PHYSICAL REVIEW A VOLUME 23, NUMBER 6 JUNE 1981

Quantum theory of radiative transfer

E. C. G. Sudarshan
Center for Particle Theory, University of Texas, Austin, Texas 78712
(Received 25 February 1980)

Stemming from the classic work of Planck, classical radiative transfer theory works with pencils of light rays streaming in straight lines in any homogeneous medium. But light is described by Maxwell's equations in classical and quantum electrodynamics, and so we need to find a bundle of rays equivalent to the electromagnetic waves. In this paper we realize such a pencils of light rays. Their properties and the equations obeyed by the ray density function are deduced. We need to generalize radiative transfer theory for this purpose appropriately. For homogeneous statistical wave fields radiative transfer theory obtains rigorously. Particular attention is paid to the polarization properties of light rays. These concepts have some relation with the phase-space picture of quantum mechanics. Also, other general questions including possible further developments are discussed.

I. RADIATIVE TRANSFER, ELECTROMAGNETIC WAVES, AND LIGHT RAYS

Quantum electrodynamics has reconciled the old corpuscular and wave theories of light. Our present understanding of light is that it is a quantum wave field and that in free space the propagation is described by Maxwell's equations.[1] But, as Wolf[2] has so eloquently expressed, radiative energy transfer by electromagnetic waves is still treated by a phenomenological theory, virtually unchanged in its essentials from the work of Planck[3] except for the phenomenological developments by Chandrasekhar.[4] Since electrodynamics is the fundamental theory, to the extent phenomenological radiative transfer theory is valid it should be derivable from electrodynamics.

The notion of pencils of rays[5] has been very useful in optical theory in a variety of contexts. In free space light rays travel in straight lines, they are reflected and refracted according to simple laws and provide a graphic illustration of the dynamic propagation of light. Such rays are natural in a corpuscular context with individual photons having simple trajectories. Reflection is easily explained in terms of elastic collisions with unyielding boundaries; but refraction already generates some problems. For point photons one would expect light to travel faster in water than in air in direct contradiction to experiment.[6] So a formalism in which photons are treated as classical point particles is not acceptable. Yet, since in photoelectric effect[7] and Compton effect[8] photons are undoubtedly making themselves seen, how could we abandon the corpuscular picture, and along with it the concept of light rays?

Wave fields can have rays associated with them through the familiar methods of the eikonal approximation. In this method we consider the Sommerfeld-Runge eikonal function S in the phase of the wave amplitude considered as a path length. This function obeys a simple differential equation under the assumption of slow variation of the amplitude and may be considered as the "wave fronts of geometrical optics." The rays are then taken along the gradients to the eikonal. This derivation, originally given for a Kirchoff scalar wave is an approximation procedure not necessarily applicable to all relevant electromagnetic problems. The constancy of the amplitude may be relevant to a plane-wave field but not even to relatively simple problems of propagation in unbounded media. Further, in interpreting the results obtained in this approximation method each ray is treated as if it is distinct from every other ray and a pencil of rays is just a jumble and not a correlated bunch.

We must, therefore, strive to obtain an *exact ray description* of wave fields free of approximations and applicable to all wave propagations including radiative transfer and typical interference and diffraction phenomena. We expect qualitatively new features to emerge in the latter case, though it will share some properties with the Sommerfeld-Runge eikonal method.

If we are successful in reintroducing the notion of light rays into the wave theory, a consequence will be the outline of a theory of radiative transfer. Some generalizations would naturally arise in any such theory to reproduce the typical wave behavior in such situations as diffraction and interference. Further, the light rays will be endowed with polarization. We would expect *light rays* to *travel in free space more or less in straight lines, yet the interference phenomenon of light superposed on light producing darkness must emerge from the theory.*[9] We must, therefore, be prepared to have some surprises about light rays in electrodynamics.

Phenomenological radiative transfer theory[2-4] uses the concept of the specific intensity of radiation $I_\nu(\vec{r}, \vec{s})$ which is related to the amount of radiant energy transported across a unit area per unit time per unit solid angle in the neighborhood of a point with coordinates $\vec{r}$ in a direction $\vec{s}$. According to Lambert's law this should be proportional to $\cos\theta$, where θ is the angle between $\vec{s}$ and the normal to the surface. The coefficient of proportionality per unit frequency interval is called the specific intensity $I_\nu(\vec{r}, \vec{s})$. By definition it is a scalar function depending on the vector $\vec{r}$ and the unit vector $\vec{s}$.

Given the specific intensity we can define other related quantities relevant to radiative transfer. The specific net flux at an arbitrary point (per unit frequency interval) is

$$\vec{F}_\nu(\vec{r}) = \int d\Omega\, I_\nu(\vec{r}, \vec{s})\vec{s}, \tag{1.1}$$

the integration being over the whole 4π solid angle. Similarly, the specific energy density of radiation is given by the formula

$$u_\nu(\vec{r}) = \int d\Omega\, I_\nu(\vec{r}, \vec{s}). \tag{1.2}$$

By virtue of the physical interpretation of the specific intensity $I_\nu(\vec{r}, \vec{s})$ as the energy carried by the light rays in the direction $\vec{s}$ at the point $\vec{r}$ we must expect

$$I_\nu(\vec{r}, \vec{s}) = I_\nu^*(\vec{r}, \vec{s}), \tag{1.3}$$

$$I_\nu(\vec{r}, \vec{s}) \geq 0, \tag{1.4}$$

as well as

$$\vec{s} \cdot \vec{\nabla} I_\nu(\vec{r}, \vec{s}) = 0. \tag{1.5}$$

with the $\vec{\nabla}$ operator acting on the $\vec{r}$ dependence. The last equation corresponds to the fact that *in free space the energy is carried by the light rays and it is conserved.*

Implicit in radiative transfer theory is the assertion that *energy associated with different frequencies is independently propagated.* One could therefore carry out the discussion for different frequencies completely independently. It follows that if we are to establish a correspondence with wave theory we must arrange to have waves of different frequencies to add in intensities, rather than in amplitudes. This would come about if the different frequency components are phase incoherent. Such a wave ensemble would be completely equivalent to a stationary ensemble. We must, therefore, look for a *correspondence between time stationary ensembles in wave theory and radiative transfer theory.* This important observation was made by Wolf[2] in his fundamental study of radiative transfer.

Classical electrodynamics in empty space is described by Maxwell's equations

$$\begin{aligned} &\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}, \\ &\vec{\nabla} \times \vec{B} = +\frac{\partial}{\partial t}\vec{E}, \\ &\vec{\nabla} \cdot \vec{E} = 0, \\ &\vec{\nabla} \cdot \vec{B} = 0. \end{aligned} \tag{1.6}$$

By virtue of these equations both $\vec{E}$ and $\vec{B}$ propagate as waves:

$$\begin{aligned} \nabla^2\vec{E} &= \frac{\partial^2}{\partial t^2}\vec{E} = -\nu^2\vec{E}, \\ \nabla^2\vec{B} &= \frac{\partial^2}{\partial t^2}\vec{B} = -\nu^2\vec{B}. \end{aligned} \tag{1.7}$$

In general, both $\vec{E}$ and $\vec{B}$ contain both positive- and negative-frequency components. Following Gabor[10] and Wolf[11] we may introduce the analytic signals which contain only positive frequencies:

$$\begin{aligned} \frac{i\partial}{\partial t}\vec{E} &= \nu\vec{E}, \\ \frac{i\partial}{\partial t}\vec{B} &= \nu\vec{B}. \end{aligned} \tag{1.8}$$

The analytic signals are always complex, and are the positive-frequency parts of the real fields with their complex conjugates containing only negative frequencies.

In classical electrodynamics it is possible to demonstrate that the energy and momentum densities are given by the expressions

$$\begin{aligned} T_{00}(\vec{r}) &= \tfrac{1}{2}[\vec{E}^2(\vec{r}) + B^2(\vec{r})], \\ T_{0j}(\vec{r}) &= E_{jkl}E_k(\vec{r})B_l(\vec{r}). \end{aligned} \tag{1.9}$$

The momentum density $T_{0j}(\vec{r})$ may also be identified with the flow of energy measured by the Poynting[12] vector

$$\vec{S}(r) = \vec{E}(r) \times \vec{B}(r), \tag{1.10}$$

which describes the flow of energy. (In all these formulas we have adopted natural units so that the velocity of light in vacuum is unity.) In terms of analytic signals we must write the energy density $W(\vec{r})$ and the energy flow $\vec{S}(\vec{r})$ in the form

$$W(\vec{r}) = \tfrac{1}{2}[\vec{E}^*(\vec{r}) \cdot \vec{E}(\vec{r}) + \vec{B}^*(\vec{r}) \cdot \vec{B}(\vec{r})], \tag{1.11}$$

$$\vec{S}(\vec{r}) = \tfrac{1}{2}[\vec{E}^*(\vec{r}) \times \vec{B}(\vec{r}) - \vec{B}^*(\vec{r}) \times E(\vec{r})]. \tag{1.12}$$

We would expect a close relationship between $u_\nu(\vec{r})$ defined by Eq. (1.2) and the value of $W(\vec{r})$ for monochromatic waves of frequency ν; and similarly between the specific net flux $\vec{F}_\nu(\vec{r})$, Eq. (1.1), and the Poynting vector $\vec{S}(\vec{r})$ for monochromatic waves. But before these relations can be made precise we must introduce the second-order correlation tensors for the statistical wave field.

In this paper, following Wolf's pioneering study, we demonstrate the equivalence of the wave field to a collection of rays and derive radiative transfer theory.

II. CORRELATION TENSORS IN ELECTRODYNAMICS

The electromagnetic field as encountered in classical optics is a statistical entity with an ensemble representing its state specification.[13] Rather than specify the value of the field quantities (taken to be positive-frequency analytic signals), we have an ensemble of such fields. When $\vec{E}$ and B are analytic signals which are random variables, the quantities of physical significance are the expectation values of these quantities identified with their ensemble averages. For most practical purposes of radiative transfer theory the important quantities are the second-order space-time correlation tensors of the ensemble which are second moments of the ensemble probability distributions. We may define the general second-rank tensors in classical statistical optics:

$$\begin{aligned}
\Gamma^{(ee)}_{jk}(\vec{r}_1, \vec{r}_2, t_1, t_2) &= \langle E^*_j(\vec{r}_1, t_1) E_k(\vec{r}_2, t_2)\rangle ,\\
\Gamma^{(eb)}_{jk}(\vec{r}_1, \vec{r}_2, t_1, t_2) &= \langle E^*_j(\vec{r}_1, t_1) B_k(\vec{r}_2, t_2)\rangle ,\\
\Gamma^{(be)}_{jk}(\vec{r}_1, \vec{r}_2, t_1, t_2) &= \langle B^*_j(\vec{r}_1, t_1) E_k(\vec{r}_2, t_2)\rangle ,\\
\Gamma^{(bb)}_{jk}(\vec{r}_1, \vec{r}_2, t_1, t_2) &= \langle B^*_j(\vec{r}_1, t_1) B_k(\vec{r}_2, t_2)\rangle .
\end{aligned} \tag{2.1}$$

Since the operators of ensemble averaging and space and time differentiation commute, these four-tensors obey 32 relations among themselves obtained by applying Maxwell's equations with respect to the variables $(\vec{r}_2, t_2)$ and their complex conjugates with respect to $(\vec{r}_1, t_1)$ to each of these tensors.[14] As one consequence of these relations we may deduce all the correlation tensors if only $\Gamma^{(ee)}_{jk}$ is known.

If the classical ensemble is stationary in the sense that all the second-order correlation tensors obey

$$\Gamma_{jk}(\vec{r}_1, \vec{r}_2, t_1 + t', t_2 + t') = \Gamma_{jk}(\vec{r}_1, \vec{r}_2, t_1, t_2),$$

then it is only a function of the difference of times:

$$\Gamma_{jk}(\vec{r}_1, \vec{r}_2, t_1, t_2) = \int_0^\infty d\nu\, \Gamma_{jk}(\vec{r}_1, \vec{r}_2, \nu) e^{i\nu(t_1 - t_2)} . \tag{2.2}$$

The specific correlation tensors $\Gamma_{jk}(\vec{r}_1, \vec{r}_2, \nu)$ satisfy the equations

$$(\nabla_1^2 + \nu^2)\Gamma_{jk}(\vec{r}_1, \vec{r}_2, \nu) = 0 , \tag{2.3}$$

$$(\nabla_2^2 + \nu^2)\Gamma_{jk}(\vec{r}_1, \vec{r}_2, \nu) = 0 , \tag{2.4}$$

and a number of other such relations of the form

$$\begin{aligned}
\epsilon_{jab} \frac{\partial}{\partial x_a} \Gamma^{(ee)}_{bk}(\vec{r}_1, \vec{r}_2, \nu) &= i\nu \Gamma^{(be)}_{jk}(\vec{r}_1, \vec{r}_2, \nu) ,\\
\epsilon_{jab} \frac{\partial}{\partial x_a} \Gamma^{(be)}_{bk}(\vec{r}_1, \vec{r}_2, \nu) &= -i\nu \Gamma^{(ee)}_{jk}(\vec{r}_1, \vec{r}, \nu) .
\end{aligned} \tag{2.5}$$

It is seen that for the time-stationary ensemble the second-order correlation tensors are all direct sums of specific tensors for definite frequencies. From now on we may restrict our attention to excitations of a definite frequency ν.

In quantum electrodynamics in free space the field quantities $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$ are operators (more precisely, operator-valued distributions). Following Glauber,[15] the analytic signal is now the positive-frequency part of the field operators which can be expanded in terms of annihilation operators only; and their adjoints consist of creation operators. The quantum ensemble of the states is now associated with a density matrix ρ and the second-order correlation tensors are defined by

$$\begin{aligned}
\Gamma^{(ee)}_{jk}(r_1 r_2 t_1 t_2) &= \mathrm{tr}\,[\rho E^\dagger_j(\vec{r}_1, t_1) E_k(\vec{r}_2, t_2)] ,\\
\Gamma^{(eb)}_{jk}(\vec{r}_1, \vec{r}_2, t_1, t_2) &= \mathrm{tr}\,[\rho E^\dagger_j(\vec{r}_1, t_1) B_k(\vec{r}_2, t_1)] ,\\
\Gamma^{(be)}_{jk}(\vec{r}_1, \vec{r}_2, t_1, t_2) &= \mathrm{tr}\,[\rho B^\dagger_j(\vec{r}_1, t_1) E_k(\vec{r}_2, t_2)] ,\\
\Gamma^{(bb)}_{jk}(\vec{r}_1, \vec{r}_2, t_1, t_2) &= \mathrm{tr}\,[\rho B^\dagger_j(\vec{r}_1, t_1) B_k(\vec{r}_2, t_2)] .
\end{aligned} \tag{2.6}$$

Equations (2.2)–(2.5) remain valid for the quantum electrodynamic correlation tensors. Moreover, considered as a matrix in its arguments, subscripts and superscripts, the correlation tensors in quantum electrodynamics are also nonnegative just as in the classical electrodynamic correlation tensors. In fact, given a classical ensemble of light waves which yield a set of correlation functions we can construct a quantum ensemble in quantum electrodynamics with the same correlation functions. The quantum ensemble will be defined by a density matrix ρ which enters Eq. (2.6). In this sense classical electrodynamics and quantum electrodynamics are indistinguishable as far as the second-order correlation functions are considered, a result which is of general validity in coherence optics.[15]

III. RAYS IN ELECTRODYNAMICS

Since the second-order correlation functions give a complete discription of all the classical optical phenomena (with the exception of nonlinear optics) the light rays must be deducible from the correlation function. Yet the waves are quite unlike rays: to specify the direction and location of a ray is natural but to localize a wave is to lose all ability to determine its direction of motion. On the other hand, if we know the direction more or less accurately from having a wave front, then the localization is lost. This is a familiar problem in quantum theory of particles where position and momentum cannot be simultaneously specified. Conjugate variables like position and momentum cannot be simultaneously specified. Hence, expressions containing both position and momentum variables are not unambiguously defined: we must define an "ordering." The simultaneous probability distribution in such conjugate pairs of variables is thus a new kind of mathematical object; depending upon the ordering we use for specifying functions of conjugate variables the probability distributions will take on different forms. For classical optics we face a similar problem when searching for a suitable definition of rays.

We seek a suitable association of the second-order correlation functions with the specific intensity $I_\nu(\vec{r},\vec{s})$. Such a correspondence involves the analogs of sets of conjugate variables of position and momentum for waves: and, as such, we expect some degree of ambiguity in defining the corresponding distribution function but *none of the possible definitions would satisfy all the standard requirements of a probability distribution in the two sets of variables*. The choice we must make is one that violates the fewest intuitive requirements on $I_\nu(\vec{r},\vec{s})$. *Since interference phenomena involve addition of light to light producing darkness we cannot retain the positivity of $I_\nu(r,s)$ if we are to have exact correspondence.*[9] Granted that, we should attempt to satisfy all the other requirements.

A suitable solution to the quantum-mechanical problem is to introduce the Wigner-Moyal distribution. In this case the ordering problem is solved by taking the completely symmetrized product of momentum and coordinate factors. The quantum-mechanical state, now represented by the wave function (or, more generally, by an ensemble of wave functions), is associated with a two-point density whose partial Fourier transforms give the Wigner function which is the probability distribution function of the Wigner-Moyal theory. This suggests to us that the proper quantity to deal with is the second-order correlation function and its partial Fourier tranformation. This we proceed to do.

To get the specific intensity we need a function of the nature of an energy density which depends on six variables. We are, therefore, naturally led to choose the generalization of the energy density

$$M(\vec{r}_1,\vec{r}_2)=\tfrac{1}{2}\langle[\vec{E}^*(r_1)\cdot\vec{E}(r_2)+\vec{B}^*(\vec{r}_1)\cdot\vec{B}(\vec{r}_2)]\rangle$$

$$=\tfrac{1}{2}[\Gamma_{jj}^{(ee)}(\vec{r}_1,\vec{r}_2)+\Gamma_{jj}^{(bb)}(\vec{r}_1,\vec{r}_2)] \qquad (3.1)$$

and its transform, $W(\vec{r},\vec{k})$, defined by[9]

$$W(\vec{r},\vec{k})=(2\pi)^{-3}\int d^3\sigma\, e^{-ik\cdot\sigma}M(\vec{r}+\tfrac{1}{2}\vec{\sigma},\vec{r}-\tfrac{1}{2}\vec{\sigma})$$

$$=(2\pi)^{-3}\int d^3\sigma\, e^{-ik\cdot\sigma}\tfrac{1}{2}\langle\vec{E}^*(\vec{r}+\tfrac{1}{2}\vec{\sigma})\cdot\vec{E}(\vec{r}-\tfrac{1}{2}\vec{\sigma})+\vec{B}^*(\vec{r}+\tfrac{1}{2}\vec{\sigma})\cdot\vec{B}(\vec{r}-\tfrac{1}{2}\vec{\sigma})\rangle\,. \qquad (3.2)$$

By construction $W(\vec{r},\vec{k})$ is real for all values of $\vec{r}$ and $\vec{k}$ and may be identified with the specific intensity of radiation at point $\vec{r}$ in the direction $\vec{k}$. By virtue of Eqs. (2.3) and (2.4) of the preceding section we may write

$$[(\tfrac{1}{2}\vec{\nabla}+i\vec{k})^2+\nu^2]\,W(\vec{r},\vec{k})=0\,,$$

$$[(\tfrac{1}{2}\vec{\nabla}-i\vec{k})^2+\nu^2]\,W(\vec{r},\vec{k})=0\,.$$

These, in turn, imply

$$k^2W(\vec{r},\vec{k})=(\nu^2+\tfrac{1}{4}\nabla^2)W(\vec{r},\vec{k}) \qquad (3.3)$$

and

$$\vec{k}\cdot\nabla W(\vec{r},\vec{k})=0\,. \qquad (3.4)$$

For fields which are not rapidly varying, that is, not varying appreciably over the distance of several wavelengths, the second term on the right-hand side of Eq. (3.3) is negligible and we get the approximate equality

$$k^2\simeq\nu^2\,.$$

Hence, $|k|$ may be taken to be the frequency (or, more precisely, the wave number) of the photons associated with the rays whose specific density is given by the function $W(\vec{r},\vec{k})$. Equation (3.4) is identical with Eq. (1.5) of Sec. I if we identify $I_\nu(\vec{r},\vec{s})$ with $\int dp\,p^2W(\vec{r},p\vec{s})$ with $\vec{s}$ being identified with the unit vector in the direction $\vec{k}$:

$$I_\nu(\vec{r},\vec{s})=\int W(\vec{r},p\vec{s})\,p^2dp\,. \qquad (3.5)$$

By construction we also note that the integral of the W function over all wave numbers is just the energy density of the electromagnetic field:

$$M(\vec{r},\vec{r})=\tfrac{1}{2}\langle[\vec{E}^*(\vec{r})\cdot\vec{E}(\vec{r})+\vec{B}^*(\vec{r})\cdot\vec{B}(\vec{r})]\rangle$$

$$=\int d^3k\,W(\vec{r},\vec{k})=u_\nu(\vec{r}) \qquad (3.6)$$

by virtue of Eqs. (1.2) and (3.5).

Let us now consider the next $\vec{F}$ defined by Eq. (1.1). The flow of energy in the electromagnetic field is associated with the Poynting vector $\vec{S}$. We must take the two-point generalization of the Poynting vector:

$$\vec{S}(\vec{r}_1,\vec{r}_2)=\tfrac{1}{2}\langle[\vec{E}^*(\vec{r}_1)\times\vec{B}(\vec{r}_2)-\vec{B}^*(\vec{r}_1)\times\vec{E}(\vec{r}_2)]\rangle \qquad (3.7)$$

or

$$S_j(\vec{r}_1, \vec{r}_2) = \tfrac{1}{2}\epsilon_{jkl}[\Gamma_{kl}^{(eb)}(\vec{r}_1, \vec{r}_2) - \Gamma_{kl}^{(be)}(\vec{r}_1, \vec{r}_2)] .$$

Making use of the (generalized) Fourier expansions

$$\vec{E}(\vec{r}) = \int d^3p\, e^{i\vec{p}\cdot\vec{r}}\, \tilde{\vec{E}}(\vec{p}),$$

$$\vec{B}(\vec{r}) = \int d^3p\, e^{i\vec{p}\cdot\vec{r}}\, \tilde{\vec{B}}(\vec{p}),$$

and making use of Maxwell's equation for the monochromatic waves, we may write

$$\tilde{\vec{B}}(\vec{p}) = \frac{1}{\nu}\vec{p}\times\tilde{\vec{E}}(\vec{p}),$$

$$\tilde{\vec{E}}(\vec{p}) = -\frac{1}{\nu}\vec{p}\times\tilde{\vec{B}}(\vec{p}).$$

We may, therefore, conclude

$$\begin{aligned}\nu\int d^3\sigma\, e^{-i\vec{k}\cdot\vec{\sigma}}\,\vec{S}(\vec{r}+\tfrac{1}{2}\vec{\sigma}, \vec{r}-\tfrac{1}{2}\vec{\sigma}) &= \vec{k}\int d^3\sigma\, e^{-i\vec{k}\cdot\vec{\sigma}} M(\vec{r}+\tfrac{1}{2}\vec{\sigma}, \vec{r}-\tfrac{1}{2}\vec{\sigma}) \\ &\quad + \vec{\nabla}\times\int d^3\sigma\, e^{-i\vec{k}\cdot\vec{\sigma}}\,\vec{N}(\vec{r}+\tfrac{1}{2}\vec{\sigma}, \vec{r}-\tfrac{1}{2}\vec{\sigma}) \\ &= \vec{k}\,W(\vec{r},\vec{k}) + \vec{\nabla}\times V(\vec{r},\vec{k}),\end{aligned} \tag{3.8}$$

where

$$\vec{N}(\vec{r}_1, \vec{r}_2) = \tfrac{1}{2}\langle[\vec{E}^*(\vec{r}_1)\times\vec{E}(\vec{r}_2) + \vec{B}^*(\vec{r}_1)\times\vec{B}(\vec{r}_2)]\rangle . \tag{3.9}$$

Let us write

$$\nu^{-1}\vec{k}\,W(\vec{r},\vec{k}) = \vec{S}_1(\vec{r},\vec{k}), \tag{3.10}$$

$$\nu^{-1}\vec{\nabla}\times V(\vec{r},\vec{k}) = \vec{S}_2(\vec{r},\vec{k}). \tag{3.11}$$

Then $\vec{S}_2$ is negligibly small compared with $\vec{S}_1$ for most practical situations where the correlation tensors are only slowly varying. Moreover, we see that $\vec{S}_2$ being the curl of a vector does not contribute to the net flux in any case. We may now identify the net flux of radiation with the integral of $\vec{S}_1(\vec{r},\vec{k})$:

$$\begin{aligned}\vec{F}_\nu(r) &= \int d\Omega\, \vec{S}_1(\vec{r},\vec{k}) \\ &\simeq \int d\Omega \int d^2\sigma\, e^{i\vec{k}\cdot\vec{\sigma}}\,\vec{S}(\vec{r}+\tfrac{1}{2}\vec{\sigma}, \vec{r}-\tfrac{1}{2}\vec{\sigma}).\end{aligned} \tag{3.12}$$

We have thus arrived at a picture of energy and energy flow in the electromagnetic field consistent with the intuitive notions: the specific flow of energy is in the direction of the momentum and numerically equal to the specific energy density (velocity of light =1!) except for a fine circulatory motion extending over distances of the order of a wavelength.[17]

In the phenomenological (scalar) theory the primary quantity was the specific intensity $I_\nu(\vec{r},\vec{s})$ in terms of which the energy density $U_\nu(\vec{r})$ and the specific net flux $\vec{F}_\nu(\vec{r})$ were defined by Eqs. (1.1) and (1.2). If we compare them to Eqs. (3.10) and (3.12) we see that we may make the identifications

$$\begin{aligned} I_\nu(\vec{r},\vec{s}) &= W(\vec{r},\vec{k}), \\ U_\nu(\vec{r}) &= \int d\Omega_k\, W(\vec{r},\vec{k}), \\ \vec{F}_\nu(r) &= \nu^{-1}\int d\Omega_k \vec{k}\, W(\vec{r},\vec{k}) \end{aligned} \tag{3.13}$$

with the neglect of $\vec{S}_2(\vec{r},\vec{k})$ and the identification of ν with $|\vec{k}|$. We shall see below that for statistically homogeneous fields no approximation is involved.

If the field becomes statistically homogeneous, the results assume an even more appealing form. A statistically homogeneous field is invariant under space translations:

$$\Gamma_{jk}(\vec{r}_1, \vec{r}_2, \nu) = \Gamma_{jk}(\vec{r}_1+\vec{\rho}, \vec{r}_2+\vec{\rho}, \nu).$$

In this case the functions $W(\vec{r},\vec{k})$, $\vec{S}_1(\vec{r},k)$, and $\vec{S}_2(\vec{r},\vec{k})$ assume especially simple forms:

$$\begin{aligned} W(\vec{r},\vec{k}) &= W(k), \\ \vec{S}_1(\vec{r},\vec{k}) &= \vec{S}_1(\vec{k}), \\ \vec{S}_2(\vec{r},\vec{k}) &= 0, \end{aligned} \tag{3.14}$$

and hence (3.12) becomes an exact equality. Moreover,

$$W(\vec{k}) = \tfrac{1}{2}\int d^3\sigma\, e^{i\vec{k}\cdot\vec{\sigma}}[\Gamma_{jj}^{(ee)}(\tfrac{1}{2}\vec{\sigma}, -\tfrac{1}{2}\vec{\sigma}) + \Gamma_{jj}^{(bb)}(\tfrac{1}{2}\vec{\sigma}, -\tfrac{1}{2}\vec{\sigma})] .$$

It follows that

$$W(\vec{k}) \geq 0 . \tag{3.15}$$

This reproduces Eq. (1.4) by virtue of the identification (3.12).

In a homogeneous stationary electromagnetic field, classical or quantized, the primary quantities of radiative transfer theory obtain.

IV. POLARIZATION PROPERTIES OF LIGHT RAYS

Light is polarizable. Light rays must therefore be endowed with polarization. The electromagnetic field equations (1.6) and their consequences for the second-order correlation functions imply that there are two constraints on each of the correlation tensors. These are, for example,

$$\frac{\partial}{\partial x_{1j}}\Gamma^{(ee)}_{jk}(\vec{r}_1,\vec{r}_2)=0\,,\qquad \frac{\partial}{\partial x_{2k}}\Gamma^{(ee)}_{jk}(\vec{r}_1,\vec{r}_2)=0\,, \tag{4.1}$$

and similar equations for the (eb), (be), (bb) correlation tensors. In terms of tensor W functions this may be written in the form

$$\left(k_i-\frac{i}{2}\frac{\partial}{\partial x_i}\right)W^{(ee)}_{ij}(\vec{r},\vec{k})=0\,,\qquad \left(k_j+\frac{i}{2}\frac{\partial}{\partial x_j}\right)W^{(ee)}_{ij}(\vec{r},\vec{k})=0\,, \tag{4.2}$$

where

$$W^{(ee)}_{jk}(\vec{r},\vec{k})=(2\pi)^{-3}\times\int d^3\sigma\,\Gamma^{(ee)}_{jk}(\vec{r}+\tfrac{1}{2}\vec{\sigma},\vec{r}-\tfrac{1}{2}\vec{\sigma})e^{i\vec{k}\cdot\vec{\sigma}} \tag{4.3}$$

and similar equations for the (eb), (be), (bb) correlation tensors. For a homogeneous illumination which has no dependence on $\vec{r}$, these equations simplify to assert the transversality of each of the $W_{ij}(\vec{k})$. In most cases where the variation with $\vec{r}$ of W_{ij} is small over several wavelengths we may neglect the gradient terms in (4.2). The wave equations (1.7) and hence (3.3) and (3.4) apply equally to $W^{(ee)}_{ij}(\vec{r})$, etc. Hence, in particular,

$$\vec{k}\cdot\vec{\nabla}W_{ij}(\vec{r},\vec{k})=0\,. \tag{4.4}$$

To the extent that the gradient terms can be neglected and the correlation tensors are taken to be transverse to the wave number, we can expand the tensor Wolf's function in terms of a complete set of 2×2 matrices.[18,4,13] A natural method is in terms of the Stokes' parameters. In as much as

$$\vec{k}\times\vec{E}=\nu\vec{B}\,,\qquad \vec{k}\times\vec{B}=-\nu\vec{E}\,, \tag{4.5}$$

the combinations $\vec{E}\pm i\vec{B}$ correspond to modes with positive (negative) helicity. Denote these modes by indices 1 and 2. Then the tensor W functions can be written in the form

$$W_{ij}(\vec{r},\vec{k})=I_{ij}(\vec{r},\vec{k})+U_{ij}(\vec{r},\vec{k})+V_{ij}(\vec{r},\vec{k})+Q_{ij}(\vec{r},\vec{k}) \tag{4.6}$$

with

$$I=W_{11}+W_{22}\,,\quad U=W_{12}+W_{21}\,,\quad V=i(W_{12}-W_{21})\,,\quad Q=W_{11}-W_{22}\,. \tag{4.7}$$

Associated with each location $\vec{r}$ and wave number $\vec{k}$ there are four Stokes parameters I, U, V, and Q, which describe the polarization properties of the rays. Since any 2×2 non-negative Hermitian matrix has two orthogonal eigenvectors and hence a decomposition into two opposite (elliptic) polarizations, *we may identify at each location for each wave number these specific polarizations and their respective weights*. If the rays gradually bend, these polarizations will also tilt gradually. In empty space and for homogeneous fields these specific polarizations are constant along the ray.

V. DISCUSSION

The contrasting pictures of light as consisting of electromagnetic waves and as consisting of pencils of rays are synthesized in this paper. With every beam of light waves we may associate a system of generalized rays of light with wave number $\vec{k}$ at the location $\vec{r}$. In terms of the density function $W(\vec{r},\vec{k})$, we can define the energy density and energy flow. These derivations provide a basis for the phenomenology of radiative transfer theory. The Poynting vector decomposes into two components, one of which is the convective radiation transport, the other being a divergence-free flow with a fine structure with the scale of the wavelength of light.

The function $W(\vec{r},\vec{k})$ which is identified with the specific intensity $I_\nu(\vec{r},\vec{s})$, apart from a factor of k^2, is a real quantity (associated with the excitation of a field). For a homogeneous field it becomes independent of $\vec{r}$ and is then guaranteed to be non-negative. This function bears a close resemblance to the Wigner-Moyal phase-space density[19]:

$$\rho(\vec{q},\vec{p})=(2\pi)^{-3}\int d^3\sigma\,e^{i\vec{\sigma}\cdot\vec{p}}\psi^*(\vec{q}+\tfrac{1}{2}\vec{\sigma})\psi(\vec{q}-\tfrac{1}{2}\vec{\sigma}) \tag{5.1}$$

associated with a wave function ψ in quantum mechanics. While (5.1) has a similar structure to the W function (3.2) there are some essential differences. First, the Wigner-Moyal phase-space density is normalized:

$$\int\int d^3q\,d^3p\,\rho(\vec{q},\vec{p})=1\,,$$

while no such restriction obtains for the W function. A proper identification of $W(\vec{r},\vec{k})$ is obtained by the eigenmode decomposition[16]

$$\Gamma_{jk}(\vec{r}_1,\vec{r}_2)=\sum_n\gamma_n u^*_{nj}(\vec{r}_1)u_{nk}(\vec{r}_2) \tag{5.2}$$

appropriate for any second-order function. Without loss of generality we may choose the mode functions to be normalized:

$$\int d^3\sigma_1 u^*_{nj}(r_1)u_{mj}(r_1)=\delta_{m,n}\,. \tag{5.3}$$

Then

$$\int d^3\sigma_1 \Gamma_{jj}(r_1, r_1) = \sum_n \gamma_n$$

and

$$W(\vec{r}, \vec{k}) = \sum \gamma_n W_n(\vec{r}, \vec{k}),$$

with

$$W_n(r, k) = (2\pi)^{-3} \int d^3\sigma\, e^{i\vec{k}\cdot\sigma}[u_{nj}^{(e)*}(\vec{r} + \tfrac{1}{2}\vec{\sigma})u_{nj}^{(e)}(\vec{r} - \tfrac{1}{2}\vec{\sigma}) + u_{nj}^{(b)*}(\vec{r} + \tfrac{1}{2}\vec{\sigma})u_{nj}^{(b)}(\vec{r} - \tfrac{1}{2}\vec{\sigma})] . \quad (5.4)$$

In other words, *the W function is the weighted sum of a collection of Wigner-Moyal phase-space densities.* In the unique case of a mode-pure illumination[16,20] in which n takes only one value it is a simple multiple of it. The nonpositive definiteness is a reflection of the corresponding behavior of the Wigner-Moyal densities.

There are essential limitations on the validity of the ray picture of wave fields. In geometric optics, as well as in radiative transfer theory, there can be arbitrarily narrow pencils and arbitrary variations in the specific intensity with both direction and location. But in wave optics arbitrarily narrow pencils diffract and rapid variations in space are inconsistent with the fixed-frequency wave equations (3.3) and (3.4). Associated with such unphysical geometrical optics idealizations the W function becomes nonpositive.

It is to be emphasized that the ray bundles are linear in the second-order correlation tensors and, as such, incoherent beam mixing corresponds to addition of the rays. A simple example is the mixture of the eigenmodes in the correlation tensors discussed above in relation to the Wigner-Moyal distribution. But *in any phase-coherent superposition of light waves the resultant set of generalized rays of light is not just the sum of the set of rays*: there is an interference term. Elsewhere[9] I have analyzed a number of wave optical phenomena on this basis. In the particular case of a two-slit interference pattern one sees that the rays consist of three pencils, one from each of the slits with positive-definite W functions and a fictitious central pencil with intensity varying from positive to negative values as the angles change. The set of the three pencils reproduces the geometry of the interference pattern; but the important point is that the superposition of the amplitudes from the two slits generates a third pencil of rays. It is worth noting that *in most discussions of the two-slit interference phenomenon in terms of trajectories of photons this crucial fact is not recognized.*

It is possible to trace the behavior of the rays in the elementary processes of reflection and refraction. As expected, asymptotically we have parallel rays of light along the geometric directions of incidence, reflection, and refraction. In addition we have a number of interference contributions which alter the W function in the neighborhood of the interface between the two media. It would be interesting to study reflection and refraction using curved mirrors and lenses, and study, in particular, the path of rays in the context of focusing and image formation.

The rays of light in the present theory are endowed with polarization. To the extent that the illumination is slowly varying the polarization is at right angles to the direction of the ray. But this simple picture of transverse polarization cases to be valid in regions where the illumination is rapidly varying. This is only to be expected since under these circumstances wavelike behavior is going to be pronounced and the rays are bending to accommodate Eq. (3.4).

The importance of radiative transfer theory is in the context of propagation of light in a medium. The problem is definitely more complex than the one treated here. We have made a beginning in this direction. A more systematic effort is in progress in the work of Zubairy and Wolf.[21] It is appropriate that at this time when astrophysics and particle physics have joined hands, quantum electrodynamics and radiative transfer theory should do likewise.

ACKNOWLEDGMENT

I thank Narasimha Mukunda for constructive criticism of the ideas presented in this paper.

[1]J. C. Maxwell, Philos. Trans. R. Soc. London 158, 643 (1868).

[2]E. Wolf, Phys. Rev. D 13, 869 (1976); M. S. Zubairy and E. Wolf, Opt. Commun. 20, 321 (1977); E. Wolf, J. Opt. Soc. Am. 68, 6 (1978).

[3]M. Planck, *The Theory of Heat Radiation* (Dover, New York, 1959); E. A. Milne, Handb. Astrophys. 3, Pt. 1, 65 (1930).

[4]S. Chandrasekhar, *Radiative Transfer* (Dover, New York, 1950).

[5]F. A. Jenkins and H. E. White, *Fundamentals of Optics* (McGraw-Hill, New York, 1957).

[6]L. Foucault, C. R. 30, 551 (1850); H. L. Fizeau, *ibid.* 30, 562 (1850).

[7]A. Einstein, Ann. Phys. (Leipzig) 17, 132 (1905).
[8]A. H. Compton, Phys. Rev. 21, 483 (1923); 22, 409 (1923).
[9]E. C. G. Sudarshan, Phys. Lett. 73A, 269 (1979).
[10]D. Gabor, J. Inst. Electr. Eng. 93, 429 (1964).
[11]E. Wolf, Proc. R. Soc. London A230, 246 (1955).
[12]J. H. Poynting, Philos. Trans. R. Soc. London 175, 343 (1884); O. Heaviside, Electrician 14, 178 (1885); 14, 306 (1885).
[13]M. Born and E. Wolf, *Principles of Optics* (Pergamon, Oxford, 1964); J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968).
[14]P. Roman and E. Wolf, Nuovo Cimento 17, 462 (1960).
[15]R. J. Glauber, Phys. Rev. Lett. 10, 84 (1963); Phys. Rev. 130, 2529 (1963); E. C. G. Sudarshan, Phys. Rev. Lett. 10, 277 (1963).
[16]E. C. G. Sudarshan, J. Math. Phys. Sci. 3, 121 (1969).
[17]Both the definition of the W function and the decomposition of the Poynting vector differ from those given in Wolf's paper, Ref. 2. The present definition automatically yields a real ray density $W(\vec{r},\vec{k})$. The decomposition of the Poynting vector for monochromatic radiation given in this paper is in terms of the ray wave number and the gradient of the ray density, reminiscent of the decomposition of the electromagnetic vertex of the nucleon.
[18]G. G. Stokes, Trans. Cambridge Philos. Soc. 9, 399 (1852); N. Wiener, Acta Math. 55, 117 (1930).
[19]E. P. Wigner, Phys. Rev. 40, 749 (1932); J. E. Moyal, Proc. Cambridge Philos. Soc. 45, 99 (1949).
[20]E. C. G. Sudarshan and N. Mukunda, J. Math. Phys. 20, 1801 (1979).
[21]Private communication.