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A QUALITATIVE UNCERTAINTY PRINCIPLE FOR SEMISIMPLE LIE GROUPS

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Dedicated to Robert Edwards in recognition of 25 years' distinguished contribution to mathematics in Australia, on the occasion of his retirement

Abstract

Recently M. Benedicks showed that if a function $f \in L^2(\mathbf{R}^d)$ and its Fourier transform both have supports of finite measure, then $f = 0$ almost everywhere. In this paper we give a version of this result for all noncompact semisimple connected Lie groups with finite centres.

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0. Introduction

Let G be a locally compact group equipped with left Haar measure dm and $\hat{G}$ its unitary dual (that is, a maximal set of pairwise inequivalent unitary irreducible continuous representations of G). For $f \in L^1(G)$ and $\pi \in \hat{G}$, define the operator $\pi(f) = \int_G f(x)\pi(x)\,dm(x)$ (which acts on the underlying Hilbert space for π). The assignment $\pi \to \pi(f)$ can be thought of as the (group theoretic) analogue of the classical Fourier transform $\hat{f}$ of an integrable function on $\mathbf{R}$. It has long been recognized that if f is 'concentrated' near a point, then $\hat{f}$ has to be 'spread out' and vice versa. A quantitative expressions of this principle leads to the Heisenberg uncertainty principle—see for example [5].

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Another expression of this principle is the following result of Benedicks [2]: If $f \in L^2(\mathbf{R}^d)$ with $m(\operatorname{supp} f) < \infty$ and $m(\operatorname{supp} \hat{f}) < \infty$, then $f = 0$ a.e. (If $\operatorname{supp} f$ is compact then the above result collapses to an easy exercise in introductory Fourier analysis. However with only the assumption $m(\operatorname{supp} f) < \infty$, the result quoted above is more substantial.) In view of this a natural question to ask is whether the above principle can be formulated for a locally compact group G. In this paper we show that a principle very close to the one of Benedicks holds for all noncompact semisimple connected Lie groups with finite centres. Earlier this kind of principle had been established for a wide variety of groups including $\mathrm{SL}(2, \mathbf{R})$ ([7]). However, for general semisimple Lie groups rather severe restrictions had to be placed on the kind of L^1 functions being dealt with. For quantitative versions of this principle for certain groups see [4], [8] and [9].

1. Notation and preliminaries

Throughout this paper G will denote a connected noncompact semisimple Lie group with finite centre. (For unexplained terminology and results, see [11].) Fix a maximal compact subgroup K of G. Let $\hat{G}$ denote the unitary dual of G and $\hat{K}$ the unitary dual of K. Fix a Haar measure m on G—as is well known G is unimodular—and let μ be the (corresponding) Plancherel measure on $\hat{G}$. In this section we describe the structure and representation theory of G that will be needed in the next section.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G with Cartan involution θ. Here $\mathfrak{k}$ is the Lie algebra of K. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$ and let

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

be a decomposition of $\mathfrak{g}$ into real root spaces for $\mathfrak{a}$, where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and R is the set of nonzero real roots. Fix once and for all a set of positive roots R^+, and let S be the set of simple positive roots. We write $\mathfrak{n}$ for $\sum_{\alpha \in R^+} \mathfrak{g}_\alpha$. At the group level, we write K, A and N for the connected subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ respectively, and M for the centralizer of A in K. Then MAN is a so-called minimal parabolic subgroup, hereafter denoted P_0. (It is unique up to conjugation.)

The other "parabolic subgroups" of G (up to conjugation) all arise in the following way. Pick a subset S_i of S and let R_i be the set of roots which are linear combination of roots in S_i. There is a unique closed subgroup of G, denoted by P_i (known as a parabolic subgroup), which contains P_0 and whose

Lie algebra is

$$\bigoplus_{\alpha \in R_i} \mathfrak{g}_\alpha + \mathfrak{m} + \mathfrak{a} + \mathfrak{n}.$$

This group has a Langlands decomposition $P_i = M_i A_i N_i$, where M_i is reductive, A_i is abelian and N_i is nilpotent. The Lie algebra $\mathfrak{m}_i$ of M_i is generated by $\mathfrak{m} \oplus \sum_{\alpha \in R_i} \mathfrak{g}_\alpha$; A_i and N_i are $\exp(\mathfrak{a}_i)$ and $\exp(\mathfrak{n}_i)$ respectively (exp denoting the exponential map), where $\mathfrak{a}_i$ is the orthogonal complement to $\mathfrak{m}_i \cap \mathfrak{a}$ in $\mathfrak{a}$, relative to the inner product on $\mathfrak{a}$ induced by the Killing form, and $\mathfrak{n}_i = \sum_{\alpha \in R^+ \setminus R_i} \mathfrak{g}_\alpha$. If M_i contains a compact Cartan subgroup, then P_i is said to be cuspidal. We let $\{P_j : j \in J\}$ be a maximal set of (nonconjugate) cuspidal parabolic subgroups constructed as above.

Harish-Chandra showed that sufficiently many irreducible unitary representations of G to decompose $L^2(G)$ may be obtained by taking a cuspidal parabolic subgroup P_j, a discrete series representation δ of M_j (that is, $\delta \in (\hat{M}_j)_d$) and a character $\chi_\lambda \colon \exp(H) \to \exp(i\lambda(H))$ of A_j (where $\lambda \in \mathfrak{a}_j^*$, the real dual of $\mathfrak{a}_j$), forming the unitary representation (denoted abusively) $\delta \otimes \chi_\lambda \otimes 1$ of P_j (where $\delta \otimes \chi_\lambda \otimes 1(man) = \delta(m)\chi_\lambda(a)$, $m \in M_j$, $a \in A_j$, $n \in N_j$) and inducing unitarily to G. We write $\pi^{(j)}_{\delta,\lambda} = \mathrm{ind}^G_{P_j} \delta \otimes \chi_\lambda \otimes 1$. The representations $\pi^{(j)}_{\delta,\lambda}$ and $\pi^{(i)}_{\varepsilon,\mu}$ can be equivalent only if $P_i = P_j$, and then if and only if (δ, λ) and (ε, μ) are conjugate under an appropriate (finite) Weyl group action. In [6] (see in particular Sections 25 and 36) Harish-Chandra calculated explicitly the Plancherel measure associated with the various series of representations of G. Except in the case when $P_i = G$ (that is, when G is a cuspidal parabolic subgroup of itself), A_i is a nontrivial vector group, and then for fixed δ in $(\hat{M}_i)_d$, the Plancherel measure $\mu(i, \delta, \lambda)$ is a smooth function of λ, which actually extends to an analytic function in a tube containing $\mathfrak{a}_i^*$ in $(\mathfrak{a}_i^*)_{\mathbb{C}}$ (the complexification of $\mathfrak{a}_i$) and is of polynomial growth in λ in almost all directions in $\mathfrak{a}_i^*$. An easy consequence of Harish-Chandra's calculation is $\mu(i, \delta, \mathfrak{a}_i^*) = \infty$.

Now we need to study the representations $\pi^{(i)}_{\delta,\lambda}$ in more detail. We fix a *proper* parabolic subgroup P_i of G and $\delta \in (\hat{M}_i)_d$. Let H_δ be the Hilbert space of δ. Define $H^{(i)}_\delta$ as the space of measurable H_δ-valued functions v on K which satisfy the conditions

$$v(km) = \delta(m^{-1})v(k), \qquad k \in K,\ m \in K \cap M_i,$$

and

$$\int_K |v(k)|^2 \, dk < \infty.$$

The induced representations $\pi^{(i)}_{\delta,\lambda}$ may be considered to act unitarily on $H^{(i)}_\delta$ by the formula

$$[\pi^{(i)}_{\delta,\lambda}(g)v](k) = \delta(m^{-1})v(k') \exp[(i\lambda + \rho_i)H_i(g^{-1}k)]$$

where

$$\rho_i = \frac{1}{2} \sum_{\alpha \in R^+ \setminus R_i} (\dim \mathfrak{g}_\alpha)\alpha \qquad \text{and} \qquad g^{-1}k = k'm \ \exp(H_i(g^{-1}k))n,$$

with $k' \in K$, $m \in M_i \cap AN$, $n \in N_i$ and $H_i(g^{-1}k) \in \mathfrak{a}_i$. (It should be noted that every element of G can be expressed uniquely in the form $kman$ where $k \in K$, $m \in M_i \cap AN$, $a \in A_i$ and $n \in N_i$. Indeed the 'Iwasawa decomposition' gives a unique decomposition of the form kan with $k \in K$, $a \in A$, $n \in N$. Furthermore an then factorizes uniquely as $a'n'a_in_i$ with $a'n' \in AN \cap M_i$ and $a_in_i \in A_iN_i$.) We note that the action of K on $H_\delta^{(i)}$ is just left translation and this is independent of λ in $\mathfrak{a}_i^*$.

Fix a basis $\{e_j : j \in \mathbf{N}\}$ of $H_\delta^{(i)}$ consisting of K-finite vectors. We have the following result:

PROPOSITION. *Fix a proper cuspidal parabolic subgroup P_i and choose δ in $(\hat{M}_i)_d$. Given $f \in L^1(G)$ and j, $h \in \mathbf{N}$, the function*

$$\lambda \to \int_G f(x) \langle \pi_{\sigma,\lambda}^{(i)}(x) e_j, e_h \rangle dx$$

on a_i^* extends to a holomorphic function in a tube in $(\mathfrak{a}_i^*)_{\mathbf{C}}$ which contains $\mathfrak{a}_i^*$.

PROOF. Using the techniques of Cowling [3] and of Anker [1], it is straightforward to show that, if the imaginary part of λ in $(\mathfrak{a}_i^*)_{\mathbf{C}}$ is not too big, then the representation $\pi_{\delta,\lambda}^{(i)}$ of the analytic continuation acts isometrically on a mixed L^p-space which we denote $L^p(K)$. The basis vectors e_j and e_h being smooth lie in $L^p(K)$ and its dual $L^q(K)$ for all p. In fact

$$\begin{aligned} |\langle \pi_{\delta,\lambda}^{(i)}(x) e_j, e_h \rangle| &\le \|\pi_{\delta,\lambda}^{(i)}(x) e_j\|_p \|e_h\|_q = \|e_j\|_p \|e_j\|_q \le \|e_j\|_\infty \|e_h\|_\infty \\ &= \sup_{k \in K} \|e_j(k)\|_{HS} \sup_{k \in K} \|e_h(k)\|_{HS} < \infty. \end{aligned}$$

The proposition follows immediately.

2. The main results

We are now in a position to state and prove the following theorem.

THEOREM. *Let G, $\hat{G}$, K, μ and m be as in the introduction. Let $f \in L^1(G)$ and let $A_f = \{x : f(x) \neq 0\}$ and $B_f = \{\pi \in \hat{G} : \pi(f) \neq 0\}$. If $m(KA_fK) < \infty$ and $\mu(B_f) < \infty$, then $f = 0$ a.e.*

PROOF. The statement that $\mu\{\pi\colon \pi(f) \neq 0\} < \infty$ implies that the Plancherel measure of each set $\{\tau \in \hat{A}_i\colon \pi^{(i)}_{\sigma,\tau}(f) \neq 0\}$ is finite. Since the Plancherel measure of $\{\pi_{\sigma,\tau}\}_{\tau\in\hat{A}_i}$ for fixed σ and i is infinite, it follows that $\{\tau \in \hat{A}_i\colon \pi^{(i)}_{\sigma,\tau}(f) = 0\}$ has positive Plancherel measure and hence has positive Lebesgue measure. This last statement follows from the fact that the Plancherel measure on the series $\{\pi^{(i)}_{\sigma,\tau}\}_{\tau\in\hat{A}_i}$, σ, i fixed, is absolutely continuous with respect to Lebesgue measure on $\hat{A}_i$. Now one knows that (at least in the sense of distributions) $f = \sum_{\nu\in\hat{K}}\sum_{\mu\in\hat{K}} d(\mu)d(\nu)\chi_\mu * f * \chi_\nu$. (Here we are identifying the characters χ_μ and χ_ν of the representations μ and ν of the compact group K with the (singular) measures $\chi_\mu\, dk$ and $\chi_\nu\, dk$ on G. Also for each $\mu \in \hat{K}$, $d(\mu)$ is its dimension.) Fix $\delta_1, \delta_2 \in \hat{K}$ and consider $h = \chi_{\delta_1} * f * \chi_{\delta_2}$. Let $E^{(i)}_\sigma = \{\tau \in \hat{A}_i\colon \pi^{(i)}_{\sigma,\tau}(f) = 0\}$. From what we said above $E^{(i)}_\sigma$ has positive Lebesgue measure. Now notice that if $\tau \in E^{(i)}_\sigma$, then $\pi^{(i)}_{\sigma,\tau}(h)$ is also zero. Thus $\pi^{(i)}_{\sigma,\tau}(h)$ is zero on a set of positive Lebesgue measure in $\hat{A}_i$. Let $u_1, \ldots, u_m$ be those basis vectors in $H^{(i)}_\sigma$ which transform according to δ_1 for $\pi^{(i)}_{\sigma,\tau}|_K$ and $w_1, \ldots, w_n$ be those basis vectors in $H^{(i)}_\sigma$ which transform according to δ_2 for $\pi^{(i)}_{\sigma,\tau}|_K$. (Notice these are independent of τ for σ and i fixed.) Since h satisfies $h = d(\delta_1)d(\delta_2)\chi_{\delta_1} * h * \chi_{\delta_2}$, it follows that $\pi^{(i)}_{\sigma,\tau}(h)$ is completely determined by the scalars $\langle \pi^{(i)}_{\sigma,\tau}(h)w_s, u_t\rangle$, $1 \le s \le n$ and $1 \le t \le m$. However $\langle \pi^{(i)}_{\sigma,\tau}(h)w_i, u_t\rangle = \int_G \langle \pi^{(i)}_{\sigma,\tau}(x)w_i, u_t\rangle h(x)\, dx$. By the proposition, as a function of τ the above function is holomorphic in a strip containing $\hat{A}_i$. Thus the fact that this vanishes in a set of positive Lebesgue measure on $\hat{A}_i$ forces it to be identically zero on $\hat{A}_i$. Hence for fixed i and fixed $\sigma \in (\hat{M}_i)_d$, $\pi^{(i)}_{\sigma,\tau}(h) = 0$ for all τ. Thus for all i and $\sigma \in (\hat{M}_i)_d$, $\pi^{(i)}_{\sigma,\tau}(h) = 0$ for all $\tau \in \hat{A}_i$. This means that $\pi^{(i)}_{\sigma,\tau}(h) = 0$ unless $P_i = G$ so that the Fourier transform of h is supported by the discrete series of G. From D. Vogan's theory of minimal K-types [10], it is clear that only finitely many discrete series representations of G when restricted to K can contain the representations δ_1 and δ_2. On the other hand, it is routine to show that if $\sigma \in \hat{G}_d$, then $\sigma(h) = 0$ unless $\sigma|_K$ contains δ_1 and δ_2. Consequently h is a finite linear combination of matrix elements of the discrete series of G, and is therefore real analytic on G. Now since $h = \chi_{\delta_1} * f * \chi_{\delta_2}$, if $x \notin KA_fK$, $h(x) = 0$. But by our assumption $m(KA_fK) < \infty$ and so $m(KA_fK)^c > 0$. Thus since G is connected and h is real analytic, this forces $h \equiv 0$. However $f \sim \sum\sum d(\delta)d(\nu)\chi_\delta * f * \chi_\nu$ and we have just shown that each term on the right side is zero. Hence $f = 0$ as a distribution, that is, $f = 0$ a.e.

An examination of the proof shows we have actually proved the following stronger result.

COROLLARY (to proof of Theorem). *Let $f \in L^1(G)$ with $m(KA_fK)^C > 0$. Assume for each fixed i, σ that the set $\{\tau \in \hat{A}_i, \pi^{(i)}_{\sigma,\tau}(f) = 0\}$ has positive Lebesgue measure. Then $f = 0$ a.e.*

REMARK. The property discussed in this paper (and in [2] and [7]) fails completely in many situations. For example, the Fourier transform of the characteristic function of a compact open subgroup of the p-adic numbers is another such characteristic function. Also the existence of supercuspidal representations for reductive p-adic groups gives rise to further counterexamples [7].

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