ON THE ISOMORPHISM OF DISCRETE SUBGROUPS OF SL(2, R)

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The purpose of this paper is to give a sufficient condition for the isomorphism of two discrete subgroups Γ and Γ' of SL(2, R) in terms of the norms of the primitive hyperbolic elements of Γ and Γ'. The proof exploits some well-known properties of the Selberg zeta function [3].

1. INTRODUCTION

Let G be the group SL(2, R) (that is, the group of two-by-two real matrices of determinant 1). Let Γ be a discrete subgroup of G such that $\Gamma \backslash G$ is compact. We shall also assume that Γ contains no elements of finite order. It is known (see [1, p. 11]) that under these assumptions Γ contains only hyperbolic elements. (An element $\gamma \in \Gamma$ is said to be *hyperbolic* if it has distinct, real eigenvalues.) An element $\gamma \in \Gamma$ is said to be *primitive* if it is not a positive power of any other element of Γ; clearly, each conjugate of γ will also be primitive. Let $\{P_\alpha\}$ ($\alpha = 1, 2, \cdots$) be a complete set of representatives of the primitive hyperbolic conjugacy classes of Γ, and let $N\{P_\alpha\}$ be the norm of P_α (if λ_1 and λ_2 are the eigenvalues of P_α, then $N\{P_\alpha\} = [\max(|\lambda_1|, |\lambda_2|)]^2$). We are now in a position to state our theorem:

THEOREM. *Let Γ and Γ' be discrete subgroups of* G (= SL(2, R)), *without elements of finite order, and such that $\Gamma \backslash G$ and $\Gamma' \backslash G$ are compact. Let $n_1, n_2, \cdots$ be the norms of the primitive hyperbolic classes of Γ, where $n_1 < n_2 < \cdots$, and let m_i be equal to the number of primitive classes whose norm is exactly n_i. Let n_i' and m_i' be similarly defined for Γ'. Then, if $n_i = n_i'$ and $m_i = m_i'$ for all* i, *the subgroups Γ and Γ' are isomorphic (as abstract groups).*

(*Note.* It is known [2], [4] that $N\{P_\alpha\}_{\alpha=1}^{\infty}$ has no accumulation point, and therefore we can write the norms of the hyperbolic elements in the order of increasing magnitude.)

Before giving the proof, we require some more facts. Let

$$H = \{Z;\ Z \in \mathbb{C} \text{ and } \Im Z > 0\}.$$

Then, since Γ is a subgroup of SL(2, R), Γ acts on the upper half-plane H, and under the assumptions on Γ, $\Gamma \backslash H$ is a compact Riemann surface and Γ is its fundamental group. Let p be the genus of $\Gamma \backslash H$. Then, again under the assumptions on Γ, $p > 1$. In [3], A. Selberg introduces the zeta function

$$Z_\Gamma(s) = \prod_\alpha \prod_{n=0}^{\infty} [1 - (N\{P_\alpha\})^{-s-n}].$$

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The following properties of $Z_\Gamma(s)$ are of interest to us (for proofs, see [2] or [4]).

(A) The expression on the right converges to an analytic function for $\Re s > 1$. However, $Z_\Gamma(s)$ has an analytic continuation to the entire complex plane. (In fact Z_Γ is an entire function of finite order.)

(B) If k is a nonnegative integer, then $s = -k$ is a zero of Z_Γ of multiplicity $(2k + 1)(2p - 2)$ (these are the so-called *trivial* zeros of Z_Γ; recall that p = genus of $\Gamma \backslash H$, and in our case $p > 1$).

2. PROOF OF THE THEOREM

The fact that $n_i = n_i'$ and $m_i = m_i'$ for all i implies $Z_\Gamma(s) = Z_{\Gamma'}(s)$. In particular, $Z_\Gamma(s)$ and $Z_{\Gamma'}(s)$ have the same zeros, and by comparing the formula for the multiplicities of the zeros at $s = -k$, we find that $p = p'$, where

$$p = \text{genus of } \Gamma \backslash H \quad \text{and} \quad p' = \text{genus of } \Gamma' \backslash H.$$

Since $\Gamma \backslash H$ and $\Gamma' \backslash H$ are compact Riemann surfaces, this implies that $\Gamma \backslash H$ and $\Gamma' \backslash H$ are homeomorphic. But, as we have already observed, Γ is the fundamental group of $\Gamma \backslash H$ and Γ' is the fundamental group of $\Gamma' \backslash H$, and therefore Γ and Γ' are isomorphic (as abstract groups).

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