ACCRETION OF MATTER BY A SATELLITE

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Abstract. The changes in the motion of a satellite resulting from encounter with interplanetary material are investigated. Solutions are derived when the material moves in the plane of the circular satellite orbit and when the material at great distances approaches normally to the orbit.

In this paper we discuss changes in the mean motion of a planetary satellite arising from its encounter with gaseous or dust particles in space. We assume that the particles do not interact with one another and have, at infinity, a uniform density, ρ, and a uniform velocity, V_∞, relative to the planet.

If the gravitational effects of the planet and satellite were neglected, the capture cross-section of the satellite would be simply the geometrical cross-section πR^2, where R is the radius. The gravitational attraction of the planet, however, modifies the density and velocity of particles in its neighborhood and so modifies also the effective capture cross-section and rate of accretion. We shall neglect the gravitational effect of the satellite on the particles.

In Sections (*a*) to (*d*) we consider the case where the direction of V_∞ is parallel to the plane of the satellite orbit and calculate:

(*a*) The rate of accretion of particles by the planet;
(*b*) The rate of accretion by the satellite;
(*c*) The rate of change of mean motion of the satellite; and
(*d*) The numerical value of the rate for Jupiter V.

In Section (*e*) we discuss the case where V_∞ is perpendicular to the plane of the satellite orbit.

In the following paper, F. Kerr and F. L. Whipple apply the theory developed here in order to find out whether the reported accelerations of Phobos and Jupiter V can be explained as the effects of a resisting medium or of tidal friction.

(*a*). As reference line we choose the one from the planet in the direction of $-V_\infty$. In the gravitational field of the planet of mass M, the orbit of a particle is an hyperbola confined to some plane through the reference line. Any particular orbit in a given plane can be specified by the perpendicular distance, p, of the planet from either asymptote of the hyperbola; equivalently the orbit may be specified by the angle 2δ between the asymptotes. Then

$$\cos\delta = -\left[1 + \left(\frac{pV_\infty^2}{\mu}\right)^2\right]^{-\frac{1}{2}},$$

where $\mu = MG$ and G is the constant of gravitation.

In such an orbital plane, let us introduce polar coordinates (r, φ) with the planet as origin and with the reference line as initial line. The equation of a particle orbit with parameter p (or δ) is then

$$\frac{1}{r} = \frac{\mu}{p^2V_\infty^2} \times \left\{1 + \left[1 + \left(\frac{pV_\infty^2}{\mu}\right)^2\right]^{\frac{1}{2}} \cos(\delta - \varphi)\right\}, \quad (1)$$

or

$$\frac{1}{r} = \frac{\mu}{p^2V_\infty^2}\left(1 - \cos\varphi + \frac{pV_\infty^2}{\mu}\sin\varphi\right). \quad (2)$$

At distance r from the planet, the speed V of a particle, from conservation of energy, is given by:

$$V^2 = V_\infty^2 + \frac{2\mu}{r}. \quad (3)$$

Let P be the value of p for which the hyperbola just touches the planet of radius A. Then, from (1) with $\varphi = \delta$,

$$P = A\left(1 + \frac{2\mu}{AV_\infty^2}\right)^{\frac{1}{2}}. \quad (4)$$

All particles whose orbits have $p \leq P$ will strike, and hence be captured by, the planet. Neglecting the effects of motion of the planet round the sun, and integrating over all orientations of the orbital plane, we find that the rate of increase of planetary mass is

$$\frac{dM}{dt} = \pi P^2 V_\infty \rho = \pi A^2 V_\infty \rho \left(1 + \frac{2\mu}{AV_\infty^2}\right). \quad (5)$$

From now on, except for Section (*e*), we consider explicitly the case where V_∞ is parallel to the plane of the satellite orbit. If the satellite orbit be taken to be a circle of radius a, the hyperbola with parameter P in its plane will intersect the orbit in two points (a, φ_1) and (a, φ_2), with

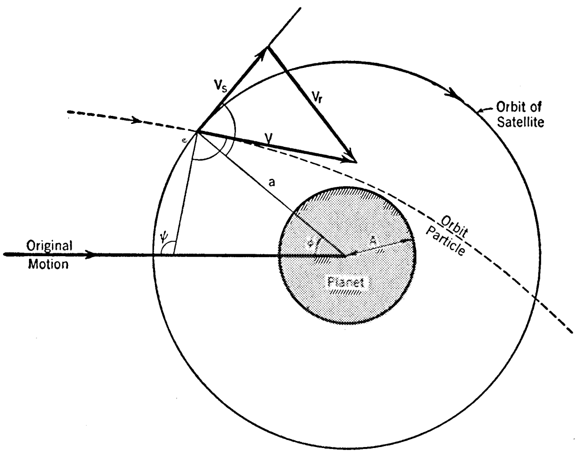

Figure 1. Geometry of particle, satellite and planet.

$\varphi_1 < \varphi_2$, given by

$$\cos(\delta_1 - \varphi_1) = \frac{\dfrac{2A}{a} + \dfrac{A^2V_\infty^2}{a\mu} - 1}{\dfrac{AV_\infty^2}{\mu} + 1},$$

$$\varphi_2 = 2\,\delta_1 - \varphi_1,$$

where

$$\cos\delta_1 = \frac{-1}{\dfrac{AV_\infty^2}{\mu} + 1}.$$

Here two cases must be distinguished: (i) $\varphi_2 > \pi$, and (ii) $\varphi_2 < \pi$. These are characterized by the relation of V_∞ to a critical value

$$V_\infty^* = \left[\frac{2a\mu}{A^2}\left(1 - \frac{A}{a}\right)\right]^{\frac{1}{2}}. \qquad (6)$$

If $V_\infty < V_\infty^*$, then $\varphi_2 > \pi$ and the satellite receives material from both sides of the initial line so long as $(2\pi - \varphi_2) < \varphi < \varphi_2$. If $V_\infty > V_\infty^*$, then $\varphi_2 < \pi$ and the planet shields the satellite from accretion when $\varphi_2 < \varphi < (2\pi - \varphi_2)$.

(*b*). Let p_φ, p_{φ_1}, p_{φ_2} refer, respectively, to hyperbolas passing through the center of the satellite (a,φ), and touching its disc on the sides near and farther from the initial line. Since the satellite radius R is very small compared to a, the hyperbolas touch the satellite at the points whose Cartesian coordinates are very nearly

$$(a\ \cos\varphi \mp R\ \cos\psi,\ a\ \sin\varphi \mp R\ \sin\psi),$$

the upper and lower signs referring, respectively, to the p_{φ_1}, p_{φ_2} hyperbolas. Here ψ is the angle between the initial line and the normal at (a,φ) to the hyperbola p_φ, (Fig. 1):

$$\psi = \tan^{-1}\left(\frac{\dfrac{p_\varphi V_\infty^2}{\mu} + \sin\varphi}{\cos\varphi - 1}\right). \qquad (7)$$

Defining

$$w = \frac{2\mu}{aV_\infty^2}, \quad \text{and} \quad h = \frac{R}{a},$$

we have by (2), after some manipulation,

$$p_\varphi = \frac{a}{2}\{\sin\varphi + [\sin^2\varphi + 2w(1 - \cos\varphi)]^{\frac{1}{2}}\}, \qquad (8)$$

$$p_{\varphi_{1,2}} = \frac{a}{2}\{(\sin\varphi \mp h\sin\psi) + [(\sin\varphi \mp h\sin\psi)^2 + 2w(1 \mp 2h\cos(\psi - \varphi) + h^2)^{\frac{1}{2}} - 2w(\cos\varphi \mp h\cos\psi)]^{\frac{1}{2}}\}. \qquad (9)$$

All particles with trajectories in the orbital plane of the satellite and with parameter p in the range $p_{\varphi_1} \leq p \leq p_{\varphi_2}$ will strike the satellite.

To study the encounters of particles not in the orbital plane of the satellite, let us consider a plane passing through the initial line and a point H of the satellite most distant from its orbital plane. If the dihedral angle between these planes is χ, then

$$\tan\chi = h/\sin\varphi.$$

The rectangular coordinates of H in the inclined plane are now $a\cos\varphi$ and $a(h^2 + \sin^2\varphi)^{\frac{1}{2}}$ so that the hyperbola passing through H will correspond to $p = p_{\varphi H}$, which can be derived, as before, from (2) to give

$$p_{\varphi H} = \frac{a}{2}(\sin^2\varphi + h^2)^{\frac{1}{2}}\left\{1 + \left[1 + 2w\frac{(1 + h^2)^{\frac{1}{2}} - \cos\varphi}{\sin^2\varphi + h^2}\right]^{\frac{1}{2}}\right\}. \qquad (10)$$

The distance, β, of the asymptote to this hyperbola from the orbital plane will be $p_{\varphi H}\sin\chi$, or

$$\beta = \frac{R}{2}\left\{1 + \left[1 + 2w\frac{(1 + h^2)^{\frac{1}{2}} - \cos\varphi}{\sin^2\varphi + h^2}\right]^{\frac{1}{2}}\right\}. \qquad (11)$$

Evidently the distance from the orbital plane of the asymptote of any hyperbola intersecting the satellite will be less than β. Therefore, the area at infinity through which the material passes normally to reach the satellite can be taken approximately as an ellipse whose semi-axes are $\alpha = (p_{\varphi_2} - p_{\varphi_1})/2$ and β.

The velocity V_r, relative to the satellite, of the material approaching the center of the satellite can be seen from Figure 1 to be

$$V_r = [V^2 + V_S^2 + 2VV_S\cos(\psi - \varphi)]^{\frac{1}{2}}, \quad (12)$$

where $V_S = (\mu/a)^{\frac{1}{2}}$ is the orbital velocity of the satellite and $V = (V_\infty^2 + 2\mu/a)^{\frac{1}{2}}$ is the total velocity of the material.

To take account of the material approaching the satellite from the opposite side of the initial line, we measure φ in the reverse direction from the initial line. Then the expressions for α and β remain unchanged, but the relative velocity, V_r, now becomes

$$V_r' = [V^2 + V_S^2 + 2VV_S\cos(\psi - \varphi)]^{\frac{1}{2}}. \quad (13)$$

The total mass gained by the satellite during one orbital period, $T = 2\pi a^{\frac{3}{2}}\mu^{-\frac{1}{2}}$, then becomes

$$\pi\int_0^{\theta_2}\alpha\beta\rho\frac{V_r}{V}V_\infty\frac{T}{2\pi}d\varphi + \pi\int_0^{\theta_2}\alpha\beta\rho\frac{V_r'}{V}V_\infty\frac{T}{2\pi}d\varphi = \frac{T\rho V_\infty}{2V}\int_0^{\theta_2}\alpha\beta(V_r + V_r')d\varphi \quad (14)$$

on the assumption that the encounters are inelastic.

(*c*). The mean angular motion of the satellite about the planet is

$$n = (MG)^{\frac{1}{2}}/a^{\frac{3}{2}},$$

so that, differentiating logarithmically, we find

$$\frac{1}{n}\frac{dn}{dt} = \frac{1}{2}\frac{1}{M}\frac{dM}{dt} - \frac{3}{2}\frac{1}{a}\frac{da}{dt}.$$

In the case of a circular motion with radius a, a tangential acceleration changes the motion as follows:

$$\frac{1}{a}\frac{da}{dt} = \frac{T}{\pi a}S,$$

where T is the period and S is the forward component of external acceleration perpendicular to the radius vector.

Hence, from (14), for a satellite of mass m, we find for the rate of change of mean motion

$$T\left(\frac{1}{n}\frac{dn}{dt}\right) = \frac{1}{2}\frac{1}{M}\frac{dM}{dt}T - \frac{3T^2\rho V_\infty}{4\pi am}\left\{\int_0^{\theta_2}\alpha\beta V_r\left[\cos(\psi-\varphi) - \frac{V_S}{V}\right]d\varphi - \int_0^{\theta_2}\alpha\beta V_r'\left[\cos(\psi-\varphi) + \frac{V_S}{V}\right]d\varphi\right\}. \quad (15)$$

(*d*). Since $h = R/a \ll 1$, we may neglect h^2 and higher powers in numerical calculations. Therefore, except in the close neighborhood of $\varphi = 0$ and π, we may calculate α, β, and ψ from the following expressions which are correct to this approximation:

$$\alpha = \frac{R}{2}\left\{\sin\psi + \frac{\sin\psi + w[\cos(\psi-\varphi) - \cos\psi]/\sin\varphi}{\left(1 + w\sec^2\frac{\varphi}{2}\right)^{\frac{1}{2}}}\right\}, \quad (16)$$

$$\beta = \frac{R}{2}\left\{1 + \left(1 + w\sec^2\frac{\varphi}{2}\right)^{\frac{1}{2}}\right\}, \quad (17)$$

and

$$\tan\psi \simeq -\cot\frac{\varphi}{2}\left(1 + \frac{2\beta}{Rw}\right). \quad (18)$$

Near $\varphi = 180°$, but not precisely at $\varphi = 180°$, the quantites within the integrals of (15) may be approximated by constants except for β, which is given by

$$\beta = \frac{Rw^{\frac{1}{2}}}{(\epsilon^2 + h^2)^{\frac{1}{2}}}, \quad (19)$$

where $\epsilon = |\varphi - \pi|$ and $h = R/a \ll 1$.

Hence, in (15) the following integral may be used over the range in φ from $\pi - \epsilon_0$ to $\pi + \epsilon_0$:

$$\int_{-\epsilon_0}^{+\epsilon_0}\beta d\epsilon = 2Rw^{\frac{1}{2}}\ln\left[\frac{\epsilon_0}{h} + \left(1 + \frac{\epsilon_0^2}{h^2}\right)^{\frac{1}{2}}\right]. \quad (20)$$

Although (20) is a satisfactory numerical approximation over a range of a degree or so in φ, it fails in a minute range ($\epsilon \sim h$) about $\varphi = \pi$. We have derived the precise expression for β in this range but the applicable range in φ is so small that the integral (20) is not much affected so long as $\epsilon_0/h \gg 1$.

In the numerical work applicable to Jupiter