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On the BGK Collision Model for a Two-Component Assembly

By

P. L. Bhatnagar

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The BGK collision model for a one-component assembly of neutral particles has been extended to two-component assemblies by Gross and Krook. They have evaluated only six phenomenological constants out of the fourteen introduced in the definition of the model. In this note, the relaxation problem has been completely solved and all but two constants have been evaluated. These remaining two constants must be determined by invoking other physical considerations like Jeans' relation for collisional momentum transfer and Newton's law of heat transfer between the two components. The resulting equations of transfer and the expressions for collisional momentum and energy transfer between the two components have been evaluated explicitly.

1. Introduction

Bhatnagar, Gross and Krook [1] introduced a collision-model — called BGK model — which was generalized later on by Gross and Krook [4] for a two-component assembly. In the latter work, the authors have proceeded up to a certain stage in evaluating the phenomenalogical constants introduced in the theory by the consideration of the relaxation problem. It appears that their arguments require deeper analysis and justification. In the present paper, we have worked out the relaxation problem completely and shown that it is possible to determine all the constants introduced from phenomenalogical considerations except two. If we further accept Jeans' [5] relation for the collisional momentum transfer and Newton's law of heat transfer between the components, we can fix also the other two constants. We have evaluated all these constants and deduced the corresponding transport equations. The form of the contributions of the collisions to the equations of transfer seem to justify the proposed model. In a subsequent paper we utilize a similar model for studying the small amplitude oscillations of an assembly consisting of positive ions, electrons and neutral particles.

2. Collision Model

Let us denote the two components of the assembly by the suffixes 1 and 2. The distribution functions $f_1(\vec{\xi}_1, \vec{r}, t)$ and $f_2(\vec{\xi}_2, \vec{r}, t)$ for these components are determined by the Boltzmann equations

$$\begin{aligned}\frac{\partial f_1}{\partial t} + \xi_{1i}\frac{\partial f_1}{\partial x_i} + \frac{F_{1i}}{m_1}\frac{\partial f_1}{\partial \xi_{1i}} = &- f_1 \iint f_2\, g_{21} b\, db\, d\epsilon\, d\vec{\xi}_2 - \\ &- f_1 \iint f_1\, g_{11} b\, db\, d\epsilon\, d\vec{\xi}_1 + \\ &+ \iint f_2'(\vec{\xi}_2', \vec{r}, t)\, f_1'(\vec{\xi}_1', \vec{r}, t)\, g_{21} b\, db\, d\epsilon\, d\vec{\xi}_2 + \\ &+ \iint f_1'(\vec{\xi}_1', \vec{r}, t)\, f_1'(\vec{\xi}_1', \vec{r}, t)\, g_{11} b\, db\, d\epsilon\, d\vec{\xi}_1 ,\end{aligned} \tag{1}$$

and

$$\begin{aligned}\frac{\partial f_2}{\partial t} + \xi_{2i}\frac{\partial f_2}{\partial x_i} + \frac{F_{2i}}{m_2}\frac{\partial f_2}{\partial \xi_{2i}} = &- f_2 \iint f_1\, g_{12} b\, db\, d\epsilon\, d\vec{\xi}_1 - \\ &- f_2 \iint f_2\, g_{22} b\, db\, d\epsilon\, d\vec{\xi}_2 + \\ &+ \iint f_1'(\vec{\xi}_1', \vec{r}, t)\, f_2'(\vec{\xi}_2', \vec{r}, t)\, g_{12} b\, db\, d\epsilon\, d\vec{\xi}_1 + \\ &+ \iint f_2'(\vec{\xi}_2', \vec{r}, t)\, f_2'(\vec{\xi}_2', \vec{r}, t)\, g_{22} b\, db\, d\epsilon\, d\vec{\xi}_2 ,\end{aligned} \tag{2}$$

in terms of the standard notation of CHAPMAN and COWLING [2].

Interpreting the first integral in (1) as the number of particles of the first type removed from a given velocity range $(\vec{\xi}_1, d\vec{\xi}_1)$ due to collisions with the second type of particles, GROSS and KROOK [4], following the suggestion of [1], approximated it by

$$-\frac{N_2(\vec{r}, t)}{\sigma_{21}} f_1 . \tag{3}$$

Similarly the third term, being the number of particles of the first type brought into that range by collisions with particles of the second type, is approximated by

$$\frac{N_2(\vec{r}, t)\, N_1(\vec{r}, t)}{\sigma_{21}} \Phi_{21}(\vec{\xi}_1, \vec{r}, t) . \tag{4}$$

In the absence of any knowledge of the distribution of the first type of particles after collision we assume a local Maxwellian distribution

$$\Phi_{21}(\vec{\xi}_1, \vec{r}, t) = \left(\frac{m_1}{2\pi k T_{21}}\right)^{3/2} \exp\left\{-\frac{m_1}{2kT_{21}}(\vec{\xi}_1 - \vec{u}_{21})^2\right\}. \tag{5}$$

Similar assumptions are made for $\Phi_{11}(\vec{\xi}_1, \vec{r}, t)$, $\Phi_{12}(\vec{\xi}_2, \vec{r}, t)$ and $\Phi_{22}(\vec{\xi}_2, \vec{r}, t)$, in the other collision terms in (1) and (2). According to the present collision model, the kinetic equations then reduce to

$$\frac{\partial f_1}{\partial t} + \xi_{1i}\frac{\partial f_1}{\partial x_i} + \frac{F_{1i}}{m_1}\frac{\partial f_1}{\partial \xi_{1i}} = -\left(\frac{N_1}{\sigma_{11}} + \frac{N_2}{\sigma_{21}}\right) f_1 + \frac{N_1^2}{\sigma_{11}}\Phi_{11} + \frac{N_2 N_1}{\sigma_{21}}\Phi_{21} , \tag{6}$$

and

$$\frac{\partial f_2}{\partial t} + \xi_{2i}\frac{\partial f_2}{\partial x_i} + \frac{F_{2i}}{m_2}\frac{\partial f_2}{\partial \xi_{2i}} = -\left(\frac{N_1}{\sigma_{12}} + \frac{N_2}{\sigma_{22}}\right) f_2 + \frac{N_1 N_2}{\sigma_{12}}\Phi_{21} + \frac{N_2^2}{\sigma_{22}}\Phi_{22} , \tag{7}$$

where

$$N_1 = \int f_1 d\vec{\xi}_1 , \qquad N_2 = \int f_2 d\vec{\xi}_2 , \tag{8}$$

$$\vec{u}_{11} = \frac{1}{N_1} f_1 \vec{\xi}_1 d\vec{\xi}_1 , \qquad \vec{u}_{22} = \frac{1}{N_2} \int f_2 \vec{\xi}_2 d\vec{\xi}_2 , \tag{9}$$

$$\frac{3kT_{11}}{m_1} = \frac{1}{N_1} \int f_1 (\vec{\xi}_1 - \vec{u}_{11})^2 d\vec{\xi}_1 , \qquad \frac{3kT_{22}}{m_2} = \frac{1}{N_2} \int f_2 (\vec{\xi}_2 - \vec{u}_{22})^2 d\vec{\xi}_2 . \tag{10}$$

It is usually stated than these equations are linear partial differential equations. But this is so only apparently. When we substitute for the particle density N_1 and N_2 and for the selfconsistent electromagnetic field, the nonlinearity and the integral terms become explicit. The assumed mean velocities and the temperatures $\vec{u}_{12}$, $\vec{u}_{21}$, T_{12}, T_{21} of the scattered particles should satisfy the conservation laws of mass, momentum and energy in addition to yielding the correct initial and asymptotic behaviour of the assembly.

In order to study the implications of the conservation laws, we deduce the following transfer equations in the usual manner:

$$\frac{\partial N_1}{\partial t} + \operatorname{div}(N_1 \vec{u}_{11}) = 0 , \tag{11}$$

$$\frac{\partial N_2}{\partial t} + \operatorname{div}(N_2 \vec{u}_{22}) = 0 , \tag{12}$$

$$\frac{\partial}{\partial t}(N_1 \vec{u}_{11}) + \operatorname{div}(N_1 \vec{u}_{11} \vec{u}_{11}) - \frac{e_1 N_1 \vec{E}}{m_1} = \frac{N_2 N_1}{\sigma_{21}} (\vec{u}_{21} - \vec{u}_{11}) , \tag{13}$$

$$\frac{\partial}{\partial t}(N_2 \vec{u}_{22}) + \operatorname{div}(N_2 \vec{u}_{22} \vec{u}_{22}) - \frac{e_2 N_2 \vec{E}}{m_2} = \frac{N_1 N_2}{\sigma_{12}} (\vec{u}_{12} - \vec{u}_{22}) , \tag{14}$$

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{3kN_1 T_{11}}{m_1}\right) + \operatorname{div}(N_1 \vec{Q}_1) - \int \{\operatorname{grad}(\vec{\xi}_1 - \vec{u}_{11})^2\} \cdot \vec{\xi}_1 f_1 d\vec{\xi}_1 \\ = \frac{N_2 N_1}{\sigma_{21}} \left[\frac{3k}{m_1}(T_{21} - T_{11}) + (\vec{u}_{21} - \vec{u}_{11})^2\right], \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{3kN_2 T_{22}}{m_2}\right) + \operatorname{div}(N_2 \vec{Q}_2) - \int \{\operatorname{grad}(\vec{\xi}_2 - \vec{u}_{22})^2\} \cdot \vec{\xi}_2 f_2 d\vec{\xi}_2 \\ = \frac{N_1 N_2}{\sigma_{12}} \left[\frac{3k}{m_2}(T_{12} - T_{22}) + (\vec{u}_{12} - \vec{u}_{22})^2\right], \end{aligned} \tag{16}$$

where

$$\vec{Q}_1 = \frac{1}{N_1} \int f_1 (\vec{\xi}_1 - \vec{u}_{11})^2 \vec{\xi}_1 d\vec{\xi}_1 , \quad Q_2 = \frac{1}{N_2} \int f_2 (\vec{\xi}_2 - \vec{u}_{22})^2 \vec{\xi}_2 d\vec{\xi}_2 , \tag{17}$$

The self-consistent electric field is given by

$$\operatorname{div} \vec{E} = 4\pi (e_1 N_1 + e_2 N_2) . \tag{18}$$

The equations (11) and (12) show that the mass of each component is separately conserved during collision, as it should be, since we are considering only the elastic processes. The application of this law does not provide any relation between $\vec{u}_{12}$, $\vec{u}_{21}$, T_{12} and T_{21}. The right hand sides of equations (13) and (14) represent the change in momentum per unit mass of the first component due to collisions with the second component and the change in the momentum per unit mass of the second component due to its collision with the first component. The total momentum change of the assembly due to collision must vanish. This yields the relation

$$m_1(\vec{u}_{21}-\vec{u}_{11}) + m_2(\vec{u}_{12}-\vec{u}_{22}) = 0 . \tag{19}$$

We now multiply the right hand sides of (6) and (7) by $\frac{m_1\vec{\xi}_1^2}{2}$ and $\frac{m_2\vec{\xi}_2^2}{2}$ respectively and add. Equating the resulting expression to zero we get the condition for the conservation of energy:

$$3k(T_{21}-T_{11}) + 3k(T_{12}-T_{22}) + m_1(\vec{u}_{21}^2-\vec{u}_{11}^2) + m_2(\vec{u}_{12}^2-\vec{u}_{22}^2) = 0 . \tag{20}$$

The equations (19) and (20) provide 4 relations among the 8 unknowns $\vec{u}_{12}$, $\vec{u}_{21}$, T_{12} and T_{21}. Hence we have to supply 4 more equations between them from other physical considerations.

From (5) it is clear that $\vec{u}_{21}$ and T_{21} are the mean velocities and temperatures near which the first type of scattered particles are distributed after collisions with particles of type two. We may, therefore, assume that $\vec{u}_{21}$ is some function of $\vec{u}_{11}$ and $\vec{u}_{22}$ while T_{21} is some function of T_{11}, T_{22}, $\vec{u}_{11}$ and $\vec{u}_{22}$. Following II, we assume the following phenomenological relations:

$$\vec{u}_{21} = a_{21}\vec{u}_{11} + a_{22}\vec{u}_{22} , \tag{21}$$

$$T_{21} = b_{21}T_{11} + b_{22}T_{22} + K\vec{u}_{22}^2 + L\vec{u}_{11}\cdot\vec{u}_{22} + M\vec{u}_{11}^2 . \tag{22}$$

and similarly

$$\vec{u}_{12} = a_{11}\vec{u}_{11} + a_{12}\vec{u}_{22} , \tag{23}$$

$$T_{12} = b_{11}T_{11} + b_{12}T_{22} + D\vec{u}_{22}^2 + E\vec{u}_{11}\cdot\vec{u}_{22} + F\vec{u}_{11}^2 , \tag{24}$$

containing altogether fourteen unknown constants. The assumption of linearity is apparently very restrictive, but these terms may be considered as the first two terms of the expansion of $\vec{u}_{21}$, $\vec{u}_{12}$, T_{21} and T_{12}. Additional support in favour of these assumptions is obtained when we calculate the rate of transfer of momentum and energy from one component to another after evaluating these constants.

3. Relaxation Problem

To determine the unknown constants, we consider the relaxation (non-equilibrium) problem. We shall neglect the spatial derivatives and external forces. Thus

$$\frac{dN_1}{dt} = 0 , \quad \frac{dN_2}{dt} = 0 , \tag{25}$$

so that

$$N_1 = \text{constant} \quad \text{and} \quad N_2 = \text{constant}. \tag{26}$$

Finally the whole assembly is bound to be neutral and

$$\therefore\ e_1 N_1 + e_2 N_2 = 0\,, \tag{27}$$

so that for all times t,

$$\vec{E} = 0\,. \tag{28}$$

The rest of the transfer equations can be reduced to

$$\frac{\partial \vec{u}_{11}}{\partial t} = \frac{N_2}{\sigma}(\vec{u}_{21} - \vec{u}_{11}) = -\mu_{21}\vec{u}_{11} + \mu_2 \vec{u}_{22}\,, \tag{29}$$

$$\frac{\partial \vec{u}_{22}}{\partial t} = \frac{N_1}{\sigma}(\vec{u}_{12} - \vec{u}_{22}) = \mu_1 \vec{u}_{11} - \mu_{12}\vec{u}_{22}\,, \tag{30}$$

$$\begin{aligned}\frac{\partial T_{11}}{\partial t} = -\nu_{21} T_{11} + \nu_2 T_{22} + \lambda_2 \Big[\Big(K + \frac{m_1 \mu_2^2}{3k\lambda_2^2}\Big)\vec{u}_{22}^2 + \\ + \Big(L - \frac{2m_1\mu_2^2}{3k\lambda_2^2}\Big)\vec{u}_{11}\cdot\vec{u}_{22} + \Big(M + \frac{m_1\mu_2^2}{3k\lambda_2^2}\Big)\vec{u}_{11}^2\Big],\end{aligned} \tag{31}$$

$$\begin{aligned}\frac{\partial T_{22}}{\partial t} = -\nu_{12} T_{22} + \nu_1 T_{11} + \lambda_1 \Big[\Big(D + \frac{m_2 \mu_1^2}{3k\lambda_1^2}\Big)\vec{u}_{22}^2 + \\ + \Big(\vec{E} - \frac{2m_2\mu_1^2}{3k\lambda_1^2}\Big)\vec{u}_{11}\cdot\vec{u}_{22} + \Big(F + \frac{m_2\mu_1^2}{3k\lambda_1^2}\Big)\vec{u}_{11}^2\Big],\end{aligned} \tag{32}$$

where

$$\sigma = \sigma_{12} = \sigma_{21}\,, \quad \lambda_1 = \frac{N_1}{\sigma}\,, \quad \lambda_2 = \frac{N_2}{\sigma}\,, \tag{33}$$

$$\left.\begin{aligned}&\mu_{12} = \lambda_1(1 - a_{12})\,, \qquad \mu_{21} = \lambda_2(1 - a_{21})\,,\\ &\nu_{12} = \lambda_1(1 - b_{12})\,, \qquad \nu_{21} = \lambda_2(1 - b_{21})\,,\\ &\mu_1 = \lambda_1 a_{11}\,, \quad \mu_2 = \lambda_2 a_{22}\,, \quad \nu_1 = \lambda_1 b_{11}\,, \quad \nu_2 = \lambda_2 b_{22}\,.\end{aligned}\right\} \tag{34}$$

We solve these equations under the following conditions:

Initial conditions. For $t = 0$,

$$\vec{u}_{11} = \vec{A}\,, \quad \vec{u}_{22} = \vec{B}\,, \quad T_{11} = T_1\,, \quad T_{22} = T_2\,. \tag{35}$$

Final conditions. For $t \to \infty$,

$$\left.\begin{aligned}&\vec{u}_{11}\,, \quad \vec{u}_{22} \to \vec{u}_\infty\,, \quad T_{11}\,, \quad T_{22} \to T_\infty\,,\\ &\frac{\partial \vec{u}_{11}}{\partial t}\,, \quad \frac{\partial \vec{u}_{22}}{\partial t}\,, \quad \frac{\partial T_{11}}{\partial t}\,, \quad \frac{\partial T_{22}}{\partial t} \to 0\,.\end{aligned}\right\} \tag{36}$$

The equations determining $\vec{u}_{11}$ and $\vec{u}_{22}$ are

$$\Big[\frac{\partial^2}{\partial t^2} + (\mu_{21} + \mu_{12})\frac{\partial}{\partial t} + (\mu_{12}\mu_{21} - \mu_1\mu_2)\Big]\vec{u}_{11 \atop 22} = 0\,. \tag{37}$$

These have solutions

$$\left.\begin{matrix}\vec{u}_{11}\\ \vec{u}_{22}\end{matrix}\right\} = \frac{1}{\mu_1 + \mu_2}\Big[\mu_1\vec{A} + \mu_2\vec{B} + \binom{\mu_2}{-\mu_1}(\vec{A} - \vec{B})\,e^{-(\mu_1 + \mu_2)t}\Big], \tag{38}$$

which tend to non-zero finite values as $t \to \infty$, satisfy the prescribed initial and final conditions and have the relaxation time $\frac{1}{\mu_1 + \mu_2}$, provided

$$\mu_{12} = \mu_1 \quad \text{and} \quad \mu_{21} = \mu_2 \,. \tag{39}$$

Substituting these in to equation (19) for the conservation of momentum and equating the independent term and the coefficient of $e^{-(\mu_1+\mu_2)t}$ separately to zero, we have

$$\frac{m_1 \lambda_1}{\mu_1} = \frac{m_2 \lambda_2}{\mu_2} . \tag{40}$$

Thus out of the four constants μ_1, μ_2, μ_{12}, μ_{21}, only one remains to be fixed. From (39) and (40) we obtain

$$a_{11} = 1 - a_{12} \,, \quad a_{22} = 1 - a_{21} \,, \quad \frac{a_{11}}{m_1} = \frac{a_{22}}{m_2} . \tag{41}$$

If in addition, we invoke the Jeans relation that

$$a_{11} = \frac{\mu_1}{\lambda_1} = \frac{m_1}{m_1 + m_2} \,, \quad a_{22} = \frac{\mu_2}{\lambda_2} = \frac{m_2}{m_1 + m_2} , \tag{42}$$

we have

$$a_{12} = \frac{m_2}{m_1 + m_2} = a_{22} \,, \quad a_{21} = \frac{m_1}{m_1 + m_2} = a_{11} \,. \tag{43}$$

We can now write down the rate of transfer of momentum to "1" due to collision with "2",

$$\frac{m_1 N_1 N_2}{\sigma} (\vec{u}_{21} - \vec{u}_{11}) = \frac{N_1 N_2}{\sigma} \frac{m_1 m_2}{m_1 + m_2} (\vec{u}_{22} - \vec{u}_{11}) \,, \tag{44}$$

while the rate of momentum transfer to "2" due to collision with "1" is

$$\frac{N_1 N_2}{\sigma} \frac{m_1 m_2}{m_1 + m_2} (\vec{u}_{11} - \vec{u}_{22}) \,. \tag{45}$$

These appear to be reasonable expressions, supporting our assumptions about $\vec{u}_{12}$ and $\vec{u}_{21}$. Similar assumptions have been made by COWLING [3] and SPITZER [6] in connection with other considerations. Also

$$\vec{u}_{12} = \frac{m_1 \vec{u}_{11} + m_2 \vec{u}_{22}}{m_1 + m_2} = \vec{u}_{21} \,. \tag{46}$$

Therefore, the particles after collision are distributed about the mean flow velocity.

We now consider the temperature equations

$$\left[\frac{\partial^2}{\partial t^2} + (\nu_{12} + \nu_{21}) \frac{\partial}{\partial t} + (\nu_{12}\nu_{21} - \nu_1 \nu_2)\right] \begin{matrix} T_{11} \\ T_{22} \end{matrix} = \begin{matrix} x_0 \\ y_0 \end{matrix} ,$$
$$\begin{matrix} x_0 \\ y_0 \end{matrix} + \begin{matrix} x_1 \\ y_1 \end{matrix} e^{-(\mu_1+\mu_2)t} + \begin{matrix} x_2 \\ y_2 \end{matrix} e^{-2(\mu_1+\mu_2)t} , \tag{47}$$

where

$$x_0 = [\lambda_1 \nu_2 (D+E+F) + \nu_{12}\lambda_2 (K+L+M)] \left(\frac{\mu_1 \vec{A} + \mu_2 \vec{B}}{\mu_1+\mu_2}\right)^2,$$

$$y_0 = [\lambda_2 \nu_1 (K+L+M) + \lambda_1 \nu_{21} (D+E+F)] \left(\frac{\mu_1 \vec{A} + \mu_2 \vec{B}}{\mu_1+\mu_2}\right)^2,$$

$$\begin{aligned} x_1 = [&\lambda_1 \nu_2 \{-2\mu_1 D + (\mu_2 - \mu_1) E + 2\mu_2 F\} + \\ &+ \lambda_2 \nu_{12} \{-2\mu_1 K + (\mu_2 - \mu_1) L + 2\mu_2 M\} + \\ &+ \lambda_2 (\mu_1+\mu_2) \{2\mu_1 K - (\mu_2-\mu_1) L - 2\mu_2 M\}] \frac{(\mu_1 \vec{A} + \mu_2 \vec{B}) \cdot (\vec{A} - \vec{B})}{(\mu_1+\mu_2)^2}, \end{aligned}$$

$$\begin{aligned} y_1 = [&\lambda_2 \nu_1 \{-2\mu_1 K + (\mu_2 - \mu_1) L + 2\mu_2 M\} + \\ &+ \lambda_1 \nu_2 \{-2\mu_1 D + (\mu_2 - \mu_1) E + 2\mu_2 F\} + \\ &+ \lambda_1 (\mu_1+\mu_2) \{2\mu_1 D - (\mu_2-\mu_1) E - 2\mu_2 F\}] \frac{(\mu_1 \vec{A} + \mu_2 \vec{B}) \cdot (\vec{A} - \vec{B})}{(\mu_1+\mu_2)^2}, \end{aligned}$$

$$\begin{aligned} x_2 = \Big[&\lambda_1 \nu_2 \Big\{\mu_1^2 D - \mu_1\mu_2 E + \mu_2^2 F + \frac{m_2 \mu_1^2}{3k\lambda_1^2} (\mu_1+\mu_2)^2\Big\} + \\ &+ \lambda_2 \nu_{12} \Big\{\mu_1^2 K - \mu_1\mu_2 L + \mu_2^2 M + \frac{m_1 \mu_2^2}{3k\lambda_2^2} (\mu_1+\mu_2)^2\Big\} + \\ &+ \lambda_2 (\mu_1+\mu_2) \Big\{-2\mu_1^2 K + 2\mu_1\mu_2 L - 2\mu_2^2 M - \\ &\qquad - \frac{2m_1\mu_2^2}{3k\lambda_2^2} (\mu_1+\mu_2)^2\Big\}\Big] \frac{(\vec{A}-\vec{B})^2}{(\mu_1+\mu_2)^2}, \end{aligned}$$

$$\begin{aligned} y_2 = \Big[&\lambda_2 \nu_1 \Big\{\mu_1^2 K - \mu_1\mu_2 L + \mu_2^2 M + \frac{m_1 \mu_2^2}{3k\lambda_2^2} (\mu_1+\mu_2)^2\Big\} + \\ &+ \lambda_1 \nu_{21} \Big\{\mu_1^2 D - \mu_1\mu_2 E + \mu_2^2 F + \frac{m_2 \mu_1^2}{3k\lambda_1^2} (\mu_1+\mu_2)^2\Big\} + \\ &+ \lambda_1 (\mu_1+\mu_2) \Big\{-2\mu_1^2 D + 2\mu_1\mu_2 E - 2\mu_2^2 F + \\ &\qquad + \frac{m_2\mu_1^2}{3k\lambda_1^2} (\mu_1+\mu_2)^2\Big\}\Big] \frac{(\vec{A}-\vec{B})^2}{(\mu_1+\mu_2)^2}. \end{aligned} \tag{48}$$

The solutions of (31) and (32) satisfying the initial and final conditions and tending to a non-zero finite temperature T_∞ as $t \to \infty$ are

$$\left.\begin{matrix} T_{11} \\ T_{22} \end{matrix}\right\} = T_\infty + \begin{matrix} S \\ Q \end{matrix}\, e^{-(\nu_1+\nu_2)t} + \begin{matrix} x_1 \\ y_1 \end{matrix}\, \frac{e^{-(\mu_1+\mu_2)t}}{(\mu_1+\mu_2)^2 - (\mu_1+\mu_2)(\nu_1+\nu_2)} + \begin{matrix} x_2 \\ y_2 \end{matrix}\, \frac{e^{-2(\mu_1+\mu_2)t}}{4(\mu_1+\mu_2)^2 - 2(\mu_1+\mu_2)(\nu_1+\nu_2)}, \tag{49}$$

provided

and

$$\left.\begin{matrix} \nu_{12} = \nu_1, \quad \nu_{21} = \nu_2 \\ \dfrac{\nu_2}{\lambda_2} = \dfrac{\nu_1}{\lambda_1}, \end{matrix}\right\} \tag{50}$$

where

$$-(\nu_1+\nu_2)\,Q=\frac{x_1}{(\mu_1+\mu_2)-(\nu_1+\nu_2)}+\frac{x_2}{2(\mu_1+\mu_2)-(\nu_1+\nu_2)}+\nu_2 T_2-\nu_{21}T_1+$$
$$+\lambda_2\left[K\vec{B}^2+L\vec{B}\cdot\vec{A}+M\vec{A}^2+\frac{m_1\mu_2^2}{3k\lambda_2^2}(\vec{B}-\vec{A})^2,\right. \tag{51}$$

$$-(\nu_1+\nu_2)\,S=\frac{y_1}{(\mu_1+\mu_2)-(\nu_1+\nu_2)}+\frac{y_2}{2(\mu_1+\mu_2)-(\nu_1+\nu_2)}+\nu_1 T_1-\nu_{12}T_2+$$
$$+\lambda_1\left[D\vec{B}^2+E\vec{B}\cdot\vec{A}+F\vec{A}^2+\frac{m_2\mu_1^2}{3k\lambda_1^2}(\vec{B}-\vec{A})^2\right].$$

Out of the four ν's, only one, say, ν_1 remains to be determined. To determine the rest of the constants D, E, F, K, L and M, we proceed as follows:

From the initial conditions $T_{11}=T_1$ and $T_{22}=T_2$ at $t=0$ and making use of the equations (49)—(51), T_∞ is calculated from T_{11} and also from T_{22}. Since the system relaxes to equilibrium, these two values of T_∞ must be the same. That yields the condition

$$(\lambda_2 K-\lambda_1 D)+(\lambda_2 L-\lambda_1 E)+(\lambda_2 M-\lambda_1 F)=0\,. \tag{52}$$

This relation holds good whatever the values of the mean flow velocities of the two components be. Also, T_{12} and T_{21} should tend to T_∞ as $t\to\infty$ irrespective of the values of the mean flow velocities; hence by (50) we have

$$K=-\frac{1}{2}L=M \tag{53}$$

and

$$D=-\frac{1}{2}E=F\,. \tag{54}$$

In view of (53) and (54), the condition (52) is automatically satisfied. Substituting the expressions for T_{12}, T_{21} and the solutions for T_{11} and T_{22} in the energy conservation equation (20) and equating the term independent of t, coefficient of $e^{-(\mu_1+\mu_2)t}$ and that of $e^{-2(\mu_1+\mu_2)t}$ to zero separately, we have

$$(K+D)+(L+E)+(M+F)=0\,,$$
$$-2\mu_1(K+D)+(\mu_2-\mu_1)(L+E)+2\mu_2(M+F)=0\,, \tag{55}$$
$$\mu_1^2(K+D)-\mu_1\mu_2(L+E)+\mu_2^2(M+F)=\frac{m_1\mu_2}{3k\lambda_2}(\mu_1+\mu_2)^2\left[2-\frac{\mu_1}{\lambda_1}-\frac{\mu_2}{\lambda_2}\right],$$

Solving these equations, we get

$$K+D=\frac{m_1\mu_2}{3k\lambda_2}\left(2-\frac{\mu_1}{\lambda_1}-\frac{\mu_2}{\lambda_2}\right). \tag{56}$$

Since

$$\begin{aligned}&\frac{3k}{m_1}\left\{T_{21}-T_{11}-\frac{\nu_1}{\lambda_1}(T_{22}-T_{11})\right\}\\&=\left(\frac{6k}{m_1^2 m_2}K\right)\frac{1}{2}\frac{m_1 m_2}{m_1+m_2}(\vec{u}_{11}-\vec{u}_{22})^2,\end{aligned} \tag{56a}$$

the factor $\left(\frac{6kK}{m_1^2 m_2}\right)$ represents the fraction of kinetic energy that has been used up to heat the scattered particles of type 1. Hence we can evaluate K by finding out the average of

$$\frac{\frac{1}{2}m_1(\vec{\xi}_1'^2-\vec{\xi}_1^2)}{\frac{1}{2}\frac{m_1 m_2}{m_1+m_2}(\vec{\xi}_2-\vec{\xi}_1)^2}.$$

When the law of interaction is COULOMB's law this average value is approximately $\frac{1}{10}$. Knowing K, (56) determines D.

The factor $\left[2-\frac{\mu_1}{\lambda_1}-\frac{\mu_2}{\lambda_2}\right]$ would reduce to unity on using Jeans' relations for $\frac{\mu_1}{\lambda_1}$ and $\frac{\mu_2}{\lambda_2}$.

Substituting the values of the constants, the expressions (24) and (22) for T_{12} and T_{21} become

$$\begin{aligned}T_{12}&=T_{22}+\frac{\nu_2}{\lambda_2}(T_{11}-T_{22})+D(\vec{u}_{22}-\vec{u}_{11})^2,\\T_{21}&=T_{11}+\frac{\nu_1}{\lambda_1}(T_{22}-T_{11})+K(\vec{u}_{11}-\vec{u}_{22})^2.\end{aligned} \tag{57}$$

In order to understand the physical significance of these terms, consider the energy change in component "1" due to collisions with particles of component "2" given by

$$\begin{aligned}&\frac{m_1 N_1 N_2}{\sigma}\frac{3k}{m_1}\left[T_{21}-T_{11}+\frac{m_1}{3k}(\vec{u}_{21}^2-\vec{u}_{11}^2)\right]\\&=\frac{m_1 N_1 N_2}{\sigma}\frac{3k}{m_1}\left[\frac{\nu_1}{\lambda_1}(T_{22}-T_{11})+K(\vec{u}_{22}-\vec{u}_{11})^2+\frac{m_1}{3k}(\vec{u}_{21}^2-\vec{u}_{11}^2)\right].\end{aligned} \tag{58}$$

The first term can be considered as arising from the temperature difference between the two components as determined by NEWTON's law of heat transfer. Hence, from experimental results the factor $\frac{\nu_1}{\lambda_1}$ can be fixed. The second term is that proportion of the kinetic energy of the relative motion of the two components which is retained in component "1". As the model assumes that after collision the particles of component "1" are distributed around $\vec{u}_{21}$ and that they are distributed around $\vec{u}_{11}$ before collision, the third term represents simply the change in kinetic energy of the scattered particles of component "1". Since radiation and other types of inelastic processes are neglected, these are the only processes by which energy can be gained and the theory adequately takes account of these mechanisms. This justifies the assumption of the particular form of T_{12} and T_{21}. Let us write down the values of the constants that have been used in (49) and (51) after simplification:

$$x_0 = y_0 = x_1 = y_1 = 0\,,$$

$$\left.\begin{aligned} x_2 &= \frac{2m_1\mu_2}{3k\lambda_2}(\mu_1+\mu_2)\,\lambda_2\left[\frac{\nu_1}{\mu_1+\mu_2} - K\frac{3k\lambda_2}{m_1\mu_2} - \frac{\mu_2}{\lambda_2}\right](\vec{A}-\vec{B})^2,\\ y_2 &= \frac{2m_1\mu_2}{3k\lambda_2}(\mu_1+\mu_2)\,\lambda_1\left[\frac{\nu_2}{\mu_1+\mu_2} - D\frac{3k\lambda_2}{m_1\mu_2} - \frac{\mu_1}{\lambda_1}\right](\vec{A}-\vec{B})^2. \end{aligned}\right\} \quad (59)$$

$$T_\infty = \frac{m_1\mu_2}{3k\lambda_2}\,\frac{\lambda_1\nu_2}{(\mu_1+\mu_2)(\nu_1+\nu_2)}\,(\vec{A}-\vec{B})^2,$$

These values of the constants give us the complete expressions for the temperatures for all times t.

If we now consider linearized problems such as plasma oscillations, we may neglect the terms containing the squares and products of the velocities. In that case

$$T_{12} = T_{22} + \frac{\nu_2}{\lambda_2}(T_{11} - T_{22})\,, \quad T_{21} = T_{11} + \frac{\nu_1}{\lambda_1}(T_{22} - T_{11})\,, \quad (60)$$

and the right hand sides of (31) and (32) become

$$\frac{N_1N_2}{\sigma}\,\frac{3k}{m_1}\,\frac{\nu_1}{\lambda_1}(T_{22} - T_{11})\,, \quad \frac{N_1N_2}{\sigma}\,\frac{3k}{m_2}\,\frac{\nu_2}{\lambda_2}(T_{11} - T_{22})\,,$$

respectively. Thus we find that the transfer equations are simple and physically meaningful. Knowing the values of $\vec{u}_{12}$, $\vec{u}_{21}$, T_{12} and T_{21} we can write down the Boltzmann equations which can be solved for specific (given) initial distributions and microscopic boundary conditions.

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Prof. P. L. Bhatnagar, Department of Applied Mathematics,
Indian Institute of Science, Bangalore 12, India