DISCRETE SYMMETRIES AND CHIRALITY INVARIANCE IN QUANTUM FIELD DYNAMICS

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ABSTRACT

The aim of the present paper is to discuss systematically the discrete symmetry operations on a quantized field in interaction; and to base the introduction of the new quantum number "chirality" for spinor fields on these symmetry properties. In the course of this investigation, several general results on the group of symmetry operations are proved and relation between certain sets of discrete symmetry operations and the spinor representation of the rotation group in 3 and 4 dimensions is established. An attempt has been made to present clearly the connection between additive and multiplicative quantum numbers, gauge transformations, unitary transformations and invariance laws. The chirality invariance of spinor fields in interaction is discussed in some detail. The emphasis throughout is on the systematic development rather than on details of application. The paper is divided into two parts, the first dealing with the general theory of discrete symmetry operations and the second concerned with chirality invariance for spinor fields.

I. GENERAL PRINCIPLES

(1) *Additive and Multiplicative Operators.*—We wish to discuss briefly the two types of quantum numbers associated with symmetry operations. On the one hand, we have unitary operators U acting on many particle states yielding an eigenvalue which is the product of the eigenvalues of the corresponding operator acting on the single particle states from which the many particle states may be supposed to be constituted. For example, since

$$U a_r^* a_s^* \mid 0\rangle = U\, a_r^*\, U^{-1}\, U a_s^*\, U^{-1}\, U \mid 0\rangle$$

introducing the eigenvalues λ_{rs}, λ_r, λ_s

$$U a_r^* a_s^* \mid 0\rangle = \lambda_{rs} a_r^* a_s^* \mid 0\rangle \; U a_r^* \mid 0\rangle = \lambda_r a_r^* \mid 0\rangle,$$
$$U a_s^* \mid 0\rangle = \lambda_s a_s^* \mid 0\rangle,$$

one has the multiplicative relation

$$\lambda_{rs} = \lambda_r \lambda_s.$$

On the other hand, additive quantum numbers like the number operator $N = \sum_n a_n{}^* a_n$ has eigenvalues N_{rs}, N_r, N_s which satisfy the additive relation

$$N_{rs} = N_r + N_s.$$

It is the latter group of operators which illustrate most directly the particle properties of a quantised field; and the familiar dynamical quantities like energy momentum and electric charge all belong to this class.

In practice, the multiplicative operators of physical significance are always unitary and hence have eigenvalues of modulus unity. While in general all unimodular eigenvalues are permissible, we shall find it convenient to confine our attention at the outset to those cases where $U^2 = +1$. Here we point out an important elementary connection between such multiplicative unitary operators and additive operators with integral eigenvalues.

Operating on one particle states the eigenvalues of operators U are always ± 1 and may hence be replaced by a matrix w with eigenvalues ± 1. Since the transformation is unitary, it maps creation operators into creation operators and we restrict ourselves to those cases where the mapping takes one particle states into one particle states. One can hence always put

$$U \mid 0\rangle = \mid 0\rangle$$

without introducing any inconsistency. We note that the effect of the operator U on a product of field operators A.B.C. ......is given by the rule:

$$A \rightarrow UAU^{-1} = (wA)$$

$$A.B.C..... \rightarrow UAU^{-1}.UBU^{-1}.UCU^{-1}....$$

$$(wA)(wB)(wC).... = \Omega_\pi A.B.C....$$

By the definition $U \mid 0\rangle = \mid 0\rangle$, acting on any state Ω_π becomes identical to U.

On the other hand, we may define an additive operator Ω_σ by the relation

$$\Omega_\sigma A.B.C.... \equiv (wA).B.C.... + A.(wB).C.... + A.B.(wC).... + ...$$

Such an operator satisfies the equation

$$\Omega_\sigma A = A\Omega_\sigma + wA \quad \text{or} \quad [\Omega_\sigma, A] = wA.$$

The solution of this equation can be constructed as follows: diagonalise w. The creation operators separate into two classes corresponding to the eigenvalues ± 1. The destruction operators also possess the same eigenvalue for w as the corresponding creation operators. For every creation operator C_n^* construct the number operator $C_n^* C_n$. The additive operator Ω_σ is then defined by the relation:

$$\Omega_\sigma = \sum_n{}^{(+)} C_n^* C_n - \sum_n{}^{(-)} C_n^* C_n$$

$\Sigma^{(+)}$ running over positive eigenvalues and $\Sigma^{(-)}$ over negative eigenvalues for w. Direct calculation shows that the operator so defined satisfies the requirements imposed on it.

With suitable modification of the steps, the restriction to the eigenvalues ± 1 may be relaxed.

(2) *Discontinuous Symmetry Operations on a Spinor Field.*—The following discontinuous operations on the field operator are standard:

$$\psi_{(x)} \to U_p \psi_{(x)} U_p^{-1} = \beta\psi(-x)$$

$$\bar{\psi}_{(x)} \to \bar{\psi}(-x)\,\beta \qquad \text{Space reflection}$$

$$\psi_{(x)} \to U_C \psi_{(x)} U_C^{-1} = C'\psi^+_{(x)} = C\bar{\psi}^T_{(x)} \qquad \text{Charge conjugation}$$

$$\bar{\psi}_{(x)} \to \bar{\psi}^T C'^{-1}$$

In each of these cases, U is a unitary operator acting on the Hilbert space which can be constructed explicitly. Notice that in all cases $U^2 = +1$ and hence the eigenvalues of U operating on any state are ± 1 and thus define "parities".

Anticipating our discussion in Chapter II, we shall now introduce an additional operation valid only for a spinor field of vanishing mass:

$$\psi_{(x)} \to U_x \psi_{(x)} U_x^{-1} = \gamma_5 \psi_{(x)}$$

$$\bar{\psi}_{(x)} \to -\bar{\psi}_{(x)}\,\gamma_5 \qquad \text{Chiral adjoint.}$$

Again one notices that $U_x^2 = +1$ and U_x operating on a one particle state yields its chiral number. A state possessing an even (odd) number of negative chiral particles has the eigenvalue $+1\,(-1)$ for U_x.

(3) *The Algebra of Symmetry Operators.*—The method of replacing unitary transformation U by their associated matrix operators w gives a

convenient method of studying the group of symmetry operations. Such a group is completely defined by specifying the transformation properties of a complete set of fundamental operators (*i.e.*, the set of operators which cannot be obtained by algebraic manipulations from other operators already considered). Since the unitary transformations considered preserve adjoint relations, such a complete set is provided by the 4 sets a_r^* (k), $b_r^*(k)$ $(r = 1, 2)$. The matrix operators w perform linear transformations in the vector space of the fundamental field operators $a_r^*(k)$, $b_r^*(k)$. From the definition of the symmetry transformations one has the following results:

Charge conjugation C: $a_{1,2}^*(k) \rightarrow b_{1,2}^*(k)$ $b^*_{1,2}(k) \rightarrow a^*_{1,2}(k)$

Space reflection P: $a^*_{1,2}(k) \rightarrow a^*_{2,1}(-k)$ $b^*_{1,2}(k) \rightarrow -b^*_{2,1}(-k)$

Chiral adjoint X: $a_2^*(k) \rightarrow -a_2^*(k)$ $b_1^*(k) \rightarrow b_1^*(k)$

$a_1^*(k) \rightarrow a_1^*(k)$ $b_2^*(k) \rightarrow -b_2^*(k)$

The corresponding unitary operators U_c, U_p, U_x can be constructed using standard methods[4,5] and yield the following expressions:

$$U_c = \pi_k \{1 - a_1^*(k) a_1(k) - b_2^*(k) b_2(k) + a_1^*(k) b_2(k) + b_2^*(k) a_1(k)\} \{1 - a_2^*(k) a_2(k) - b_1^*(k) b_1(k) + a_2^*(k) b_1(k) + b_1^*(k) a_2(k)\}.$$

$$U_p = \pi_k \{1 - a_1^*(k) a_1(k) - a_2^*(k) a_2(k) + a_1^*(k) a_2(-k) + a_2^*(-k) a_1(k)\} \{1 - b_1^*(k) b_1(k) - b_2^*(-k) b_2(-k) - b_1^*(k) b_2(-k) - b_2^*(-k) b_1(k)\}.$$

$$U_x = \pi_k \{1 - 2a_2^*(k) a_2(k)\} \{1 - 2b_2^*(k) b_2(k)\}.$$

One can also introduce a particle adjoint by the rule:

Particle adjoint ν: $a_r^*(k) \rightarrow a_r^*(k)$ $b_r^*(k) \rightarrow -b_r^*(k)$

$$U_\nu = \pi_{k,r} \{1 - 2b_r^* b_r\}.$$

Since fundamental field operators form a (closed) representation space $\mathfrak{S}$ for the non-singular operators C, P, X, ν one can consider new transformations generated by successive applications of these. Thus we are led to the following transformations:

$$\begin{array}{ccc} CP & CX & C\nu \\ PX & P\nu & X\nu \end{array}$$

and to products of more than two factors. In associating w_i with the U_i so generated, one has to see whether the eigenvalues, in $\mathfrak{S}$, of the product operation is real or imaginary. Thus for example

$$(C\nu) \quad A\,(C\nu)^{-1} = iwA \qquad w \text{ hermitian}$$

since the w matrices for C and ν anticommute.

By virtue of the fact that $U_i^2 = +1$, the algebra generated by n operators U_i consists of $n!$ distinct elements in general. However since the one particle submatrices for G, C, P, X, ν either commute or anticommute so that in $\mathfrak{S}$ the number of distinct matrices one has to consider is reduced to

$$n + \frac{n\,(n-1)}{2\,!} + \ldots . = 2^n - 1.$$

The following comments are generally valid:

(*i*) The hermitian w_i have the composition property

$$w_{R_1} w_{R_2} = k w_{R_1} w_{R_2}$$

where $k = +1$ if R_1 and R_2 commute and $k = -i$ if they anticommute (operating on $\mathfrak{S}$).

(*ii*) Any algebraically independent subset $R_1, R_2, \ldots, R_h$ generates a subgroup of order 2^h including the identity.

(*iii*) In particular, any two elements R_1, R_2 generate a group of order 4 which is isomorphic to the Abelian group generated by σ_3 and -1 if R_1, R_2 commute and isomorphic to the Pauli group generated by σ_1, σ_2 if R_1, R_2 anticommute.

The introduction of the hermitian w_i enables us to define the additive quantum numbers Ω_{σ_i} associated with U_i. The following correspondences may be directly verified:

X $\leftrightarrow$ total chiral number

ν $\leftrightarrow$ Number of particle—number of antiparticles.

P ⟷ Number of positive parity particles—Number of negative parity particles.

(4) *Conjugate Symmetry Operations.*—For a pair of anticommuting operations R, S the associated hermitian matrices obey the law of composition:

$$w_{RS} = - w_R w_S.$$

We shall say that such a pair of R and S are "conjugate" operations.[6] Examples are (C, ν), (C, $iC\nu$), (P, X), etc. One recalls that the $\mathfrak{S}$ matrices w_R, w_S, w_{RS} are isomorphic to the Pauli matrices σ_1, σ_2, σ_3. We shall now prove the following;

LEMMA I.—Under each transformation of a conjugate pair of symmetry operators, the 4 sets of field operators $a_r^* (k)$, $b_r^* (k)$ are spanned by two equal groups (of vectors in $\mathfrak{S}$) one corresponding to the eigenvalue $+ 1$ and the other to $- 1$.

The proof is immediate if one diagonalises the corresponding hermitian w_R. Since $U_R^2 \neq + 1$, $w_R^2 = + 1$ and hence its eigenvalues are ± 1. But,

$$\mathrm{Tr}\, \{w\}_S = \mathrm{Tr}\, \{i w_{RS} w_R\} = \frac{i}{2} \mathrm{Tr}\, \{w_{RS} w_R + w_R w_{RS}\} = 0$$

and hence the eigenvalues should be ± 1 in pairs.

Let the linear set in $\mathfrak{S}$ corresponding to the eigenvalue $+ 1$ of w be labelled η_λ $(\lambda = 1 \cdot 2)$ so that

$$w_R \eta_\lambda = + \eta_\lambda.$$

It then follows that ζ_λ defined by

$$\zeta_\lambda = \omega_S \eta_\lambda$$

satisfy the relation

$$w_R \zeta_\lambda = w_R w_S \eta_\lambda = - w_S w_R \eta_\lambda = - \zeta_\lambda$$

and thus belongs to the eigenvalue $- 1$. The operation of multiplication by w_S, which is equivalent to a unitary transformation by U_S, thus changes the sign of the eigenvalues. We thus have:

LEMMA II.—The unitary transformations U_S associated with one of a pair (R, S) of conjugate symmetry operations causes the eigensubspaces (in $\mathfrak{S}$) of the other operator to be conjugated.

This result is most familiar to us in the form in which R = C and S = ν, so that the particle number (" charge ") eigenstates are conjugated by the charge conjugation operator and *vice versa.*

In passing we notice that if R and S are conjugate, so are (iRS, R) and (iRS, S). Also if Q and R commute as well as Q and S, then (QR, S), (R, QS) and (QR, QS) are also conjugate pairs. This implies that, given an operator R, its conjugate S is not unique and to define S uniquely, additional requirements should be placed on it.[1]

(5) *Operator Gauge Transformations.*—Among the conserved integer-valued quantum numbers that we have considered, there are serveral which are " absolutely " conserved by any " physical " interactions. By physical, here we mean any existing interaction, however weak, consistent with our present knowledge. In such a case the states with different values of this number are physically disjoint and the well-known irrelevance of the absolute phase factor of state vectors gets extended into an arbitrary phase assignment for each of the orthogonal subspaces of the Hilbert space referring to different values for the absolutely conserved quantum number considered.[7] This corresponds to a unitary transformation in the Hilbert space of states and an associated unitary transformation on the field operators which leaves the physical content of the theory unaltered. In case the field operators have matrix elements connecting these different orthogonal subspaces (and are, by definition, not observables), this unitary transformation leads to an operator gauge transformation of the field operators. We turn to a few special cases of the same.

Consider a familiar case: as far as we know, the electric charge Q is conserved absolutely. Hence different eigensubspaces of Q are physically disjoint. Hence the phase transformation

$$| Q\rangle \rightarrow \exp. \{i\lambda (Q)\} \, | Q\rangle$$

where λ (Q) is a (numerical-valued) function of the operator Q, leaves physical predictions unaltered. On the operators ψ this lead to the operator gauge transformation

$$\psi \rightarrow e^{i\lambda (Q)} \psi e^{-i\lambda (Q)} = e^{i\{\lambda (Q) - \lambda (Q+q)\}} \psi$$

where

$$[\psi, Q] = q\psi \qquad q = \pm 1, 0$$

For the particular choice

$$\lambda (Q) = \lambda \, . \, Q$$

λ being a c-number, the operator gauge transformation degenerates into a familiar c-number gauge transformation of the first kind:

$$\psi \rightarrow e^{-i\lambda q}\psi.$$

Similar results hold for other absolutely conserved operators.

The same concepts are applicable to the symmetry operators and related additive operators Ω_σ. Consider for example, the one particle eigenstates of chirality. If the total Lagrangian absolutely conserves chirality, there is no way of measuring the relative phase factors of the different eigenstates and there is an essential arbitrariness of these phase factors. The chiral conjugation operator is also arbitrary then to the extent that interchanging states of opposite chirality with fixed (but arbitrary) phase factors is its generic function. A phase transformation of the type discussed above on the Hilbert space would then induce associated rotations of the chiral conjugation operator (in the abstract 3-space of the Pauli group generated by these conjugate operators).

(6) *Continuous Transformation Generated by Discrete Operators.*—The gauge transformations resulting from a conserved discrete operator form a continuous group with the discrete operator acting as a generator. It is the aim of this section to exploit this connection and develop the relation of (discrete) conjugate symmetry operators to a continuous rotation group.

Consider the rotation group in a 3-dimensional space and an arbitrary representation by matrices S_1, S_2, S_3 of the infinitesimal generators of rotations about 3 arbitrarily chosen (orthogonal) axes. Then the reflection operators are distinct from the spin operators S_i in all cases except for spin $\frac{1}{2}$. In this unique case, the reflection and spin operators are identical. This circumstance of the basic spinor rotation operators representing reflections is fundamental in the geometric theory of spinors.

Given a 2-component spinor

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

the reflection with respect to a plane normal to the unit vector n is defined by the linear transformation[8]:

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \rightarrow \begin{pmatrix} n_3 & n_1 - in_2 \\ n_1 + in_2 & -n_3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \quad \text{or} \quad \xi \rightarrow \sigma . n \, \xi.$$

The rotation around the axis n through an angle θ can be generated by two successive reflections in planes passing through n and inclined at an angle

$\theta/2$. The matrix operators σ thus play a dual role. Any representation of all three reflection operators automatically leads to a representation of the continuous rotation group. This property is shared by no other representation and singles out the fundamental spinor representation of the rotation group.

The discrete symmetry operations which we have considered so far have the property that they have only two distinct eigenvalues ± 1. They consequently admit a 2-component representation. We know, further, that two conjugate symmetry operations generate a group isomorphic to the Pauli group. In view of the above identification of the rotation and reflection operations for two component representations, we are led to assert that any two conjugate symmetry operations generate a group of continuous rotations in an abstract 3-dimensional space and yield a spinor representation of this group.

Take for example charge conjugation (C) and chirality (X). Call the operator iCX by Y. The creation operators of a positive chiral eigenstate a^* and its antiparticle b^* (of automatically the opposite chirality), or equivalently, the one particle states $a^* \mid 0\rangle \equiv \mid A\rangle$, $b^* \mid 0\rangle \equiv \mid B\rangle$, yield the following representation for C, Y, X:

$$C\begin{pmatrix} |A\rangle \\ |B\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} |A\rangle \\ |B\rangle \end{pmatrix} \qquad Y\begin{pmatrix} |A\rangle \\ |B\rangle \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} |A\rangle \\ |B\rangle \end{pmatrix}$$

$$X\begin{pmatrix} |A\rangle \\ |B\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} |A\rangle \\ |B\rangle \end{pmatrix}.$$

If n is a unit vector the operations

$$\begin{pmatrix} |A\rangle \\ |B\rangle \end{pmatrix} \rightarrow \begin{pmatrix} n_3 & n_1 - in_2 \\ n_1 + in_2 & -n_3 \end{pmatrix}\begin{pmatrix} |A\rangle \\ |B\rangle \end{pmatrix} = (n_1 C + n_2 Y + n_3 X)\begin{pmatrix} |A\rangle \\ |B\rangle \end{pmatrix}$$

form the group of rotations of the states in this abstract space.

A related group has been discussed by Gursey[9] where one chooses ν, C, $iC\nu$ to be the three operators generating the rotation group. They identify the structure of this group with that of the internal symmetry associated with the isotopic spin. We wish to re-emphasize here that the distinction between reflection operations and internal rotation symmetries of spin $\frac{1}{2}$ is one of terminology only.

(7) *Three- and Four-Dimensional Rotation Groups.*—From the earlier sections it is quite clear that one can construct as many 3-dimensional rotation groups as there are algebraically independent pair

of conjugate symmetry operations. The structure of the rotation group in the distinct 3-dimensional spaces constitute the complete geometrical group of rotations and reflections in a 4-dimensional space. From each distinct couple of algebraically independent pairs of conjugate operations, one can construct a 4-dimensional rotation group. In the language of the theory of group representations, the direct product of two spinor representations is equivalent to the direct-sum of two spinor representations.

The most familiar example of this type is the composition of the spatial rotation group σ with the group generated by P and X. This was employed by Dirac[10] in the construction of the γ matrices in the following form:

$$\sigma = (\sigma_1, \sigma_2, \sigma_3) \quad \rho = (\rho_1, \rho_2, \rho_3)$$

$$\rho_3 = \mathrm{X} \quad \rho_1 = \mathrm{P} \quad \rho_2 = i\mathrm{PX} \quad i.e., \quad \rho_1 = \beta, \quad \rho_2 = i\beta\gamma_5, \quad \rho_3 = \gamma_5.$$

In terms of these the vector matrices (corresponding to reflection operators) assume the form:

$$\gamma^1 = \rho_3\sigma_1 \qquad \gamma^2 = \rho_3\sigma_2 \qquad \gamma^3 = \rho_3\sigma_3 \qquad \gamma^4 = \rho_1$$

where, for convenience, we have constructed the matrices applicable to a Euclidean space (all γ^μ hermitian) rather than for a Lorentz space.[8] A similar method has been employed by Schwinger in his formulation of a 4-dimensional internal symmetry group.

In our case such a group can be constructed, for example, by choosing

$$\sigma = (\nu, \mathrm{CP}, i\mathrm{CP}\nu) \qquad\qquad \rho = (\mathrm{X}, i\mathrm{XP}, \mathrm{P}).$$

The four vector matrices are:

$$\gamma^1 = \mathrm{P}\nu \quad \gamma^2 = \mathrm{C} \quad \gamma^3 = i\mathrm{C}\nu \quad \gamma^4 = \mathrm{X}$$

with corresponding rotation operators

$$\sigma^{12} = i\mathrm{CP}\nu \qquad \sigma^{23} = \nu \qquad \sigma^{31} = \mathrm{CP}$$

$$\sigma^{34} = \mathrm{CX}\nu \qquad \sigma^{14} = i\mathrm{XP}\nu \qquad \sigma^{42} = i\mathrm{CX}$$

and the set of disjoint operators $\sigma^{\pm}$

$$\sigma_+^{\,1} = \tfrac{1}{2}\,\nu\,(1+i\mathrm{XP}) \quad \sigma_+^{\,2} = \tfrac{1}{2}\,\mathrm{C}\,(\mathrm{P}+i\mathrm{X}) \quad \sigma_+^{\,3} = \tfrac{1}{2}\mathrm{C}\nu\,(\mathrm{X}+i\mathrm{P})$$

$$\sigma_-^{\,1} = \tfrac{1}{2}\nu\,(1-i\mathrm{XP}) \quad \sigma^{-2} = \tfrac{1}{2}\,\mathrm{C}\,(\mathrm{P}-i\mathrm{X}) \quad \sigma_-^{\,3} = \tfrac{1}{2}\mathrm{C}\nu\,(-\mathrm{X}+i\mathrm{P})$$

from independent rotations.

A physically more interesting case is got by combining a natural 3-dimensional rotation group with a set of independent conjugate symmetry operations. Such a group is generated by combining the isotopic spin symmetry with the operations X and P. We put

$$\sigma = (\tau_1, \tau_2, \tau_3) \qquad \rho = (i\mathrm{PX}, \mathrm{X}, \mathrm{P}).$$

The four vector matrices and rotation operators are

$$\gamma^1 = \mathrm{P}\tau_1 \qquad \gamma^2 = \mathrm{P}\tau_2 \qquad \gamma^3 = \mathrm{P}\tau_3 \qquad \gamma^4 = i\mathrm{PX}$$

and

$$\sigma^{23} = \tau_1 \qquad \sigma^{31} = \tau_2 \qquad \sigma^{12} = \tau_3$$

$$\sigma^{14} = \tau_1\mathrm{X} \qquad \sigma^{42} = \tau_2\mathrm{X} \qquad \sigma^{34} = \tau_3\mathrm{X}.$$

The rotation operators thus correspond to ordinary and chiral isotopic rotations. The independent rotations are:

$$\sigma_{\pm}{}^1 = \tfrac{1}{2}(1 \pm \gamma_5)\,\tau_1, \;\; \sigma_{\pm}{}^2 = \tfrac{1}{2}(1 \pm \gamma_5)\,\tau_2, \;\; \sigma_{\pm}{}^3 = \tfrac{1}{2}(1 \pm \gamma_5)\,\tau_3$$

and thus correspond to the rotation of either chiral state separately. This case exhibits the decomposition of the direct product into a direct sum most clearly.

II. CHIRALITY INVARIANCE IN SPINOR FIELD DYNAMICS

(1) *Structure of Spinor Field Dynamics and Chirality Invariance.*—The field theory of elementary particles is patterned after the dynamics of systems with a finite number of degrees of freedom and, likewise, can be given a systematic development based on the fundamental action principle. Naturally the enumeration of the independent degrees of freedom, *i.e.*, the kinematic characterization of the dynamical system, should in turn decide the *structure* of its equations of motion, though the actual *temporal development* depends on the details of the *dynamics*. In a covariant field theory, where space and time derivatives appear on an equal footing, this intrinsic property of dynamical systems is nevertheless preserved. The complete specification of the system is contained in the action principle; and this in particular implies the generators of changes in the field operators and, from these, the commutation relations.[11]

The spinor field is the simplest dynamical system from this view-point in the sense that all the field components are independent dynamical variables. In the present paper we shall be mainly concerned with the chiral symmetry of spinor fields and a natural starting point for a systematic development of this is to write down the Lagrangian density:

$$L(x) = \tfrac{1}{2}[\psi^*, \; i\beta\gamma^\mu\partial_\mu\psi] + H(\psi)$$

where H (ψ) does not involve any derivatives of ψ and is called the dynamical term. The generator of changes in the field operators is

$$G_\psi = i \int d^3x \psi^*_{(x)} \, \delta\psi_{(x)} \qquad [\psi_{(x)}, \; G_\psi] = i\delta\psi_{(x)}$$

and, consequently, the commutation relations assume the form

$$\{\psi^*_{(x)}, \; \psi_{(x)}'\}_+ = \delta^{(3)}(x - x').$$

It is to be explicitly noticed that these commutation relations *do not* depend upon the structure of H (ψ) as long as no derivatives of ψ are involved in H (ψ). We have taken ψ to be non-hermitian and hence we have 2×4 independent sets of dynamical variables and consequently, for each value of x, 4 independent degrees of freedom.

We now notice that the two projection operators

$$\frac{1 \pm \gamma_5}{2}$$

have the property of preserving adjoint relations and lead to uncoupled degrees of freedom, when operating on the field operators ψ, ψ^*. Define:

$$\psi_\pm(x) = \frac{1 \pm \gamma_5}{2}\psi(x) \qquad \psi_\pm{}^*(x) = \psi^*(x)\frac{1 \pm \gamma_5}{2}.$$

Then the adjoint relations and commutation relations become:

$$\psi_\pm{}^* = (\psi_\pm)^* \qquad \{\psi_\pm{}^*(x), \; \psi_\mp(x')\}_+ = 0.$$

$$\{\psi_\pm{}^*(x), \; \psi_\pm(x')\}_+ = \delta^{(3)}(x - x')$$

where, as above, we have suppressed the indices. We shall call $\psi_\pm$ as the two *chiral* fields and, by convention, call $\psi_\pm$ the positive (and $\psi_{...}$ the negative) chiral field.[3, 6] This possibility of exhibiting the two degrees of freedom separately is closely connected with the *Chirality invariance* of the kinematic part

$$L_{kin} = \tfrac{1}{2}[\psi^*, \; i\beta\gamma^\mu\partial_\mu\psi]$$

of the Lagrangian. By this we mean that the chiral transformation

$$\psi_{(x)} \to \gamma_5\psi_{(x)} \qquad \psi^*_{(x)} \to (\gamma_5\psi_{(x)})^* = \psi^*_{(x)}\gamma_5$$

of the field operator ψ, leaves L_{kin} invariant. It is essential to recognize, that no specific assumption about the invariance of H (ψ) has been made so far. The structure of the kinematic term alone makes it legitimate to talk about the chiral label (just like the spin label) though before it can be identified with a quantum number, the properties of the dynamical term H (ψ) must be ascertained.

In fact, the kinematic independence of the chiral degrees of freedom ceases to be a dynamic independence if H (ψ) is not invariant under the chiral transformation, the two chiral modes become coupled in such a case. The best example is the mass term $\frac{1}{2}m\,[\psi^*, \beta\psi]$. Consequently, to identify the quantum number associated with the kinematic independence, it is necessary to take a system where it becomes a dynamic independence, *i.e.*, one in which H $(\psi) =$ H $(\gamma_5\psi)$. A trivial case of this type is got by putting H $(\psi) = 0$ obtaining a spinor field with zero mass. In this case the *conserved quantum number* associated with chirality invariance is simply the sign of the polarisation in the direction of its momentum (sometimes called "helicity"). But we employ the term chirality in its fundamental dynamical significance as a covariant label for two classes of independent modes, even in those cases where the corresponding quantum number is not conserved. Thus chirality invariance and chiral transformation are concepts independent of whether or not the corresponding quantum number is conserved or not.

In fact, it is meaningful to consider the kinematic part alone as the "free field" Lagrangian and consider all the rest to be interactions. The mass terms in H (ψ) would then become a "true" interaction: and the free field would correspond to quanta of zero mass. Such a separation into "free" and "interaction" parts is best suited for discussions of chirality invariance; and is not inconsistent with the point of view that the "mass" is the phenomenological expression for the modification of space-time behaviour at very short separations.

The linear transformation of the field operator which we have called the chiral transformation can be generated by a unitary transformation on the field operator. The interaction representation corresponding to the above-discussed separation into free and interaction parts L_{kin} and H (ψ) respectively permits one to write down this unitary operator U_x:

$$U_x = \pi_{k} \{1 - 2a_2^*(k)\, a_2(k)\}\, \{1 - 2b_1^*(k)\, b_1(k)\}.$$

Here the a_r, b_r, $a_r{}^*$, $b_r{}^*$ are creation and destruction operators satisfying usual commutation rules in terms of which the field operators may be expanded:

$$\{a_r,\ a_s{}^*\}_+ = \delta_{rs};\quad \{b_r,\ b_s{}^*\}_+ = \delta_{rs};$$

all other anticommutators = 0

$$\psi_+ = \sum_k \{a_1(k)\, u_1(k)\, e^{ikx} + b_1{}^*(k)\, v_1(k)\, e^{-ik.x}\}$$

$$\psi_- = \sum_k \{a_2(k)\, u_2(k)\, e^{ikx} + b^*{}_2(k)\, v_2(k)\, e^{-ik.x}\}.$$

Notice that the labels 1, 2 here refer to chiral eigenstates (and not to spin eigenstates). To verify that the expression for U_x is the correct one, one simply writes down the relations:

$$\gamma_5 u_1(k) = +\, u_1(k) \qquad \gamma_5 v_1(k) = -\, v_1(k)$$

$$\gamma_5 u_2(k) = -\, u_2(k) \qquad \gamma_5 v_2(k) = +\, v_2(k),$$

and recognises that

$$U_x a_1 U_x^{-1} = +\, a_1 \qquad U_x b_1 U_x^{-1} = -\, b_1$$

$$U_x a_2 U_x^{-1} = -\, a_2 \qquad U_x b_2 U_x^{-1} = +\, b_2.$$

In passing we also notice the result:

$$U_x H(\psi) U_x^{-1} = H(\gamma_5 \psi).$$

It is instructive to write down the expressions for a few related operators in this representation. The familiar space reflection operation (parity) has the corresponding unitary operator $U_p{}^{el}$:

$$\begin{aligned} U_p{}^{el} = \pi_k \{1 &- a_1{}^*(k)\, a_1(k) - a_2{}^*(-k)\, a_2(-k) \\ &+ a_1{}^*(k)\, a_2(-k) + a_2{}^*(-k)\, a_1(k)\} \\ \{1 &- b_1{}^*(k)\, b_1(k) - b_2{}^*(-k)\, b_2(-k) \\ &- b_1{}^*(k)\, b_2(-k) - b_2{}^*(-k)\, b_1(k)\}. \end{aligned}$$

We also introduce two integral valued additive operators

$$N_1 = \sum_k \{a_1{}^*(k)\, a_1(k) - b_1{}^*(k)\, b_1(k)\}$$

$$N_2 = \sum_k \{a_2{}^*(k)\, a_2(k) - b_2{}^*(k)\, b_2(k)\}$$

which represent the number operators for the positive and negative chiral modes of the field operator ψ. They are the space integrated 4th components of conserved chiral currents:

$$J_1^{\mu} = \tfrac{1}{2}[\psi_+^*, \ \beta\gamma^{\mu}\psi_+] \qquad J_2^{\mu} = \tfrac{1}{2}[\psi_-^*, \ \beta\gamma^{\mu}\psi_-].$$

Alternatively one may define the chiral current and the total current:

$$j^{\mu} = J_1^{\mu} - J_2^{\mu} \qquad J^{\mu} = J_1^{\mu} + J_2^{\mu}$$

and the operators of *chiral number* and fermion number:

$$N_x = N_1 - N_2$$
$$N = N_1 + N_2.$$

In particular the one particle state $a_r^* \mid 0\rangle$, $b_r^* \mid 0\rangle$ have the following chiral numbers:

$$N_x a_1^* \mid 0\rangle = + a_1^* \mid 0\rangle \qquad N_x b_1^* \mid 0\rangle = - b_1^* \mid 0\rangle$$
$$N_x a_2^* \mid 0\rangle = - a_2^* \mid 0\rangle \qquad N_x b_2^* \mid 0\rangle = + b_2^* \mid 0\rangle.$$

Particles and antiparticles have thus opposite chiral numbers.

The conservation (or lack of it) of these dynamical quantities would depend upon the nature of the dynamical term. Our programme in the succeeding sections is to see to what extent various interactions permit the invariance under the chiral transformation. The point of view of first looking for invariance properties of the kinematic Lagrangian and then studying the transformation properties of the dynamic term under the corresponding transformations is perfectly general and very fruitful in the systematic study of symmetry properties and conservation laws. We hope to come back to this point of view elsewhere.

(2) *Electromagnetic Interaction.*—A non-trivial example of the chirality invariance of the dynamical term is the coupling of a (zero mass) spinor field to the electromagnetic field. The interaction term is obtained by replacing ∂_{μ} by $-ieA_{\mu}$ in the kinematic Lagrangian:

$$H(\psi) = \tfrac{1}{2}eA\mu\,[\psi^*, \ \beta\gamma^{\mu}\psi] \qquad H(\gamma_5\psi) = U_x H(\psi) U_x^{-1} = H(\psi).$$

Thus chirality invariance now becomes a dynamical invariance and the associated quantum number like chiral number are constants of the motion; this is most clearly seen by rewriting $H(\psi)$ in terms of $\psi_{\pm}$:

$$H(\psi) = \tfrac{1}{2}eA\mu\,\{[\psi_+^*, \ \beta\gamma^{\mu}\psi_+] + [\psi_-^*, \ \beta\gamma^{\mu}\psi_-]\}$$
$$= \tfrac{1}{2}e\,\{A_0([\psi_+^*, \ \psi_+] + [\psi_-^*, \ \psi_-]) + A_k([\psi_+^*, \ a^k\psi_+]$$
$$+ [\psi_-^*, \ a^k\psi_-])\}.$$

Thus ψ_+ and ψ_- act as completely independent fields and their number operators are separately conserved. Thus the electromagnetic interaction is chirality invariant, but *both* chiral components are coupled (with definite phase factors).

As is well known, the electromagnetic interaction is also space reflection (parity) invariant. The parity operation has the unitary operator U_p with the properties:

$$U_p\psi_{(x)}\, U_p^{-1} = \beta\psi\,(-x) \qquad U_p A_\mu\,(x)\, U_p^{-1} = A^\mu\,(-x)$$

$$g^{\mu\nu} = 1,\ -1,\ -1,\ -1$$

and since $\beta\gamma^\mu\,\beta = \gamma_\mu$, it follows that

$$U_p H\,(\psi_{(x)})\, U_p^{-1} = H\,(\psi_{(-x)}) \qquad U_p L_{kin}\,(x)\, U_p^{-1} = L_{kin}\,(-x)$$

which verifies the space reflection invariance. This possibility of the same interaction being chirality and parity invariant [despite the anticommutation of the corresponding linear operators γ_5 and β, when acting on the field operator ψ] stems from the fact that one of the interactions occurs in *both* the chiral eigenstates with a definite phase factor. In fact, if the phase factor is altered or the coupling constants altered in any fashion, the space reflection invariance would be destroyed.

Another conjugate operation to chirality is the well-known charge conjugation operator. [Two symmetry operations, whose squares are the identity and whose representative matrices on the field operators anticommute, are called *conjugate operations.*] Charge-conjugation invariance expresses the basic symmetry of the theory under the interchange of positive and negative electric charges. The corresponding unitary operator in our notation is:[4, 5]

$$U_c = U_c^{el}.U_c^{ph} \qquad U_c^{ph} eA_\mu\,(x)\, U_c^{ph-1} = -\, eA_\mu\,(x).$$

$$\begin{aligned} U_c^{el} = \pi\,\{&1 - a_1^*\,(k)\, a_1\,(k) - b_2^*\,(k)\, b_2\,(k) \\ &+ a_1^*\,(k)\, b_2\,(k) + b_2^*\,(k)\, a_1\,(k)\} \\ \times\,\{&1 - a_2^*\,(k)\, a_2\,(k) - b_1^*\,(k)\, b_1\,(k) + a_2^*\,(k)\, b_1\,(k) \\ &+ b_1^*\,(k)\, a_2\,(k)\}. \end{aligned}$$

The invariance may now be explicitly verified. But we also notice that

$$U_c a^*_{1,2}\,(k)\, U_c^{-1} = b_{1,2}^*\,(k) \qquad U_c b_{1,2}^*\, U_c^{-1} = a_{1,2}^*\,(k)$$

and hence under charge conjugation, the chiral numbers change sign:

$$N_x U_c\,|\,..> \;=\; -\, U_c N_x\,|\,..>.$$

Again, the possibility of invariance under the two conjugate operations follows from the fact that both the particle and antiparticle states are coupled symmetrically (for each chiral component of ψ).

Since the spinor field in interaction with the electromagnetic field possesses invariance under chirality (X), parity (P) and charge conjugation (C), it is invariant under the compound operations built from these, namely, second chirality (XP), conjugate chirality (XC), combined inversion (CP) and third chirality (CPX). The explicit construction of the unitary operators U_i can be carried out using standard methods but we shall omit further discussion of these. But suffices it to note that the electromagnetic interaction manifests a remarkable degree of symmetry.

So far, we have had no need to specify whether A_μ was a quantized dynamical field, or a prescribed external field. In all cases the chiral number was a constant of motion. But if A_μ is a quantized field, while leaving the chiral number unaltered, it alters the "composition" of the state and this makes itself felt in the enhancement (or damping) of chirality destroying couplings. For example, if one introduces a mass term this can be thought of as a (chirality destroying) coupling to a uniform external field of zero frequency. The amplification of this "coupling" by the electromagnetic interaction exhibits itself as an electromagnetic contribution to the mass.

(3) *V–A Fourfermion Interaction.*—Among the interactions of spinor fields which are chirality invariant, the Fourfermion interactions, which are the weakest of all elementary particle interactions, is most notable.[3] The remarkable near equality of this effective weak coupling constant in various weak interactions led several authors to postulate a Universal interaction of definite pairs of charged and neutral fields. The maximal parity violations associated with minimal chiral couplings were instrumental in bringing order into the structure of these interactions. Our present knowledge is consistent with a coupling of only the positive chiral states of the pairs of fields np, $\mu\nu$, $e\nu$ (and possibly pairs like Λp, Ξp involving strange particles). The interaction is such as to conserve the number of leptons. For a review of the experimental data and the success of the predictions of the theory, reference may be made to two earlier papers.[12]

If (A_i, B_i) refer to the various pairs of charged and neutral fields taking part in the interaction, the interaction Lagrangian has the form:

$$L_{int} = \tfrac{1}{4} G \sum_{i,j} [\bar{A}_i, \ \gamma^\mu (1 + \gamma_5) B_i]^+ [\bar{A}_j, \ \gamma_\mu (1 + \gamma_5) B_j] + h.c.$$

An alternative way of rewriting this interaction is in the form:

$$L_{int} = (\tfrac{1}{2})\, G J'_{\mu}\, J'^{\mu} \qquad J'_{\mu} = \tfrac{1}{2} \sum_{i} [\bar{A}_i, \gamma_{\mu} (1 + \gamma_5) B_i]$$

G is a coupling constant with the dimensions of (length).^{2}

The following properties of this interaction can be noticed immediately. First, only one chiral eigenstate is coupled. This has the consequences of (maximal) violation of space reflection invariance [*cf.* the electromagnetic interaction] and of having for all practical purposes a neutrino with only positive chirality (and an antineutrino with only negative chirality). For other particles, the mass terms " create " negative chiral states from positive chiral states and *vice versa*, so that, for example, the electron from β-decay exhibits its full set of states. Second, since the interaction was so chosen as to be bilinear in spinor fields and their adjoints, the total fermion number, summed over all the fields, is conserved and this is unaltered even by the presence of (chirality non-conserving) mass terms in the dynamical term. (Law of Conservation of Leptons.) Third, the form we have adopted for the Fourfermion interactions leads to a self-coupling of a pair of fields with half the strength, the experimental effects, say on electron-neutrino scattering, are quite small and not inconsistent with our present knowledge. Finally, for any two pairs of fields (say $\mu\nu$, $e\nu$) the present form exhibits a symmetry on interchanging charged and neutral fields: (Equivalence of Charge-Exchange and Charge-Retention orders:)

$$[\bar{\mu}, \gamma^{\mu} (1 + \gamma_5)\, \nu]\ [\bar{e}, \gamma_{\mu} (1 + \gamma_5)\, \nu]$$

$$= [\bar{\nu}, \gamma^{\mu} (1 + \gamma_5)\, \nu]\ [\bar{e}, \gamma_{\mu} (1 + \gamma_5)\, \mu].$$

From the remark made in the earlier section regarding the connection between helicity and chirality for a " free " zero mass spinor field, we expect the neutrinos from the weak interactions to possess *negative* helicity. In the case of particles like the muon and electron which are " free " but have finite masses, the chiral number is not conserved but there will result a finite degree of polarisation [with the same sign as the chira number of the probability for zero mass case is finite and opposite otherwise *cf.* the polarisation of electrons from β-decay and muons from π-decay].

Since both the electromagnetic field and the Fourfermion interaction are chirality invariant, the conservation laws of chiral number, etc., are unaltered. From the general connection between conserved quantum number and gauge transformations, we are led to expect associated gauge transformations under which the theory is invariant. The most general gauge transformation for the present case is

$$\psi_{(\gamma)} \rightarrow \exp. (ia_{(\gamma)})\, \psi_{(\gamma)} \qquad a_{(\gamma)} = \lambda_\gamma (1 - \gamma_5) + \theta_1 (1 + \gamma_5) + \theta_2 q_{(\gamma)} + \theta_3 b_{(\gamma)}$$

where λ_γ, θ_1, θ_2, θ_3 are arbitrary real numbers (and λ_γ may be different for different fields). $q_{(\gamma)}$ and $b_{(\gamma)}$ are the electric charge and baryon number of the field $\psi_{(\gamma)}$. It is trivial to verify that the currents generated by the 4 parts of this gauge transformations are, respectively, the (individual) current of negative chiral particles, the (total) current of positive chiral particles, the (total) electric current and the (total) baryon current. We emphasise the fact that only the transformation $\psi_{(\gamma)} \rightarrow e^{i\lambda_\gamma (1-\gamma_5)}.\psi$ may be applied to each field separately while all other transformations should be applied to all the fields in interaction simultaneously.

The conservation laws are maintained if vector or axial vector couplings with boson fields are included. In particular, since the chiral transformations affect only the spinor fields, only the coupling matrix (and not the structure of the boson fields[13]) is relevant. In particular, a pseudovector coupling of pseudoscalar mesons with "nucleons" of zero mass leaves the abovementioned symmetries in tact.

4. *Mesic Couplings and Internal Symmetries.*—We briefly remarked in an earlier section that chirality invariances of spinor fields coupled to boson fields depended only on the covariant structure of the dynamical (coupling) term and not on the kinematic structure of the boson field. It then follows that a spin O meson field can be coupled to the spinor field in a chirality invariant fashion. Such a coupling of the pion and nucleon fields is the pseudovector interaction:

$$\tfrac{1}{2} g\, [\bar{\psi},\ \gamma_5\gamma_\mu\tau \cdot \phi^\mu\psi] = g\, (\mathbf{J}_1{}^\mu - \mathbf{J}_2{}^\mu) \cdot \phi_\mu$$

where $\mathbf{J}_1{}^\mu$ and $\mathbf{J}_2{}^\mu$ are the chiral isotopic spin currents. The minus sign is due to the occurrence of γ_5 in the coupling. Clearly, in spite of the isotopic spin structure, the basic chirality invariance is preserved and the number of chiral particles (summed over protons and neutrons) is conserved.* In addition, the isotopic spin symmetry leads to conservation of the *total* isotopic spin current $\mathbf{J}_1{}^\mu + \mathbf{J}_2{}^\mu$ but not of the two currents separately. However, if the pion mass is zero, the sum of the chiral current $\mathbf{J}_1{}^\mu - \mathbf{J}_2{}^\mu$ and $ig^{-1}\partial_\mu\phi$ is conserved by virtue of the equations of motion of the pion field.

* Despite these advantages, the role of the pion as the dynamical agency defining the "nucleonic charge" no longer obtains with a pseudovector coupling.

When the pion nucleon coupling is extended to include the various baryons and mesons, similar relation hold. If one sums over the various baryons, the number of positive and negative chiral particles are conserved; as well as the total isotopic spin current.

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