T}^P = Tr((Z_T^P)^* Z_S^P) = Tr(Z_S^P(Z_T^P)^*) \ , \tag{1.1}$$

where Tr denotes the usual trace of an endomorphism of a finite-dimensional space. We shall think of $\chi_{S,T}$ as the following 'closed picture' - i.e., a picture with *no 'free' inputs or outputs - and shall refer to χ as the* **character of the planar algebra**.



(We shall later make precise what we mean by equality of 'picture invariants' and what $T^\dagger$ in the above picture stands for.)

We are now in a position to state the main result of this paper.

THEOREM 2 *Two subfactor planar algebras are isomorphic if and only if they have the same character.*

Or in more detail, subfactor planar algebras P and Q are isomorphic if and only if $\chi^P_{S,T} = \chi^Q_{S,T}$ for every pair S, T of planar tangles of the 'same type'.

In §2 we prove that isomorphisms of subfactor planar algebras may be chosen to be $*$-preserving and use this to show that isomorphic planar algebras have the same character. In §3 a general result of the invariant theory of unitary groups is proved which is applied in §4 to the subfactor context to show the converse implication.

Finally, we shall consistently use the sybmol $[N]$ to denote the set $\{1, 2, \cdots, N\}$ whenever $N \in \mathbb{Z}_+$ (with the convention that $[0] = \emptyset$).

2 Isomorphism vs. $*$-isomorphism

We recall, for the reader's sake, that planar algebras $P = \{P_k : k \in \ Col\}$ and $Q = \{Q_k : k \in \ Col\}$ are said to be isomorphic if there exist linear isomorphisms

$\pi_k : P_k \rightarrow Q_k$ which are compatible with the action of planar tangles in the sense that if T is any planar tangle with b internal discs of colours $k_1, \cdots, k_b$ respectively and the external disc of colour k_0, then

$$\pi_{k_0} \circ Z_T^P = Z_T^Q \circ \left(\otimes_{i=1}^b \pi_{k_i} \right) \ .$$

We begin this section by proving that subfactor planar algebras are isomorphic as planar algebras if and only if they are isomorphic as subfactor planar algebras. [2] More precisely, we shall prove the following technical fact.

THEOREM 3 *Suppose P and Q are subfactor planar algebras. Suppose there exist linear maps $\pi_k : P_k \rightarrow Q_k$ such that the family $\pi = \{\pi_k : k \in Col\}$ implements a planar algebra isomorphism of P onto Q (in the sense of being compatible with the actions of the planar operad of coloured tangles, as described in the first paragraph of this section). Then there exists another planar algebra isomorphism $\omega = \{\omega_k : k \in Col\}$ of P onto Q with the additional property that each ω_k is a $*$-isomorphism (of finite-dimensional C^*-algebras).*

Proof: The idea of the proof is as follows. To start with, fix a $k \in Col$, and consider the polar decomposition $\pi_k = \omega_k \alpha_k$ of π_k regarded as a linear operator between the Hilbert spaces P_k and Q_k (equipped with the 'Hilbert-Schmidt' norm induced by the trace). So $\omega_k : P_k \rightarrow Q_k$ (resp. $\alpha_k : P_k \rightarrow P_k$) is a unitary operator (resp., is a positive operator). We shall show that $\omega = \{\omega_k : k \in Col\}$ (and hence also $\alpha = \{\alpha_k : k \in Col\}$) implements a planar algebra isomorphism of P onto Q (of P onto P, respectively). Finally, we shall show that each ω_k is a $*$-isomorphism.

In order to prove that the family $\omega = \{\omega_k : k \in Col\}$ implements a planar algebra isomorphism, let us look at

$$\mathcal{T}_\omega = \{T \in \mathcal{T} : \omega_{k_0(T)} \circ Z_T^P = Z_T^Q \circ \left(\otimes_{i=1}^{b(T)} \omega_{k_i(T)} \right) \} \ ,$$

(where, as in [KLS], we have used the notation $\mathcal{T}$ for the collection of all tangles, $b(T)$ for the number of internal discs (or 'boxes') of T, and $k_i(T)$ for the colour of the i-th internal disc of T).

It is easy to see that $\mathcal{T}_\omega$ is 'closed under composition' in the sense that if (i) $T, S \in \mathcal{T}_\omega$, and (ii) if the colour of the external disc of S agrees with that of the i-th internal disc of T, then also the 'composite tangle' $T \circ_{D_i} S \ \in \mathcal{T}_\omega$.

[2] This should be compared with the fact that normal operators are similar if and only if they are unitarily equivalent. Both results essentially say that there is enough algebraic structure around to ensure that 'the $*$-structure is unique, if it exists'.

Since we need to show that $\mathcal{T}_\omega = \mathcal{T}$, it suffices, in view of Theorem 3.3 of [KS][3], to show that

$$\begin{aligned} 1_{0_+}, 1_{0_-} &\in \mathcal{T}_\omega & (2.2)\\ \forall k \geq 2\ ,\ R_k &\in \mathcal{T}_\omega & (2.3)\\ \forall k \in \ Col\ ,\ E^k_{k+1}, I^{k+1}_k &\in \mathcal{T}_\omega & (2.4)\\ \text{and } \forall k \in \ Col\ ,\ M_k &\in \mathcal{T}_\omega. & (2.5) \end{aligned}$$

For (2.2), note that $P_{0_\pm}$ and $Q_{0_\pm}$ are unital 1-dimensional algebras and hence canonically identified with $\mathbb{C}$; consequently the algebra isomorphisms $\pi_{0_\pm}$ get identified with $id_\mathbb{C}$; since this operator is unitary, we find that both $\omega_{0_\pm}$ and $\alpha_{0_\pm}$ also get identified with $id_\mathbb{C}$, and we see that indeed $1_{0_+}, 1_{0_-} \in \mathcal{T}_\omega$.

Suppose next that T is an 'annular tangle'- i.e., $b(T) = 1$ - and that $k_1(T) = k_1$ and $k_0(T) = k_0$. It then follows that $T^\dagger$ - see Remark 9 - may also be viewed as an (annular) tangle, and that $k_1(T^\dagger) = k_0$ and $k_0(T^\dagger) = k_1$. Hence, by assumption, we see that $\pi_{k_1} \circ Z^P_{T^\dagger} = Z^Q_{T^\dagger} \circ \pi_{k_0}$, and that $\pi_{k_0} \circ Z^P_T = Z^Q_T \circ \pi_{k_1}$. Since $Z^*_{T^\dagger} = Z_T$, we deduce by taking adjoints that $Z^P_T \circ \pi^*_{k_1} = \pi^*_{k_0} \circ Z^Q_T$, and then conclude that

$$\begin{aligned} Z^P_T \circ \alpha^2_{k_1} &= Z^P_T \circ \pi^*_{k_1} \circ \pi_{k_1}\\ &= \pi^*_{k_0} \circ Z^Q_T \circ \pi_{k_1}\\ &= \pi^*_{k_0} \circ \pi_{k_0} \circ Z^P_T\\ &= \alpha^2_{k_0} \circ Z^P_T\ ; \end{aligned}$$

Next, choose a a one-variable polynomial p without constant term, such that $p(t) = \sqrt{t}$ whenever t is an eigenvalue of either $\alpha^2_{k_0}$ or $\alpha^2_{k_1}$; then it follows that $p(\alpha^2_{k_i}) = \alpha_i$ for $i = 0, 1$, and hence

$$\begin{aligned} Z^P_T \circ \alpha_{k_1} &= Z^P_T \circ p(\alpha^2_{k_1})\\ &= p(\alpha^2_{k_0}) \circ Z^P_T\\ &= \alpha_{k_0} \circ Z^P_T\ . \end{aligned}$$

Since the α_{k_i} are invertible, we may finally conclude that

$$\begin{aligned} \omega_{k_0} \circ Z^P_T &= \pi_{k_0} \circ \alpha^{-1}_{k_0} \circ Z^P_T\\ &= \pi_{k_0} \circ Z^P_T \circ \alpha^{-1}_{k_1}\\ &= Z^Q_T \circ \pi_{k_1} \circ \alpha^{-1}_{k_1}\\ &= Z^Q_T \circ \omega_{k_1}\ . \end{aligned}$$

[3]This theorem says that the tangles displayed in equations (2.2), (2.3), (2.4) and (2.5) 'generate' the class of all tangles with respect to 'composition of tangles' defined as above.

In other words, we have shown that $T \in \mathcal{T}_\omega$ for every annular tangle T; in particular, this establishes (2.3) and (2.4).

In order to complete the proof of the theorem, it remains only to prove that (a) in addition to (2.2), (2.3) and (2.4), the statement ((2.5) is also valid, so that ω (as well as α) implements an isomorphism of planar algebras; and (b) each ω_k is a $*$-homomorphism. Both these statements will follow from the lemma below, and the proof of the theorem and the next lemma would be simultaneously complete. □

LEMMA 4 *Suppose $\pi : A \to B$ is an algebra isomorphism between two finite-dimensional C^*-algebras. Suppose σ and τ are faithful traces on A and B, respectively, such that $\sigma = \tau \circ \pi$. Consider the polar decomposition $\pi = \omega\alpha$ of π regarded as an operator between the Hilbert spaces A and B (where $\|a\| = (\sigma(a^*a))^{\frac{1}{2}}$ and $\|b\| = (\tau(b^*b))^{\frac{1}{2}}$).*

Then (the unitary operator) ω defines a $*$-isomorphism of the C^*-algebra A onto the C^*-algebra B (and hence also α defines an algebra automorphism of A).

Proof: Case 1: $A = B \sim M_n(\mathbb{C})$

In this case, π is an algebra automorphism of A and the Skolem-Noether theorem guarantees the existence of an invertible element $x \in A$ such that $\pi(a) = xax^{-1}\ \forall a \in A$. Also σ and τ must be multiples of the normal trace on the matrix algebra, and the comptibility condition $\sigma = \tau \circ \pi$ forces them to be equal. Now, if $x = u|x|$ is the polar decomposition of x, with u unitary and $|x| \geq 0$, it is fairly easy to see, using the faithfulness of the trace, that $\pi^*(b) = x^*bx^{*-1}$, $\omega(a) = uau^{-1}$ and $\alpha(a) = |x|a|x|^{-1}\ \forall a \in A$; and the desired conclusions follow.

Case 2: $A \sim M_n(\mathbb{C})$.

In this case, $Z(A) = \mathbb{C}1_A$, and it follows that also $Z(B) = \pi(Z(A)) = \mathbb{C}1_B$. So B is a finite factor of dimension n^2 and hence there exists a $*$-isomorphism - say γ - of B onto A; and in view of the uniqueness, up to scaling, of the trace on a finite factor, we see that γ is also 'trace-preserving', and consequently 'unitary' when viewed as an operator between Hilbert spaces. Note then that $\gamma \circ \pi = (\gamma \circ \omega) \circ \alpha$ is *the* polar decomposition of the algebra isomorphism $\gamma \circ \pi$ of A onto itself. It follows from Case 1 that $\gamma \circ \omega$ is a $*$-automorphism of A, and hence also $\omega = \gamma^{-1} \circ (\gamma \circ \omega)$ is a $*$-isomorphism (being a composite of such maps).

Case 3: A arbitrary (but still finite-dimensional)

Notice first that if C is any finite-dimensional C^*-algebra, and if $p \in Z(C)$, then $p = p^2$ if and only if $p = p^2 = p^*$. Hence, e is a minimal central projection of A if and only if $f = \pi(e)$ is a minimal central projection of B. Therefore if $\{e_i : 1 \leq i \leq n\}$ is the set of all minimal projections of A, then $\{f_i = \pi(e_i) : 1 \leq i \leq n\}$ is the set of all minimal projections of B. Then notice that $A = \oplus Ae_i$ and $B = \oplus Bf_i$ while π restricts to a trace-preserving (algebra-)isomorphism - call it π_i - of the factor Ae_i onto Bf_i. Since the e_i are central and σ is a trace, we see that $A = \oplus Ae_i$ is an orthogonal decomposition of the Hilbert space A; and similarly, $B = \oplus Bf_i$ is also an orthogonal decomposition. Hence the polar decomposition also factors as a direct sum

$$\omega\alpha = \pi = \oplus_i \pi_i = \oplus_i \omega_i \alpha_i$$

so that

$$\omega = \oplus_i \omega_i \ , \ \alpha = \oplus_i \alpha_i \ ;$$

and since we may deduce from Case 2 that each ω_i is a $*$-isomorphism, we conclude finally that $\omega = \oplus_i \omega_i$ is a $*$-isomorphism. □

COROLLARY 5 *If two subfactor planar algebras P and Q are isomorphic, then $\chi^P_{S,T} = \chi^Q_{S,T}$ for every pair S, T of planar tangles of the 'same type'.*

Proof: Suppose that the linear maps $\pi_k : P_k \to Q_k$ for $k \in Col$ implement a planar algebra isomorphism of P onto Q. Choose, by Theorem 3, maps $\omega_k : P_k \to Q_k$ that are unitary and implement a planar algebra isomorphism.

Fix any pair S and T of tangles each of which has an external disc of colour k_0 and b internal discs of colours $k_1, \cdots, k_b$. Since the ω_k's implement a planar algebra isomorphism, we have that for $X \in \{S, T\}$,

$$\omega_{k_0} \circ Z^P_X = Z^Q_X \circ \left(\otimes_{i=1}^b \omega_{k_i} \right) \ .$$

But now the unitarity of the ω_k is easily seen to imply that

$$\chi^Q_{S,T} = Tr(Z^Q_S (Z^Q_T)^*) = Tr(\omega_{k_0} Z^P_S (Z^P_T)^* \omega^*_{k_0}) = Tr(Z^P_S (Z^P_T)^*) = \chi^P_{S,T} \ ,$$

and thus P and Q have the same character. □

3 The relevant orbit space

We shall assume, throughout this section, that we have been given the following data:

a countable set Col of 'colours';
a collection $\{P_k : k \in Col\}$ of finite-dimensional Hilbert spaces;
a countable set $\mathcal{T}$; and[4]
an assignment

$$\mathcal{T} \ni T \mapsto \ (b_0(T), b_1(T), r^T, s^T, D(T), R(T)) \ ,$$

where
$b_0(T), b_1(T) \in \mathbb{Z}_+$;
$s^T : [b_0(T)] \to \ Col$ and $r^T : [b_1(T)] \to \ Col$; and we write s_i^T, r_j^T rather than $s^T(i), r^T(j)$;
$D(T) = \otimes_{i=1}^{b_0(T)} P_{s_i^T}$ and $R(T) = \otimes_{j=1}^{b_1(T)} P_{r_j^T}$.

We shall be concerned with the space

$$X = \prod_{T \in \mathcal{T}} Hom_{\mathbb{C}}(D(T), R(T)) \ , \tag{3.6}$$

the (visibly compact) group

$$G = \prod_{k \in \ Col} U(P_k) \ , \tag{3.7}$$

- where of course $U(H)$ denotes the unitary group of the Hilbert space H - and the action of G on X by[5]

$$(g \cdot x)_T \ = \ \left(\otimes_{j=1}^{b_1(T)} g_{r_j^T}\right) \circ x_T \circ \left(\otimes_{i=1}^{b_0(T)} g_{s_i^T}^{-1}\right) \tag{3.8}$$

The definition of the product topology and the assumed countability of $\mathcal{T}$ and finite-dimensionality of the P_k's show that the G-action on X is continuous and that X is a metrisable space. We shall write $C(X)$ to denote the vector space of all continuous complex-valued functions on X. (Note that X is not compact, so this is NOT naturally a normed space.)

LEMMA 6 *The fixed point algebra* $C(X)^G = \{f \in C(X) : f(g \cdot x) = f(x) \forall g \in G, x \in X\}$ *separates distinct* G*-orbits in* X.

[4] In the set-up we shall describe in this section, the symbol $\mathcal{T}$ could be thought of as the (countable) set of all coloured planar tangles, and it might help to think of the letters s, r, D, R as being suggestive of the words 'source', 'range', 'domain' and 'range' respectively. In the special case of a tangle, we would have $b_1 = 1$; but we could conceivably even have more complicated situations where neither b_i is 1, and there is no reason to rule this out.

[5] We denote typical elements of G and X by $g = (g_k)$ and $x = (x_T)$, respectively.

Proof: If $x \in X$, let us write $[x] = \{g \cdot x : g \in G\}$ to denote its G-orbit. Similarly, if $f \in C(X)$, let us write $[f]$ for its 'average over G', given by integrating with respect to (normalised) Haar measure on the compact group, thus:

$$[f](x) = \int_G f(g \cdot x) dg.$$

Clearly $[f] \in C(X)^G$.

Suppose $x_1, x_2 \in X$ and $[x_1] \neq [x_2]$. Then the $[x_i]$ are disjoint compact sets in the metrisable space X, so we can find an $f \in C(X)$ such that

$$f(x) = \begin{cases} 1 & if \; x \in [x_1] \\ 2 & if \; x \in [x_2] \end{cases}$$

It is then clear that $[f]$ is an element of $C(X)^G$ which takes the value i on $[x_i]$, thereby proving the lemma. □

Note that X is a product of $\mathbb{C}^n$'s, and so it makes sense to talk of 'polynomial functions on X'; we give a name to these functions in the following definition.

DEFINITION 7 *(a) For fixed $T \in \mathcal{T}, \xi \in D(T), \eta \in R(T)$, define the obviously continuous function $X_{T,\xi,\eta}$ on X by*

$$X_{T,\xi,\eta}(x) \; = \; \langle x_T \xi, \eta \rangle \; .$$

*(b) Let A be the unital *-subalgebra of $C(X)$ generated by $\{X_{T,\xi,\eta} : T \in \mathcal{T}, \xi \in D(T), \eta \in R(T)\}$.*

Note that A consists of precisely those functions of the form $p \circ \pi$, where π is the projection of X onto some finite set of factors, and p is a 'polynomial function' on that finite-dimensional quotient of X. Anticipating a definition to come later, we will denote the adjoint of the element $X_{T,\xi,\eta}$ by $X_{T^\dagger,\eta,\xi}$. Hence $X_{T^\dagger,\eta,\xi}(x) = \overline{X_{T,\xi,\eta}(x)} = \overline{\langle x_T \xi, \eta \rangle} = \langle x_T^* \eta, \xi \rangle$.

LEMMA 8 *(a) A is stable under the G-action on $C(X)$;*
(b) $f \in A \Rightarrow [f] \in A$; and
(c) $A^G = A \cap C(X)^G$ separates distinct G-orbits in X.

Proof: Note, to start with, that for any $T \in \mathcal{T}, \xi \in D(T), \eta \in R(T)$, we have

$$\begin{aligned} X_{T,\xi,\eta}(g^{-1} \cdot x) &= \langle (g^{-1} \cdot x)_T \xi, \eta \rangle \\ &= \langle \left(\otimes_{j=1}^{b_1(T)} g^{-1}_{r^T_j} \right) \circ x_T \circ \left(\otimes_{i=1}^{b_0(T)} g_{s^T_i} \right) \xi, \eta \rangle \\ &= \langle x_T (g \cdot \xi), (g \cdot \eta) \rangle \ , \end{aligned}$$

where we write $g \cdot \xi = \left(\otimes_{i=1}^{b_0(T)} g_{s^T_i} \right) \xi$ and $g \cdot \eta = \left(\otimes_{j=1}^{b_0(T)} g_{r^T_j} \right) \eta$. Clearly, $g \cdot \xi \in D(T), \ g \cdot \eta \in R(T)$, so this shows that our set of generators of A is stable under the G-action, and establishes (a). Also,

$$\begin{aligned} [X_{T,\xi,\eta}](x) &= \int_G X_{T,\xi,\eta}(g^{-1} \cdot x) dg \\ &= \int_G \langle x_T (g \cdot \xi), (g \cdot \eta) \rangle dg \\ &= \int_G \sum_{k,l} \langle (g \cdot \xi), \xi_k \rangle \langle \eta_l, (g \cdot \eta) \rangle \langle x_T \xi_k, \eta_l \rangle dg \\ &= \sum_{k,l} \left(\int_G \langle (g \cdot \xi), \xi_k \rangle \langle \eta_l, (g \cdot \eta) \rangle dg \right) X_{T,\xi_k,\eta_l}(x) \ , \end{aligned}$$

where $\{\xi_k\}_k$ and $\{\eta_l\}_l$ are orthonormal bases for $D(T)$ and $R(T)$ respectively, and of course we use the notation $f \mapsto [f]$, as in the proof of Lemma 6, for the process of averaging a function over G. We have thus shown that $[X_{T,\xi,\eta}] \in A$. A similar proof shows that also 'averages of monomials of degree more than one' are 'polynomials' and establishes (b).

(c) First observe that A separates points of X. (If $x \neq y$, pick $T \in \mathcal{T}$ such that $x_T \neq y_T$; then there must exist $\xi \in D(T), \eta \in R(T)$ such that $X_{T,\xi,\eta}(x) \neq X_{T,\xi,\eta}(y)$.) Now, suppose $x^0, x^1 \in X$ and $[x^0] \neq [x^1]$. Then the set $K = [x^0] \cup [x^1]$ is a compact metrisable space, and the function $f = 1_{[x^1]}$ is a continuous function on K. On the other hand, the collection $A_K = \{p|_K : p \in A\}$ is a unital *-subalgebra of $C(K)$ which separates points and, by the Stone-Weierstrass theorem, is consequently uniformly dense in $C(K)$. In particular, we may find a $p \in A$ such that $|p(x) - f(x)| < \frac{1}{2}$ for all $x \in K$; and we find that, for $i = 0, 1$, we have

$$\begin{aligned} |[p](x^i) - i| &= |[p](x^i) - f(x^i)| \\ &\leq \int |p(g \cdot x^i) - f(g \cdot x^i)| dg \end{aligned}$$

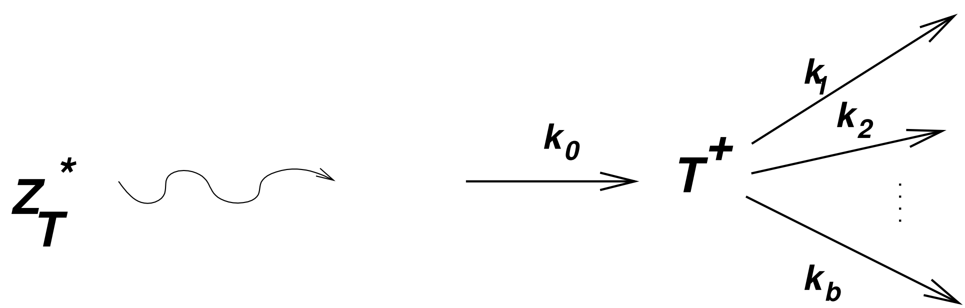
$$< \frac{1}{2} ;$$

so that

$$[p](x^0) < \frac{1}{2} < [p](x^1) \ .$$

In particular, we have found the element $[p]$ of A^G which separates the orbits $[x_0]$ and $[x_1]$. □

REMARK 9 *The reader cannot but notice an inherent asymmetry in the notion of a planar tangle; any number of 'inputs' is permitted, but there is only one 'output'. We shall partially redress this state of affairs by introducing the 'dagger' of a tangle via the following prescription:*



Thus, if the number b of internal discs of a tangle T is greater than 1, then $T^\dagger$ is not a tangle, but merely a convenient device which admits one input and several outputs; thus, in the above notation, we have

$$Z_{T^\dagger} = Z_T^* : P_{k_0} \to \otimes_{i=1}^b P_{k_i}$$

Although our subequent work with planar algebras will only need the case where $\mathcal{T}$ is precisely the class of all planar tangles, we continue to work in the more general situation of an abstract $\mathcal{T}$ in this section. We make precise the notion of the 'dagger of a member of $\mathcal{T}$' (in our possibly more general setting).

DEFINITION 10 *Define a set* $\mathcal{T}^\dagger = \{T^\dagger : T \in \mathcal{T}\}$, *and define*

$$b_0(T^\dagger) = b_1(T),\ b_1(T^\dagger) = b_0(T),\ r_i^{T^\dagger} = s_i^T,\ s_j^{T^\dagger} = r_j^T,\ D(T^\dagger) = R(T),\ R(T^\dagger) = D(T).$$

We will now proceed by formalising the notion of a closed picture and that of its associated picture invariant.

DEFINITION 11 *By a* **closed picture** *we shall mean a finite directed graph where:*

(a) each vertex v is labelled by an element $S(v) \in \mathcal{T} \cup \mathcal{T}^\dagger$ and each edge e by a colour $k(e)$;

(b) each vertex labelled by an $S \in \mathcal{T} \cup \mathcal{T}^\dagger$ comes equipped with a labelling of its in-arrows (resp., out-arrows) by $[b_0(S)]$ (resp., $[b_1(S)]$); we write $e_v(i)$ (resp., $f_v(j)$) to denote the i^{th} in-arrow (resp., the j^{th} out-arrow) at the vertex v; and

(c) the two labellings are compatible, meaning: at each vertex v, $k(e_v(i)) = s_i^{S(v)}$ and $k(f_v(j)) = r_j^{S(v)}$.

We wish to associate to a closed picture an element of the algebra A^G. This is done by a state sum approach as follows. Fix once and for all an orthonormal basis B_k for each P_k, $k \in Col$. By a state κ of a closed picture we mean an assignment of a basis vector $\kappa(e) \in B_{k(e)}$ to each edge e.

A state κ of a closed picture $\mathcal{P}$ determines an element of A, denoted $\mathcal{I}_\mathcal{P}(\kappa)$, thus: for each vertex v, define $\xi_v(\kappa) = \otimes_{l=1}^{b_0(S(v))}\kappa(e_v(l)) \in D(S(v))$ and $\eta_v(\kappa) = \otimes_{m=1}^{b_1(S(v))}\kappa(f_v(m)) \in R(S(v))$; and set $\mathcal{I}_\mathcal{P}(\kappa) = \prod_v X_{S(v),\xi_v(\kappa),\eta_v(\kappa)}$.

Finally, given a closed picture $\mathcal{P}$ we define its associated picture invariant, denoted $\mathcal{I}_\mathcal{P}$, by the equation:

$$\mathcal{I}_\mathcal{P} = \sum_\kappa \mathcal{I}_\mathcal{P}(\kappa) \tag{3.9}$$

where the sum ranges over all states of $\mathcal{P}$.

A reformulation of this definition of picture invariants will turn out to be useful; we will need a couple of definitions.

DEFINITION 12 *Let $\mathcal{F}$ denote the set of all functions $f : [N_f] \to \mathcal{T} \cup \mathcal{T}^\dagger$, where N_f is some positive integer depending on f. For each $f \in \mathcal{F}$, we define Hilbert spaces $D(f)$ and $R(f)$*

$$D(f) = \otimes_{i=1}^{N_f} \otimes_{l_i=1}^{b_0(f(i))} P_{s_{l_i}^{f(i)}}, R(f) = \otimes_{i=1}^{N_f} \otimes_{m_i=1}^{b_1(f(i))} P_{r_{m_i}^{f(i)}} ;$$

finally, for every $x \in X$, we define $Z_{f,x} : D(f) \to R(f)$ by $Z_{f,x} = \otimes_{i=1}^{N_f} x_{f(i)}$ - where $x_{T^\dagger}$ is understood to mean x_T^*.

DEFINITION 13 *Suppose that $f \in \mathcal{F}$ and suppose there exists a bijection*

$$\sigma : \coprod_{i=1}^{N_f} (\{i\} \times [b_0(f(i))]) \to \coprod_{j=1}^{N_f} (\{j\} \times [b_1(f(j))])$$

- where we use the symbol $\coprod$ to signify disjoint union - which is 'colour-preserving' in the sense that

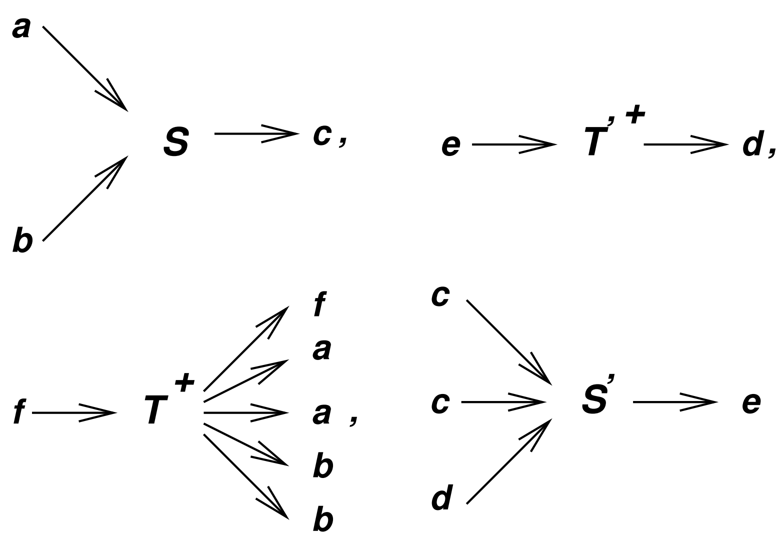
$$\sigma(i,l) = (j,m) \ \Rightarrow \ s_l^{f(i)} = r_m^{f(j)} \ .$$

Then let $V_\sigma : D(f) \to R(f)$ be the (obviously unitary) operator defined by

$$V_\sigma \left(\otimes_{i=1}^{N_f} \otimes_{l_i=1}^{b_0(f(i))} \xi_{(i,l_i)} \right) = \otimes_{j=1}^{N_f} \otimes_{m_j=1}^{b_1(f(j))} \xi_{\sigma^{-1}(j,m_j)} \ .$$

Such a pair (f, σ) can be used to construct a closed picture as follows. Take N_f vertices $v_1, \cdots, v_{N_f}$ and label v_i by $S(v_i) = f(i)$. For each $(i,l) \in \coprod_{i=1}^{N_f} (\{i\} \times [b_0(f(i))])$ put in an edge from v_j to v_i where $\sigma(i,l) = (j,m)$ and number this as the l^{th} in-arrow of v_i and as the m^{th} out-arrow of v_j. Colour this edge $s_l^{f(i)} = r_m^{f(j)}$. This gives a closed picture which we denote by $\mathcal{P}(f,\sigma)$. It should be fairly clear that by renumbering the vertices, we can get different (f,σ)'s representing the same closed picture, and that further, this renumbering is the only ambiguity in describing a closed picture by an (f,σ).

REMARK 14 *In order to demystify the surfeit of symbols in the discussion above, and to explicitly explain how the pair (f,σ) indeed leads to a* closed picture*, an example might help. So suppose the element f is given by requiring that $N_f = 5$ and that $f(1) = S, f(2) = T'^\dagger, f(3) = T^\dagger, f(4) = S', f(5) = S$, with the colours* *of the various inputs and outputs of these tangles given thus:*



We have used our convention for numbering the inputs of a tangle and the outputs of the dagger of a tangle: thus, for example, in our notation, we would have

$$s_1^S = a, s_2^S = b, r_1^S = c, r_1^{T^\dagger} = f = s_1^T,\ etc.$$

For f as above, let σ be the 'colour-preserving' bijection from the set of inputs of $S; T'^\dagger; T^\dagger; S'; S$ respectively, - labelled by the set

$$(1,1),(1,2);(2,1);(3,1);(4,1),(4,2),(4,3);(5,1),(5,2) -$$

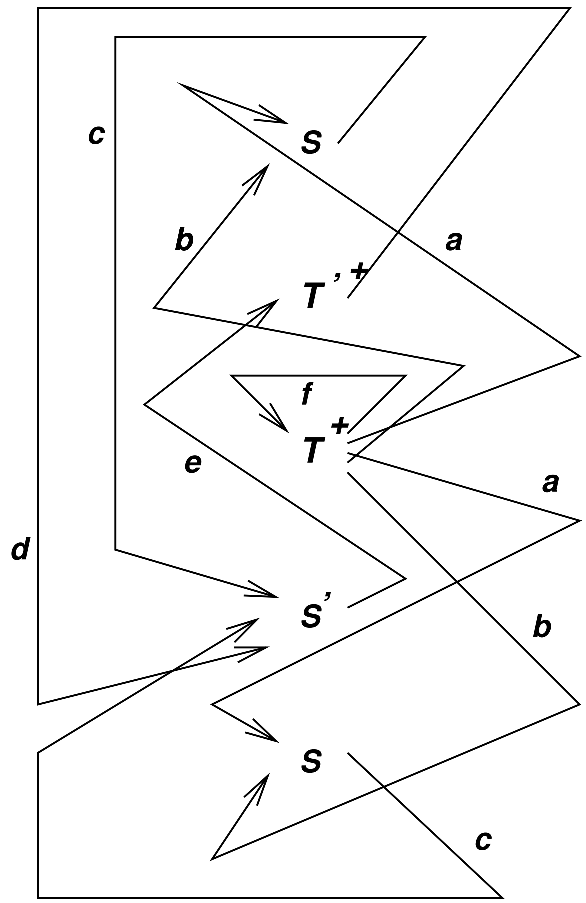
to the set of their outputs - labelled by

$$(1,1);(2,1);(3,1),(3,2),(3,3),(3,4),(3,5);(4,1);(5,1) -$$

given by the following table:

(i,l)	*(1,1)*	*(1,2)*	*(2,1)*	*(3,1)*	*(4,1)*	*(4,2)*	*(4,3)*	*(5,1)*	*(5,2)*
σ*(i,l)*	*(3,2)*	*(3,4)*	*(4,1)*	*(3,1)*	*(1,1)*	*(5,1)*	*(2,1)*	*(3,3)*	*(3,5)*

To see that things are indeed as stated, one only needs to look at the following *'picture':*



- where the five tangles and their daggers are drawn in a vertical row, and the l-th input of the i-th tangle (or dagger of a tangle) is connected to the m-th output of the j-th tangle (or dagger of a tangle) if $\sigma(i,l) = (j,m)$.

The relation between the two descriptions of closed pictures is clarified in our next proposition.

PROPOSITION 15 *Suppose that the pair (f,σ) is as in Definition 13. Then for any $x \in X$, we have $\mathcal{I}_{\mathcal{P}(f,\sigma)}(x) = Tr\,(V_\sigma^* Z_{f,x})$.*

Proof: By definition, the set $B(f) = \{\xi = \otimes_{i=1}^{N_f} \otimes_{l_i=1}^{b_0(f(i))} \xi_{(i,l_i)} : \xi_{(i,l_i)} \in B_{s_{l_i}^{f(i)}}\}$ is an orthonormal basis for $D(f)$. The equation

$$\kappa_\xi(e_{v_i}(l_i)) = \xi_{(i,l_i)} \tag{3.10}$$

is seen, once it has been unravelled, to define a bijection $\xi \leftrightarrow \kappa_\xi$ between $B(f)$ and the set of states of $\mathcal{P}(f,\sigma)$. This bijection is also seen to satisfy the equation

$$\kappa_\xi(f_{v_j}(m_j)) = \xi_{\sigma^{-1}(j,m_j)}. \tag{3.11}$$

Now note that

$$\begin{aligned}
Tr\left(V_\sigma^* Z_{f,x}\right) &= \sum_{\xi \in B(f)} \langle Z_{f,x}(\xi), V_\sigma(\xi)\rangle \\
&= \sum_{\xi \in B(f)} \prod_{i=1}^{N_f} \langle x_{f(i)}(\otimes_{l_i=1}^{b_0(f(i))} \xi_{(i,l_i)}), \otimes_{m_i=1}^{b_1(f(i))} \xi_{\sigma^{-1}(i,m_i)}\rangle \\
&= \sum_{\xi \in B(f)} \prod_{i=1}^{N_f} \langle x_{f(i)}(\otimes_{l_i=1}^{b_0(f(i))} \kappa_\xi(e_{v_i}(l_i)), \otimes_{m_i=1}^{b_1(f(i))} \kappa_\xi(f_{v_i}(m_i)\rangle \\
&= \sum_\kappa \prod_v X_{S(v),\xi_v(\kappa),\eta_v(\kappa)}(x) \\
&= \sum_\kappa \mathcal{I}_{\mathcal{P}(\sigma,f)}(\kappa_\xi)(x) \\
&= \mathcal{I}_{\mathcal{P}(f,\sigma)}(x)\ ,
\end{aligned}$$

as desired. □

Using the representations π_f^D and π_f^R of G on $D(f)$ and $R(f)$ defined, for each $f \in \mathcal{F}$, by

$$\begin{aligned}
\pi_f^D(g)\left(\otimes_{i=1}^{N_f} \otimes_{l_i=1}^{b_0(f(i))} \xi_{l_i}\right) &= \otimes_{i=1}^{N_f} \otimes_{l_i=1}^{b_0(f(i))} \left(g_{s_{l_i}^{f(i)}} \cdot \xi_{l_i}\right) \\
\pi_f^R(g)\left(\otimes_{j=1}^{N_f} \otimes_{m_j=1}^{b_1(f(j))} \eta_{m_j}\right) &= \otimes_{j=1}^{N_f} \otimes_{m_j=1}^{b_1(f(j))} \left(g_{r_{m_j}^{f(j)}} \cdot \eta_{m_j}\right)\ ,
\end{aligned}$$

we can now see - in the following Lemma, whose proof is a consequence of the definitions - that our picture invariants are indeed invariant.

LEMMA 16 *If $f \in \mathcal{F}$ and if σ is related to f as in Definition 13, then, with the notation as above and as in Definition 12, we have, for all $x \in X$, and $g \in G$,*
(a) $\pi_f^R(g) \circ Z_{f,x} \ = \ Z_{f,g\cdot x} \circ \pi_f^D(g)$;
(b) $\pi_f^R(g) \circ V_\sigma \ = \ V_\sigma \circ \pi_f^D(g)$; and
(c) $\mathcal{I}_{\mathcal{P}(f,\sigma)} \in A^G$.

□

DEFINITION 17 *Define the additive unital semigroup*

$$\Lambda = \{\lambda : \mathcal{T} \cup \mathcal{T}^\dagger \to \mathbb{Z}_+ : \lambda \text{ is finitely supported}\} ,$$

and for each $\lambda \in \Lambda$, define A_λ to be the subspace of A spanned by monomials of 'weight' λ *by which we mean a monomial of the form*

$$\prod_{T \in \mathcal{T} \cup \mathcal{T}^\dagger} \left(\prod_{i=1}^{\lambda(T)} X_{T,\xi_T^i,\eta_T^i} \right) ,$$

where $\xi_T^i \in D(T), \eta_T^i \in R(T)$.

LEMMA 18 *With the foregoing notation, we have:*
(a) the algebra A is Λ-graded (by the A_λ's);
(b) the G-action on A leaves each A_λ invariant;

Proof: (a) It follows from the definition that if f (resp., g) is one of the 'generating monomials' for A_λ (resp., A_μ), then fg is one of the generating monomials for $A_{\lambda+\mu}$; the desired conclusions follows from the fact that the 'generating monomials' linearly span the A_ν's.

(b) is clear. □

LEMMA 19 *If V is a finite-dimensional Hilbert space, then*

$$\left(V^{\otimes^k} \otimes V^{*\otimes^l}\right)^{U(V)} = \begin{cases} 0 & if\ k \neq l \\ \pi(\mathbb{C}\Sigma_k) & if\ k = l \end{cases} ;$$

where we have made the identifications

$$V^{\otimes^k} \otimes V^{*\otimes^k} = (V \otimes V^*)^{\otimes^k} = (End_{\mathbb{C}}(V))^{\otimes^k} = End_{\mathbb{C}}(V^{\otimes^k}) ,$$

and π is the obvious permutation representation of the group algebra of the symmetric group Σ_k on $V^{\otimes^k}$. Explicitly, the vector $\sum_{i_1,\cdots,i_k}(e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_k)}) \otimes (e^{i_1} \otimes \cdots \otimes e^{i_k})$ corresponds under this isomorphism to $\pi(\sigma)$, where $\{e_i\}_i$ is an(y) orthonormal basis for V, and $\{e^i\}_i$ denotes the dual (orthonormal) basis for V^*.

Proof: This result must be well-known, and documented somewhere in the literature. (It can also be deduced ([P]) from the Zariski density of $U(V)$ in $GL(V)$.) We provide a direct proof here, for the sake of completeness.

To start with, note that if $g = \omega \in \mathbb{T}$ is a complex scalar of unit modulus, then g acts as ω^{k-l} on $\left(V^{\otimes^k} \otimes V^{*\otimes^l}\right)$ and consequently the assertion regarding the case $k \neq l$ is seen to follow.

So assume $k = l$. In view of the identifications spelt out explicitly in the statement of the lemma, we need to show that if $C : GL(V) \to End(V^{\otimes^k})$ is the representation defined by $C(g)(T) = (\otimes^k g)T(\otimes^k g)^{-1}$, then

$$C(g)(T) = T \forall g \in U(V) \Leftrightarrow T \in \pi(\mathbb{C}\Sigma_k) .$$

In view of the Schur-Weyl theorem, it suffices to show that

$$C(g)(T) = T \forall g \in U(V) \Leftrightarrow C(g)(T) = T \forall g \in GL(V) ,$$

and this is what we shall do. For this, we only need to show that if T is an operator on $\left(V^{\otimes^k}\right)$ which commutes with operators of the form $\otimes^k U$ for arbitrary $U \in U(V)$, then T necessarily also commutes with operators of the form $\otimes^k S$ for all $S \in GL(V)$. Indeed, suppose

$$T(\otimes^k e^{itA}) = (\otimes^k e^{itA})T$$

for every self-adjoint operator A on V. Differentiation yields

$$T\ D(A) = \ D(A)T , \tag{3.12}$$

where $D(A) = A \otimes id_V \cdots \otimes id_V + \cdots + id_V \otimes \cdots \otimes id_V \otimes A$. Since the mapping $D :\ End(V) \to\ End\ V^{\otimes^k}$ is $\mathbb{C}$-linear, since the self-adjoint operator A was arbitrary, and since any operator is a $\mathbb{C}$-linear combination of self-adjoint operators, we find that equation (3.12) holds for every $A \in End(V)$. Exponentiating once again, we find that

$$Te^{D(A)} = e^{D(A)}T\ \forall A ,$$

and the lemma is proved. (The final statement of the lemma is verified by chasing through the string of identifications referred to.) □

PROPOSITION 20 *The space A^G_λ is linearly spanned by the set of all those picture invariants $\mathcal{I}_{\mathcal{P}(f,\sigma)}$ for which $f \in \mathcal{F}$ satisfies $\lambda(\cdot) = |f^{-1}(\cdot)|$ (and of course σ is related to f as in Definition 13).*

Proof: We prove the easier implication first. Fix an $f \in \mathcal{F}$ and a σ related to f as in Definition 13. By definition $\lambda(T)$ is the number of vertices in $\mathcal{P} = \mathcal{P}(f, \sigma)$ which are labelled T while $\lambda(T^\dagger)$ is the number of vertices labelled $T^\dagger$. It now follows from equation (3.9) and the definition of $\mathcal{I}_\mathcal{P}(\kappa)$ that $\mathcal{I}_\mathcal{P} \in A_\lambda$. Now Lemma 18(b) and Lemma 16(c) show that $\mathcal{I}_\mathcal{P} \in A_\lambda^G$.

To prove the other implication, take any λ and fix any one $f \in \mathcal{F}$ with the property that $|f^{-1}(T)| = \lambda(T)$ for all $T \in \mathcal{T} \cup \mathcal{T}^\dagger$. Consider the map from $V(f) = (\otimes_{i=1}^{N_f} \otimes_{l_i=1}^{b_0(f(i))} P_{s^{f(i)}_{l_i}}) \otimes (\otimes_{j=1}^{N_f} \otimes_{m_j=1}^{b_1(f(j))} P^*_{r^{f(j)}_{m_j}})$ to A_λ defined by

$$(\otimes_{i=1}^{N_f} \otimes_{l_i=1}^{b_0(f(i))} \xi_{(i,l_i)}) \otimes (\otimes_{j=1}^{N_f} \otimes_{m_j=1}^{b_1(f(j))} \langle \cdot, \eta_{(j,m_j)} \rangle) \stackrel{\Theta}{\mapsto} \prod_{i=1}^{N_f} X_{f(i), \xi^i_{f(i)}, \eta^i_{f(i)}} \, , \quad (3.13)$$

where $\xi^i_{f(i)} = \otimes_{l_i=1}^{b_0(f(i))} \xi_{(i,l_i)})$ and $\eta^i_{f(i)} = \otimes_{m_i=1}^{b_1(f(i))} \eta_{(i,m_i)}$. It is not hard to see that this is an epimorphism of G-modules, where the G-action on $V(f)$ is the natural one (factoring through appropriate $U(P_k)$'s).

For $k \in Col$, write $V_k(f)$ for the 'sub-product' of $V(f)$ obtained by taking only those (i, l_i) for which $s^{f(i)}_{l_i} = k$ and those (j, m_j) for which $r^{f(j)}_{m_j} = k$. Then clearly $V(f) \cong \otimes_{k \in Col} V_k(f)$. (Since f is a finitely supported function, this is actually a finite tensor product.) Further, from the product-nature of the action, it is seen that $V(f)^G \cong \otimes_{k \in Col} V_k(f)^{U(P_k)}$. On the other hand, Lemma 19 implies that a spanning set of $V_k(f)^{U(P_k)}$ is determined by the set of bijections between the sets $\{(i, l_i) : s^{f(i)}_{l_i} = k\}$ and $\{(j, m_j) : r^{f(j)}_{m_j} = k\}$; and hence a spanning set for $V(f)^G$ is given by the set of colour-preserving bijections

$$\sigma : \coprod_{i=1}^{N_f} (\{i\} \times [b_0(f(i))]) \rightarrow \coprod_{j=1}^{N_f} (\{j\} \times [b_1(f(j))]) \ ,$$

i.e., the set of $\sigma's$ as in Definition 13. Unravelling the definitions and notations, it is not too hard to see that the invariant of $V(f)$ corresponding to such a σ gets mapped under the map Θ - see equation (3.13) - to our picture invariant $\mathcal{I}_{\mathcal{P}(f,\sigma)}$. $\square$

4 The application to subfactor planar algebras

In this section, the symbol $\mathcal{T}$ will be reserved for elements of the 'coloured operad' of planar tangles - see [J1],[J2],[J3]. Given a 'subfactor planar algebra'

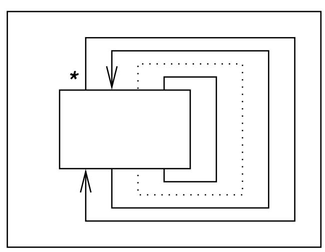
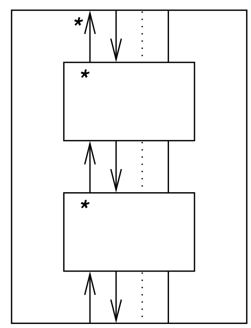
$P = \{P_k\}_{k \in Col}$, we can associate linear operators Z_T^P to tangles T as described in the 'Facts' of the Introduction.

Suppose $Q = \{Q_k\}_{k \in Col}$ is another subfactor planar algebra, which is isomorphic, as a planar algebra, to P. Then we may assume without loss of generality that $Q_k = P_k$, as a Hilbert space, for each $k \in Col$; and if the space X and the group G are as in the last section, then we may associate the point $x^Q \in X$ defined by $(x^Q)_T = Z_T^Q$. Theorem 3 shows that the planar algebras P and Q are isomorphic precisely when the points x^P and x^Q lie in the same G-orbit. Hence it follows from Proposition 20 that P and Q are isomorphic if and only if $\mathcal{I}_{\mathcal{P}(f,\sigma)}(x^P) = \mathcal{I}_{\mathcal{P}(f,\sigma)}(x^Q)$ for all picture invariants $\mathcal{I}_{\mathcal{P}(f,\sigma)}$.

Before proceeding further, we should first mention that henceforth,

$$Col = \{0_+, 0_-, 1, 2, 3, ...\} .$$

It will help if we recall the definition of some tangles that we will be using in the following analysis; for the precise definition of a tangle, see [J1]. (We shall find it convenient to use the notation of [KLS].)

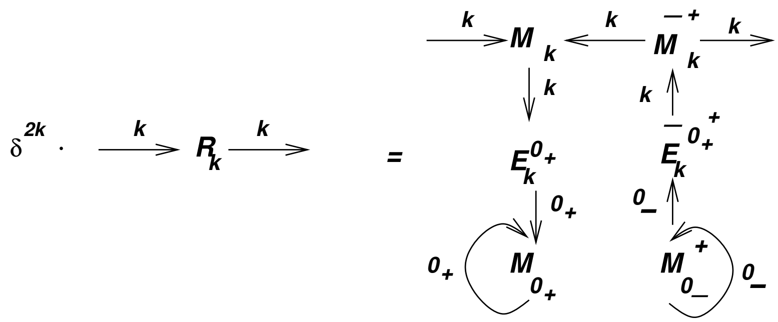
$E_k^{0_+} =$  $M_k =$ 

The above pictures are for $k > 0$ (and have $2k$ strings impinging on the internal boxes); the multiplication tangles $M_{0_\pm}$ are defined as the M_k above, except that they do not have any internal strings, and the region between the external box and the two internal boxes is shaded white, resp., black.

Further, given a tangle, say T, we shall write $\bar{T}$ to denote the tangle obtained by rotating the * of every (external, as well as internal) disc anticlockwise to the next marked point (and reversing the checkerboard shading). Hence for instance, we have $\bar{M}_{0_\pm} = M_{0_\mp}$, while $\bar{R}_k = R_k$ for $k > 1$. Here, the symbol R_k denotes the important rotation k-tangle, which is known - see [J1] - to satisfy $R_k^{(k)} = I_k^k$ (where we write $R_k^{(k)}$ for the 'k-fold iteration of R_k' and I_k^k for the identity tangle with one external and one internal disc both of colour k).

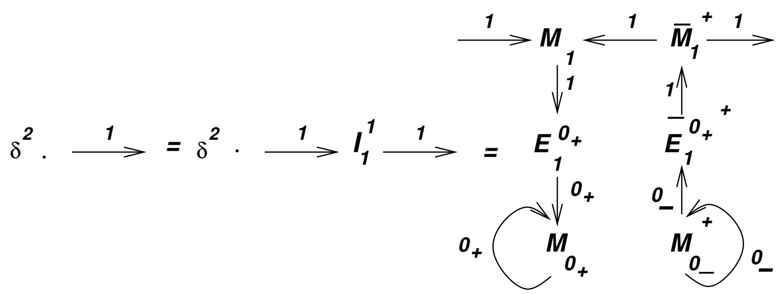
LEMMA 21 *In the pictorial notation of the earlier sections, we have (in any subfactor planar algebra) the following identities:*

(a) $\forall\, k > 1$,



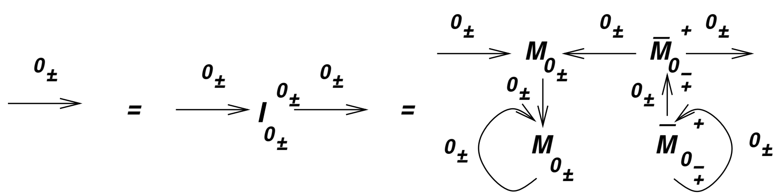
(4.14)

(b)



(4.15)

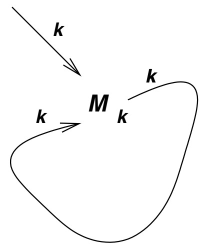
(c)



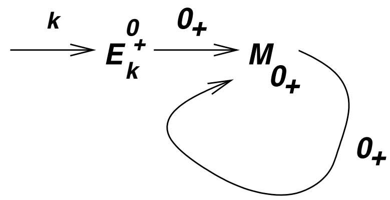
(4.16)

(In (a) and (b) above, we write the expression 'constant . picture' to mean the obvious thing - viz.*, the appropriate constant multiple of the operator represented by the picture.)*

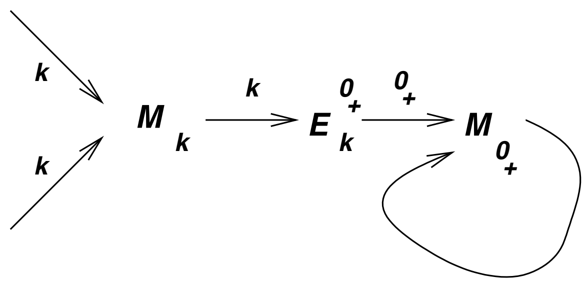
Proof: (a) To start with, it must be noted - see the explanation in [DKS], for instance - that the picture



which has one input coloured k and no outputs, is to be interpreted as the linear functional given by 'trace in the left-regular representation' of the algebra P_k. Consequently, the picture



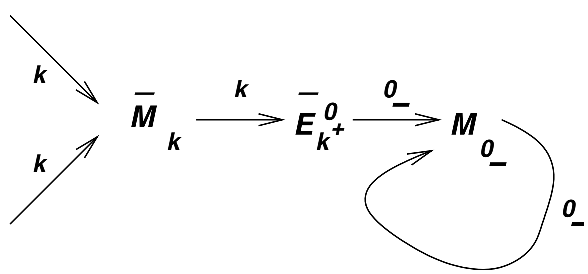
denotes the linear functional $P_k \ni x \to \delta^k \tau_k(x) \in \mathbb{C}$, where τ_k denotes the trace on P_k described in Fact (7) of the introduction. Thus, the following picture



denotes the map

$$P_k \otimes P_k \ni x \otimes y \to \delta^k \tau_k(xy) \in \mathbb{C}$$

Let $T : P_k \otimes P_k \to \mathbb{C}$ denote the map given by the picture



Notice that the adjoint operator T^* is given by the 'dagger' of this picture (i.e., the picture obtained by reversing all the arrows, and replacing all vertex labels by their daggers). Clearly, we have

$$T(\cdot) = \langle \cdot, T^*(1) \rangle$$

We shall imitate the 'Sweedler notation of Hopf algebras' and write

$$T^*(1) = \sum_{(x)} x_1 \otimes x_2 \ .$$

Notice now that both sides of the identity to be proved (i.e., equation 4.14) represent endomorphisms of P_k. Since the left side represents δ^{2k} times the unitary operator Z_{R_k}, and since τ_k is a non-degenerate trace, it will suffice to prove that if the the operator represented by the right side of equation 4.14 is B, then

$$\tau_k(B(Z^*_{R_k}(z))w) = \delta^{2k}\tau_k(zw) \ \forall \ z, w \in P_k.$$

However, the earlier analysis shows that

$$\begin{aligned} \tau_k(B(Z^*_{R_k}(z))w) &= \tau_k\left(\sum_{(x)} \delta^k \tau_k(x_1 Z^*_{R_k}(z))x_2 w\right) \\ &= \delta^k \sum_{(x)} \tau_k(x_1 Z^*_{R_k}(z))\tau_k(x_2 w) \\ &= \delta^k \sum_{(x)} \langle x_1, (Z^*_{R_k}(z))^* \rangle \langle x_2, w^* \rangle \\ &= \delta^k \sum_{(x)} \langle x_1, Z_{R_k}(z^*) \rangle \langle x_2, w^* \rangle \\ &= \delta^k \langle \sum_{(x)} x_1 \otimes x_2, Z_{R_k}(z^*) \otimes w^* \rangle \\ &= \delta^k \langle T^*(1), Z_{R_k}(z^*) \otimes w^* \rangle \\ &= \delta^k \overline{T(Z_{R_k}(z^*) \otimes w^*)} \ , \end{aligned}$$

where we have used (a) the fact that the inner product in P_k and τ_k are related by $\langle u, v \rangle = \tau_k(uv)$, in the third line above, and (b) the 'planar algebras' requirement relating the adjunction in P_k and the action of tangles in the fourth line above, which implies that

$$(Z^*_R(x))^* = Z_R(x^*) \ .$$

Now let S denote the tangle defined by

$$S = \bar{E}_k^{0+} \circ \left(\bar{M}_k \circ_{D_1} R_k \right) \ ;$$

notice that $\tau_{0_-} : P_{0_-} \to \mathbb{C}$ is the canonical identification of P_{0_-} with $\mathbb{C}$; and conclude from the definitions of S and T that $T(Z_{R_k}(z^*) \otimes w^*) = \tau_{0_-}(Z_S(z^* \otimes w^*))$; and hence

$$\begin{aligned} \overline{T(Z_{R_k}(z^*) \otimes w^*)} &= \overline{\tau_{0_-}(Z_S(z^* \otimes w^*))} \\ &= \tau_{0_-}\left((Z_S(z^* \otimes w^*))^*\right) \\ &= \tau_{0_-}\left((Z_{S^*}(z \otimes w))\right) \ , \end{aligned}$$

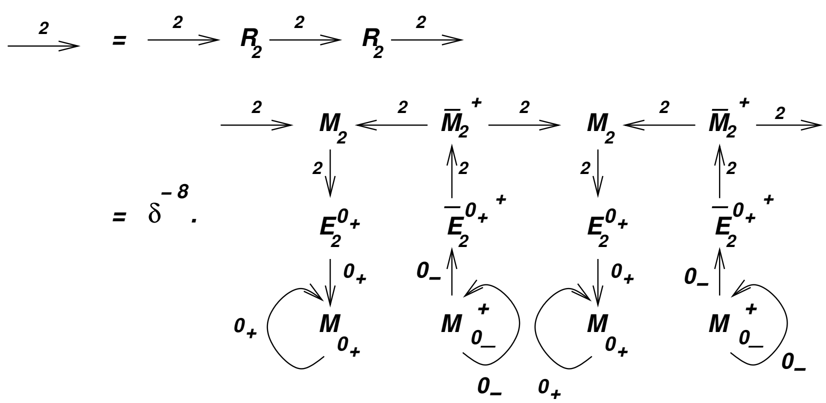
where the tangle S^* is obtained from the tangle S according to a specific procedure (see [KS]); in the case at hand, we find, after staring at a few pictures, and invoking the sphericality present in subfactor planar algebras, that

$$\tau_{0_-}\left((Z_{S^*}(z \otimes w))\right) \ = \ \delta^k \tau_k(zw) \ ,$$

thereby completing the proof of (a).

The proof of (b) and the case of 0_+ in (c) are a verbatim repetition of the above proof of (a), if one adopts the convention that $R_k = I_k^k$ for $k \leq 1$. The only modification one needs for $k = 0_-$ is that every occurrence of E_k^{0+} should also be replaced by I_k^k. □

REMARK 22 *Since $R_k^{(k)}$ - i.e., the result of iterating R_k k times - is the same as the identity tangle, we may, from Lemma 21(a), deduce an identity for I_k^k - which will be thought of as a counterpart for $k > 1$ of the identities Lemma 21 (b),(c); we illustrate the case of $k = 2$ below.*



(4.17)

The point of the foregoing exercise is the following conclusion, which will be a crucial one for us:

If in a closed picture, say $\mathcal{P}$, every arrow of colour k is 'substituted' by the analogue of the above picture with k in place of 2 (resp., the picture given by Lemma 21(b) or (c)) if $k > 1$ (resp., $k = 1$ or $0_\pm$), then the result would be a constant multiple of another picture invariant, say $\mathcal{P}_1$, which would have the crucial property that it contains no directed closed path*, and still agrees at all subfactor planar algebras with index δ.*

PROPOSITION 23 *For any picture invariant $\mathcal{I}_{\mathcal{P}(f,\sigma)}$ there exist a pair S, T of tangles of the same type, and an integer n (depending only on the picture) such that for any subfactor planar algebra P of modulus δ, we have $\mathcal{I}_{\mathcal{P}(f,\sigma)}(x^P) = \delta^n \chi^P_{S,T}$.*

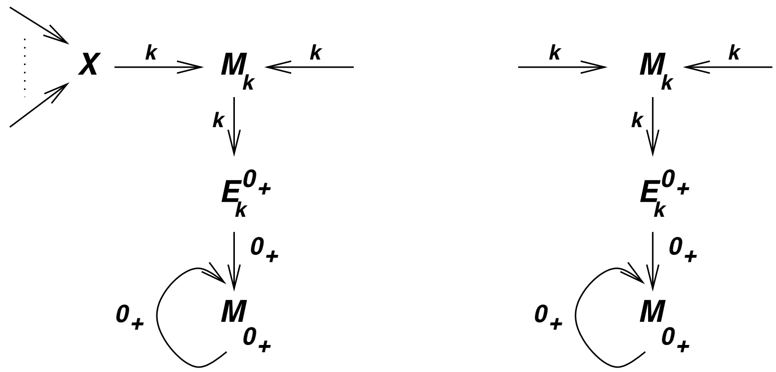
Proof: Begin with any picture invariant $\mathcal{I}_{\mathcal{P}(f,\sigma)}$ and use Remark 22 and Lemma 21 (b) and (c) (for $k > 1$, $k = 1$ and $k = 0_\pm$ respectively) to replace each arrow of colour k in its picture by subpictures that are given by the right hand sides of equations (4.17), (4.15) and (4.16). The resulting picture clearly specifies another picture invariant whose value, at any subfactor planar algebra, agrees - see Remark 22 - with that of $\mathcal{I}_{\mathcal{P}(f,\sigma)}$ at that planar algebra, up to a multiplicative factor of a power of δ.

The new picture is seen to satisfy the following properties:
(a) it contains no directed closed paths except for self loops at $M_{0_\pm}$ and $\overline{M}^\dagger_{0_\pm}$, and
(b) no arrow in it goes from an S to a $T^\dagger$ for any two tangles T and S.

Consider the connected components of the picture obtained from this one by cutting each arrow that goes from a $T^\dagger$ to an S, i.e., by replacing each such arrow by

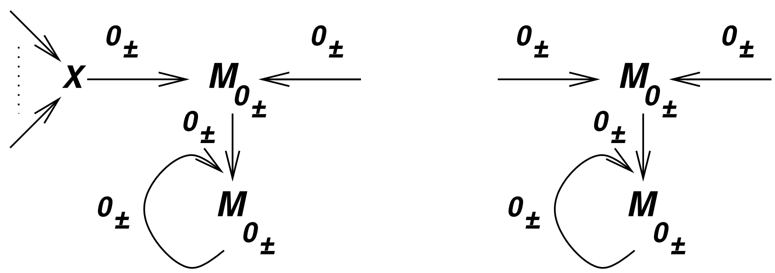
$$T^\dagger \rightarrow \quad \rightarrow S.$$

It follows from (b) above that every such connected component is of one of two types: either (i) all the vertices are labelled by elements of $\mathcal{T}$, or (ii) all the vertices are labelled by elements of $\mathcal{T}^\dagger$. Fix a component of the former type. It is easy to see, since the only arrows that remain are ones we have introduced through our substituting 'subpictures' for the original arrows, that this component is neccessarily of one of the following two forms for some $k \geq 1$:



(4.18)

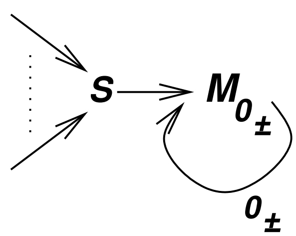
or of one of the following two forms:



(4.19)

where X is some tangle.

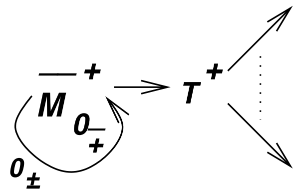
Thus in each case, there is a $0_{\pm}$-tangle S so that this connected component specifies the same function on any planar algebra as does:



(4.20)

In case of the first of the two forms for $k \geq 1$, we take S to be $E_k^{0+} \circ (M_k \circ_{D_2} X)$.

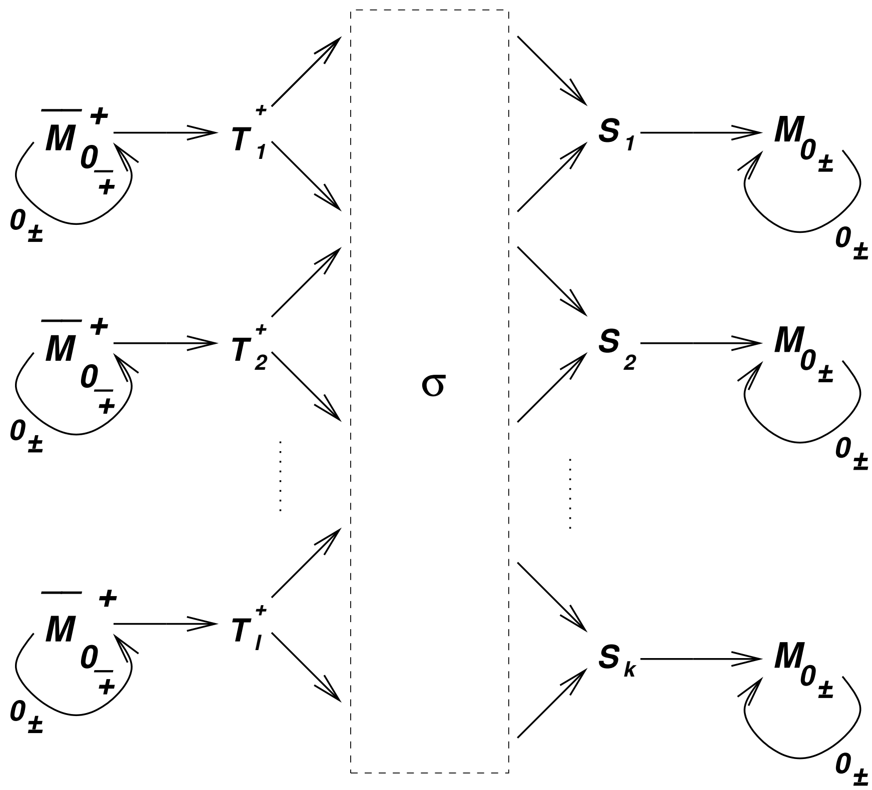
Similarly, any connected component having only elements of $\mathcal{T}^{\dagger}$ can be replaced by a picture of the following form:



(4.21)

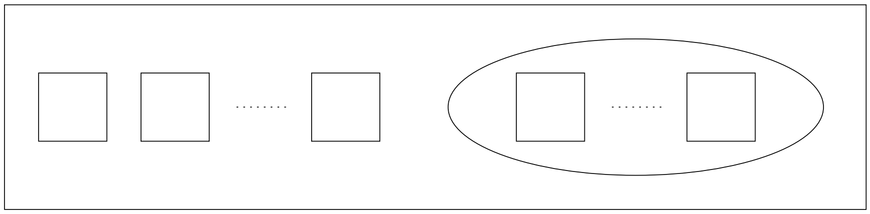
for some $0_\pm$ tangle T to get a picture whose associated picture invariant takes the same value on all subfactor planar algebras as the original one.

So if there are k connected components that involve only S's and l connected components that involve only $T^\dagger$'s then there are tangles $T_1, \cdots, T_l$ and $S_1, \cdots, S_k$ such that $\mathcal{I}_{\mathcal{P}(f,\sigma)}$ agrees with a power of δ times the invariant specified by the following picture:



take the same value at any subfactor planar algebra. Here σ represents a permutation that specifies the way that the outputs of $T_i^\dagger$'s correspond to the inputs of the S_j's. (This comes about when we 'splice' back the edges that we cut.)

It remains to construct S and T. Suppose that $S_1, \cdots, S_u$ are 0_+ tangles while $S_{u+1}, \cdots, S_k$ are 0_- tangles. Consider the 0_+ tangle W in the figure below which



has u internal 0_+ boxes and $k-u$ internal 0_- boxes (enclosed in a loop). Set $S = W \circ_{(D_1,D_2,\cdots,D_k)} (S_1, S_2, \cdots, S_k)$ - which is, by definition, the tangle obtained by substituting the 0_+-tangles $S_1, \cdots, S_u$ into the first u boxes of W and the 0_--tangles $S_{u+1}, \cdots, S_k$ into the last $k-u$ boxes of W. An entirely analogous procedure applied to $T_1, \cdots, T_l$ yields a tangle T', say. The fact that the picture is closed implies that there exists a 'colour-preserving bijection' between the sets of inoputs of T' and of S. Let T be the tangle obtained from T' by appropriately renumbering its internal boxes using the permutation σ so that the i^{th} output of $T^\dagger$ corresponds to the i^{th} input of S. This T is a tangle of the same type as S.

Suppose now that P is a subfactor planar algebra with modulus δ; and let $x_P \in X$ denote the corresponding point. The fact that there exists an integer n - depending only on $\mathcal{P}(f,\sigma)$ - such that $\delta^n \chi^P_{S,T} = \mathcal{I}_{\mathcal{P}(f,\sigma)}(x^P)$ follows from the following observations:

(a) the two pictures below represent the natural isomorphisms of $P_{0_\pm}$ onto $\mathbb{C}$ and $\mathbb{C}$ onto $P_{0_\pm}$ respectively

(4.22)

(b) $Z^P_W(a_1 1_{0_+}, \cdots, a_u 1_{0_+}, a_{u+1} 1_{0_-}, \cdots, a_k 1_{0_-}) = \delta\ a_1 a_2 \cdots a_k 1_{0_+}$ for any complex numbers $a_1, a_2, \cdots, a_k$, and

(c) for any two 0_+-tangles S, T of the same type, $Z^P_S (Z^P_T)^* (1_{0_+}) = \chi^P_{S,T} 1_{0_+}$. □

Proof of Theorem 2: That the $\chi^{\cdot}_{(S,T)}$'s are invariants of isomorphism classes of subfactor planar algebras is the content of Corollary 5. The completeness of this family of numerical invariants follows from Proposition 20 and Proposition 23. □

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