Zeitschrift für Astrophysik, Bd. 43, S. 273—288 (1957)

Radial Pulsations of an Infinite Cylinder with Finite Conductivity Immersed in Magnetic Field

By

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(Eingegangen am 10. August 1957)

In Part I we discuss the radial pulsations of an infinite cylinder immersed in a magnetic field H given by

$$H^2 = H_s^2 + (H_0^2 - H_s^2)\left\{1 - \left(\frac{r}{R}\right)^2\right\},$$

where H_0 and H_s are the magnetic fields on the axis and on the surface of the cylinder respectively and r and R are the distance from the axis and the radius of the cylinder. The particular case of a uniform magnetic field is obtained by taking $H_s = H_0$, while the particular case considered by LYTTKENS corresponds to $H_s = 0$. In LYTTKENS case, the magnetic field is proportional to the square root of the pressure at each point.

In Part II we discuss the effect of finite conductivity on the radial pulsations of an infinite cylinder immersed in uniform magnetic field. We evaluate the change in the phase of the displacement function and the amplitude of the magnetic field for $0 \leqq r \leqq R$ and the damping time of the first three modes for various magnitudes of the initial magnetic field.

1. CHANDRASEKHAR and FERMI [1], among other problems, discussed the radial pulsations of an infinite cylinder with infinite electrical conductivity and uniform density under the adiabatic conditions with a magnetic field parallel to the axis of the cylinder. They obtained an estimate for the frequency of the fundamental mode of pulsation using the variational principle. Later on LYTTKENS [3] reconsidered this problem taking the magnetic field varying as the square root of undisturbed pressure in the cylinder. He obtained the frequencies and the displacement function for the various modes in an explicit form. His main conclusion is that the displacement functions in this case are the same as those in the case of no magnetic field, while the frequencies are increased. His expression for the frequency of the n^{th} mode gives the following relation between the frequency w_n with magnetic field, frequency $w_{n\,o}$ with no magnetic field and the magnetic field H_o at the axis:

$$\frac{w_n^2 - w_{n\,o}^2}{w_{n\,o}^2} = \frac{n^2\gamma}{n^2\gamma - 1} H_0^2, \tag{1.1}$$

where γ is the ratio of specific heats. The assumed magnetic field gives rise to a volume current throughout the cylinder which vanishes at the axis but tends to infinity at the surface.

Recently, CHOPRA and TALWAR [2] have considered the problem with the field

$$H^2 = H_s^2 + (H_0^2 - H_s^2)(1 - x^2)\,, \tag{1.2}$$

where $x = \frac{r}{R}$, r denoting the distance from the axis and R the radius of the cylinder. LYTTKENS's case is a particular case of (1.2) when $H_s = 0$, while the case of uniform magnetic field is obtained by taking $H_s = H_0$. Evidently, (1.2) allows a wide choice of magnetic fields. Besides, this also gives rise to volume currents within the cylinder which remain finite even at the surface in the case when $H_s \neq 0$.

Following LYTTKENS, CHOPRA and TALWAR have terminated the series for the displacement function to obtain the frequencies for various modes. However, the series is convergent for every value of x, such that $0 \leqq x \leqq 1$, except in the case $H_s = 0$ when it becomes divergent at $x = 1$. This was the reason why LYTTKENS had to terminate the series. Consequently the displacement functions obtained by them do not satisfy the boundary condition.

$$\delta P = 0\,, \text{ at } x = 1\,, \tag{1.3}$$

which secures that the boundary of the cylinder is the surface of steady pressure. It may be noted, however, that this condition is satisfied in the case $H_s = 0$, if the series for the displacement function and its derived series are convergent at $x = 1$, which is possible only when the former is terminated. This justifies the procedure adopted by LYTTKENS. But when $H_s \neq 0$, the convergence of the series does not ensure (1.3).

In the present note we shall investigate the effect of taking the electrical conductivity σ to be finite but so large that squares and higher powers of $\frac{1}{\sigma}$ may be neglected. To be able to evaluate the effects of finiteness of conductivity we have to first satisfactorily solve the case of infinite conductivity. Consequently, we shall first discuss the radial pulsations of an infinite cylinder of infinite electrical conductivity with magnetic field (1.2).

Part I

2. *The case of infinite conductivity.* The equations governing the radial pulsations of a cylinder under adiabatic conditions, as given by CHANDRASEKHAR and FERMI [1], are:

$$\left(w^2 + \frac{4\,GM}{r^2}\right)\delta r = -\,4\,\pi^2 r \frac{d}{dm}\left[\left(\gamma p + \frac{H^2}{4\pi}\right)\varrho \frac{d}{dm}(r\,\delta r)\right], \tag{2.1}$$

$$\delta P = -\left(\gamma p + \frac{H^2}{4\pi}\right)\varrho \frac{d}{dm}(2\,\pi r\,\delta r) \tag{2.2}$$

and

$$\delta H = -\frac{H}{r}\frac{d}{dr}(r\,\delta r)\,, \tag{2.3}$$

where δr is the amplitude of displacement, w the frequency, p, ϱ, H, are the gas pressure, density, magnetic field at an internal point and m the mass per unit length of the cylinder within a radius r in the equilibrium state. In (2.2) P is the total pressure given by

$$P = p + \frac{H^2}{8\pi}, \tag{2.4}$$

and δP and δH are the amplitudes of the variations in P and H following the motion.

We have to solve (2.1) under the boundary conditions:

(i) $$\delta r = 0 \text{ at } r = 0 \tag{2.5}$$

and

(ii) $$\delta P = 0 \text{ at } r = R. \tag{2.6}$$

The total pressure P should be continuous across the boundary of the cylinder, i. e., denoting the quantities inside and outside the cylinder by the super-scripts (i) and (e) respectively

$$p^{(i)} + \frac{(H^{(i)})^2}{8\pi} = \frac{(H^{(e)})^2}{8\pi}, \tag{2.7}$$

assuming that there is no matter outside the cylinder. In the case of no surface current $H^{(i)} = H^{(e)}$ and hence, on the boundary of the cylinder $p^{(i)} = 0$. $H^{(e)}$ is either zero or uniform throughout the space outside the cylinder. Thus in the cases in which $H^{(i)}$ vanishes on the boundary of the cylinder, as in the case contemplated by LYTTKENS, one can take $H^{(e)} = 0$ to avoid surface currents and then $p^{(i)} = 0$ on the surface. But in the cases where $H^{(i)} \neq 0$ on the boundary we can avoid the occurence of surface currents by taking $H^{(e)}$ to be uniform and equal to $H^{(i)}_{\text{surface}}$. In this case also $p^{(i)} = 0$ on the boundary. In all these cases, the cylinder is in true equilibrium. But in the cases in which $H^{(e)} \neq H^{(i)}$ on the surface, we have a system of surface currents, and at the boundary $p^{(i)} \neq 0$. If $H^{(e)} > H^{(i)}_{\text{surface}}$, the continuity of P across the surface will require $p^{(i)} > 0$ there, on the other hand if $H^{(e)} < H^{(i)}$ on the surface, $p^{(i)}$ will be negative there. In the last case the cylinder is not in true equilibrium. If we take $H^{(e)} = 0$ and $H^{(i)}_{\text{surface}} \neq 0$, then we come across this situation. In this case we have surface currents, and one would fail to understand the physical significance of $p^{(i)}$ being negative. Consequently we have assumed that with the magnetic field (1.2), $H^{(e)}$ is uniform and equal to H_s, so that there are no surface currents and $p^{(i)}_{\text{surface}} = 0$, while

$$P^{(i)}_{\text{surface}} = \frac{H_s^2}{8\pi}. \tag{2.8}$$

In view of (2.8), we have, when $\varrho = \text{constant}$,

$$P = \pi G \varrho^2 R^2 (1 - x^2) + \frac{H_s^2}{8\pi} \tag{2.9}$$

so that

$$p = p_c(1 - x^2)\,, \tag{2.10}$$

where

$$p_c = \pi G \varrho^2 R^2 + \frac{H_s^2}{8\pi} - \frac{H_0^2}{8\pi}\,. \tag{2.11}$$

Using (2.10) and setting

$$\frac{r}{R} = x \text{ and } \frac{\delta r}{R} = \psi \tag{2.12}$$

in (2.1), (2.2) and (2.3), we have

$$A\psi = -\frac{d}{dx}\left[(1 - x^2 + f)\frac{1}{x}\frac{d}{dx}(x\psi)\right], \tag{2.13}$$

$$\delta P = -\left[\gamma p_c + \frac{H_0^2 - H_s^2}{4\pi}\right](1 - x^2 + f)\frac{1}{x}\frac{d}{dx}(x\psi) \tag{2.14}$$

and

$$\delta H = -\left[H_s^2 + (H_0^2 - H_s^2)(1 - x^2)\right]^{1/2}\frac{1}{x}\frac{d}{dx}(x\psi)\,, \tag{2.15}$$

where

$$A = \frac{\frac{1}{\gamma}\left(\frac{w^2}{\pi G \varrho} + 4\right)}{1 + \frac{(H_0^2 - H_s^2)(2-\gamma)}{8\pi^2 R^2 \varrho^2 G \gamma}} \tag{2.16}$$

and

$$f = \frac{H_s^2}{4\pi\gamma p_c + H_0^2 - H_s^2} = \frac{V_s^2}{S_c^2 + V_c^2 - V_s^2}\,. \tag{2.17 and 2.17'}$$

Where S_c is the sound velocity at the axis and V_c, V_s are the velocities of magneto-hydrodynamic waves at the axis and at the surface respectively.

To simplify integration we substitute

$$\psi = \frac{d\Phi}{dx} \tag{2.18}$$

in (2.13), which on integration becomes

$$A\Phi = -(1 - x^2 + f)\frac{1}{x}\frac{d}{dx}\left(x\frac{d\Phi}{dx}\right). \tag{2.19}$$

The boundary conditions (2.5) and (2.6) now become

$$\frac{d\Phi}{dx} = 0 \quad \text{at} \quad x = 0 \tag{2.20}$$

and

$$\frac{d}{dx}\left(x\frac{d\Phi}{dx}\right) = 0 \quad \text{at} \quad x = 1 \tag{2.21}$$

which on using (2.19) simplifies to

$$\frac{A\Phi}{1 - x^2 + f} = 0 \quad \text{at} \quad x = 1$$

i. e.,

$$\Phi = 0 \quad \text{at} \quad x = 1\,, \tag{2.22}$$

since at $x = 1$, the denominator does not vanish, when $f \neq 0$ i. e., $H_s \neq 0$. But when $f = 0$, from (2.14) $\delta P = 0$ in virtue of the factor $(1 - x^2)$ provided $\frac{d}{dx}\left(x \frac{d\Phi}{dx}\right)$ is finite at $x = 1$.

The condition $\Phi = 0$ at $x = 1$ enables us to determine the values of A, and hence the frequencies, corresponding to a given value of f.

We might note that in view of (2.21) $\delta H = 0$ at the surface $x = 1$. Hence we will fail to observe the magnetic pulsations of such a system except through the change in the period of pulsation.

3. Integration of (2.19). On substituting

$$\Phi = \sum_{n=0}^{\infty} a_n x^{n+c}, \; a_0 \neq 0\,, \tag{3.1}$$

in (2.19), the indicial equation gives

$$a_0 c^2 = 0 \tag{3.2}$$

i. e.,

$$c = 0\,.$$

Hence

$$\Phi = \sum_{n=0}^{\infty} a_n x^n\,. \tag{3.3}$$

In view of (2.20),

$$a_1 = 0 \tag{3.4}$$

and the recurrence formula for the coefficients is

$$a_{n+2} = \frac{n^2 - A}{(1+f)(n+2)^2} a_n\,. \tag{3.5}$$

Hence from (3.4)

$$a_{2n+1} = 0 \tag{3.6}$$

and

$$a_{2n+2} = \frac{n^2 - \frac{A}{4}}{(1+f)(n+1)^2} a_{2n}\,, \quad n = 0, 1, 2, \ldots \tag{3.7}$$

Hence

$$\Phi = \sum_{n=1}^{\infty} a_{2n} x^{2n} \tag{3.8}$$

and

$$\psi = \sum_{n=1}^{\infty} 2n\, a_{2n} x^{2n-1}\,. \tag{3.9}$$

Both of these series are convergent for $0 \leqq x \leqq 1$, provided $f \neq 0$.

Now the boundary condition (2.22) gives

$$\sum_{n=0}^{\infty} a_{2n} = 0 \tag{3.10}$$

i. e.,

$$1 + \sum_{n=1}^{\infty} \frac{\left(-\frac{A}{4}\right)\left(1^2 - \frac{A}{4}\right)\left(2^2 - \frac{A}{4}\right)\cdots\left((n-1)^2 - \frac{A}{4}\right)}{1^2 \quad 2^2 \quad 3^2 \quad \cdots \quad n^2} \frac{1}{(1+f)^n} = 0 \ldots \tag{3.11}$$

The above equation determines frequencies when f is given. We have solved this equation by numerical methods for the cases

$$f = \frac{1}{9}, 1, 9, \; 9\,9. \tag{3.12}$$

In the last case we have determined A for the first mode only, while in the remaining cases we have determined A for the first three modes. These values of A have been given in Table 1. In table 2 we give the

Table 1. *Values of $A/4$*

f	Mode 1	Mode 2	Mode 3
1/9	1.1948	5.2445	12.393
1	2.5396	12.5615	30.5654
9	14.1368	73.7582	181.023
99	144.2638		

Table 2. *Values of Φ and ψ functions for $f = 1$* (Taking $a_0 = 1$)

x	Mode 1 $\frac{A}{4} = 2.5396$		Mode 2 $\frac{A}{4} = 12.5615$		Mode 3 $\frac{A}{4} = 30.5654$	
	Φ	ψ	Φ	ψ	Φ	ψ
0.0	1	0	1	0	1	0
0.1	0.9873	—0.2530	0.9381	—1.2201	0.8527	—2.8356
0.2	0.9496	—0.5001	0.7630	—2.2301	0.4739	—4.4601
0.3	0.8877	—0.7352	0.5051	—2.8502	0.0249	—4.1900
0.4	0.8032	—0.9513	0.2101	—2.9598	—0.3057	—2.1844
0.5	0.6981	—1.1436	—0.0684	—2.5211	—0.3693	0.5268
0.6	0.5755	—1.3023	—0.2779	—1.5975	—0.2237	2.5564
0.7	0.4391	—1.4193	—0.3774	—0.3618	0.0597	2.7510
0.8	0.2934	—1.4836	—0.3489	0.9097	0.2598	0.9808
0.9	0.1446	—1.4814	—0.2061	1.8616	0.2299	—1.5360
1.0	0	—1.3935	0	2.1140	—0	—2.6478

values of the characteristic function Φ and the displacement function ψ for the first three modes for the case $f = 1$. In table 3 we give the values of

$$\left(\frac{\delta H}{H}\right)_n = \frac{A_n \Phi_n}{1 - x^2 + f}, \quad n = 1, 2, 3, \tag{3.13}$$

where n denotes the order of the mode for $f = 1$.

In passing we may note that ψ_j form an orthogonal set of functions, i. e.,

$$\int_0^1 \psi_j \psi_k x dx = 0, \text{ when } j \neq k. \tag{3.14}$$

This property is the direct outcome of our boundary condition that $\Phi = 0$ at $x = 1$. In the work in reference [2] the displacement functions

Table 3. *Variation of* $\frac{\delta H}{H}$ *with* x, *for* $f = 1$

x	Mode 1	Mode 2	Mode 3
0.0	$\frac{\delta H}{H} \propto$ 5.079	25.123	61.131
0.1	5.040	23.686	52.391
0.2	4.922	19.561	29.559
0.3	4.721	13.289	1.592
0.4	4.434	5.737	—20.310
0.5	4.053	— 1.965	—25.805
0.6	3.565	— 8.513	—16.673
0.7	2.954	—12.559	4.833
0.8	2.192	—12.891	23.356
0.9	1.234	— 8.702	23.618
1.0	0	0	0

ψ's are not orthogonal, contrary to the assertion of the authors. The statement may be verified by actually substituting in (3.14) the values for ψ_j's which they have obtained.

Let

$$B = \frac{(H_0^2 - H_s^2)\left(\frac{2}{\gamma} - 1\right)}{8\pi^2 R^2 \varrho^2 R} = \frac{H_0^2}{\left(\frac{8 M^2 G}{R^2}\right)}\left[1 - \left(\frac{H_s}{H_0}\right)^2\right]\left(\frac{2}{\gamma} - 1\right) \ldots \tag{3.15}$$

Then, when

$$H_s = H_0, \quad B = 0; \tag{3.16}$$

when $H_s \neq H_0$, from (2.17) B is determined by

$$f = \frac{B}{1 + B} \frac{2\gamma}{2 - \gamma} \frac{(H_s/H_0)^2}{1 - (H_s/H_0)^2}. \tag{3.17}$$

From (2.16)

$$\frac{T_n}{T_{n0}} = \sqrt{\frac{n^2\gamma - 1}{(1 + B)\left(\frac{A_n}{4}\right)\gamma - 1}}, \tag{3.18}$$

where T_n, T_{n0} are respectively the periods of pulsation in the n$^{\text{th}}$ mode with and without magnetic field. For a given cylinder, i. e., for fixed values of M and R, if we choose H_s/H_0, larger H_0 means larger B and larger f. From table 1 we find that as f increases, $\frac{A_n}{4}$ increases. Hence from (3.18) T_n decreases with increasing H_0.

In table 4 we collect $\frac{T_n}{T_{n0}}$ for $\gamma = 1.5$ and $H_s = H_0$ so that $f = \frac{H_0^2}{\frac{4M^2G\gamma}{R^2}}$, where M is the mass of the cylinder per unit length.

Table 4. *Showing ratio of period of pulsation T with constant magnetic field H to period of pulsation T_0 without magnetic field* (Taking 8 = 1.5)

$f = \frac{H^2}{\frac{6M^2G}{R^2}}$	Mode 1	Mode 2	Mode 3
$\frac{1}{9}$	$\frac{T}{T_0} = .794$	.853	.843
1	.422	.529	.528
9	.157	.214	.215
99	.048		

Special cases. We give below the cases in which the series (3.11) becomes finite.

(i) If we take $A = 16$, (3.11) reduces to

$$1 - \frac{4}{1+f} + \frac{3}{(1+f)^2} = 0 \quad (3.19)$$

which gives

$$f = 0 \text{ and } f = 2\,.$$

When $f = 0$ i. e. no magnetic field, $A = 16$ gives the second mode but when $f = 2$, $A = 16$ gives the first mode. In the latter case

$$\Phi = a_0\left(1 - \frac{4}{3}x^2 + \frac{1}{8}x^4\right) \quad (3.20)$$

$$\psi = a_0\left(-\frac{8}{3}x + \frac{4}{3}x^3\right) \quad (3.21)$$

$$\left(\frac{\delta H}{H}\right)_1 = \frac{16}{3}a_0(1 - x^2)\,. \quad (3.22)$$

(ii) Let $A = 36$, so that

$$1 - \frac{9}{1+f} + \frac{18}{(1+f)^3} - \frac{10}{(1+f)^3} = 0 \quad (3.23)$$

and

$$f = 0 \quad f = 3 - \sqrt{6},\ f = 3 + \sqrt{6}\,. \quad (3.24)$$

when $f = 3 + \sqrt{6}$, $A = 36$ gives the first mode, when $f = 3 - \sqrt{6}$ $A = 36$ gives the second mode, while when $f = 0$, this value of A gives the third mode, when $f = 3 + \sqrt{6}$

$$\Phi = a_0\left(1 - \frac{9x^2}{4+\sqrt{6}} + \frac{18x^4}{(4+\sqrt{6})^2} - \frac{10x^6}{(4+\sqrt{6})^3}\right) \quad (3.25)$$

$$\psi = a_0\left(-\frac{18}{4+\sqrt{6}}x + \frac{72x^3}{(4+\sqrt{6})^2} - \frac{60x^5}{(4+\sqrt{6})^3}\right) \quad (3.26)$$

$$\left(\frac{\delta H}{H}\right)_1 = \frac{36}{4+\sqrt{6}}a_0\left(1 - \frac{8}{4+\sqrt{6}}x^2 + \frac{10}{(4+\sqrt{6})^2}x^4\right)\,. \quad (3.27)$$

Similarly we can discuss the cases when $A = 64$, 100, etc.

Part II

Finite Conductivity

4. Equations of the problem. The radial pulsation of the cylinder with vanishing amplitude is determined by

$$\frac{d^2 r}{d t^2} = -\frac{2\,G m(r)}{r} - \frac{1}{\varrho}\frac{\partial p}{\partial r} + \frac{\mu}{\varrho}\left(\vec{J} \times \vec{H}\right)_r \tag{4.1}$$

and

$$m(r) = m(r_0)\,, \tag{4.2}$$

where r_0 is the equilibrium value of r.

The electromagnetic properties of the material at a point are governed by the Maxwell equations:

$$\operatorname{curl} \vec{H} = 4\,\pi \vec{j} \tag{4.3}$$

$$\operatorname{div} \vec{H} = 0 \tag{4.4}$$

$$\operatorname{curl} \vec{E} = -\frac{\mu}{c}\frac{\partial \vec{H}}{\partial t} \tag{4.5}$$

$$\operatorname{div} \vec{E} = 0 \tag{4.6}$$

and the constitutive relation

$$\vec{j} = \sigma\left(\vec{E} + \frac{\mu}{c}\frac{\partial \vec{r}}{\partial t} \times \vec{H}\right), \tag{4.7}$$

where we have neglected the displacement current and taken the free charge density to be zero and μ to be constant.

Let

$$r = r_0 + \delta r\,,\ p = p_0 + \delta p\,,\ \varrho = \varrho_0 + \delta\varrho\,,\ \text{and}\ \vec{H} = \vec{H}_0 + \delta\vec{H}\,,\ \ldots \tag{4.8}$$

In the equilibrium state, H_0 must be uniform otherwise there will volume current which cannot be in steady state in the presence of finite conductivity. We shall, consequently, take H_0 to be uniform parallel to the axis of the cylinder both inside and outside i. e. the cylinder is immersed in a uniform magnetic field and

$$\vec{j}_0 = 0 \ \text{and}\ \vec{E}_0 = 0 \tag{4.9}$$

from (4.3) and (4.7).

Retaining only the first powers of δr, δp, etc., the equation (4.2) gives

$$\frac{\delta\varrho}{\varrho_0} = -\frac{\delta r}{r_0} - \frac{\partial}{\partial r_0}(\delta r)\,. \tag{4.10}$$

The adiabatic relation gives

$$\delta p = \gamma\frac{p_0}{\varrho_0}\delta\varrho = -\gamma\frac{p_0}{r_0}\frac{\partial}{\partial r_0}(r_0\delta r)\,. \tag{4.11}$$

Also from (4.3)—(4.7), we have

$$\begin{aligned}\left(\vec{j} \times \vec{H}\right) &= \left(\vec{j} \times \vec{H}_0\right) = \frac{1}{4\pi}\operatorname{curl}\left(\delta\vec{H}\right) \times \vec{H}_0 \\ &= \frac{1}{4\pi}\left[-\operatorname{grad}\left(\delta\vec{H}\cdot\vec{H}_0\right) + \left(\vec{H}_0\cdot\operatorname{grad}\right)\delta\vec{H}\right]\end{aligned} \tag{4.12}$$

and

$$\operatorname{curl}\operatorname{curl}\left(\delta \vec{H}\right) = \frac{4\pi\mu\sigma}{c}\operatorname{curl}\left(\frac{\partial(\delta\vec{r})}{\partial t}\times\vec{H}_0\right) - \frac{\partial}{\partial t}\left(\delta\vec{H}\right). \quad (4.13)$$

In the present case variable quantities will depend only on r_0 and t, so that (4.12) and (4.13) reduce to

$$\left(\vec{j}\times\vec{H}\right)_r = -\frac{1}{4\pi}\frac{\partial}{\partial r_0}\left(H_0\delta H_z\right) \quad (4.14)$$

and

$$\frac{\partial}{\partial t}(\delta H_z) - \frac{c}{4\pi\mu\sigma}\frac{1}{r_0}\frac{\partial}{\partial r_0}\left\{r_0\frac{\partial}{\partial r_0}(\delta H_z)\right\} = -\frac{H_0}{r_0}\frac{\partial}{\partial r_0}\left\{r_0\frac{\partial}{\partial t}(\delta r)\right\}. \quad (4.15)$$

Substuting (4.10), (4.11) and (4.14) in (4.1) we have

$$\frac{\partial^2}{\partial t^2}(\delta r) = \frac{4Gm(r_0)}{r_0^2}\delta r + \frac{\gamma}{\varrho_0}\frac{\partial}{\partial r_0}\left[\frac{p_0}{r_0}\frac{\partial}{\partial r_0}(r_0\delta r)\right] - \frac{\mu}{4\pi\varrho_0}\frac{\partial}{\partial r_0}(H_0\delta H_z)\ldots. \quad (4.16)$$

Eliminating δH_z between (4.15) and (4.16) we have, on dropping the suffix 0,

$$\frac{\partial}{\partial t}\left[\varrho\left(\frac{4Gm}{r^2}\delta r - \frac{\partial^2}{\partial t^2}\delta r\right) + \frac{\partial}{\partial r}\left(\gamma p + \frac{\mu H^2}{4\pi}\right)\frac{1}{r}\frac{\partial}{\partial r}(r\,\delta r)\right] = \frac{c}{4\pi\mu\sigma}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left\{r\varrho\left[\frac{4Gm}{r^2}dr - \frac{\partial}{\partial t^2}\delta r + \frac{\gamma}{\varrho}\frac{\partial}{\partial r}\left(\frac{p}{r}\frac{\partial}{\partial r}\overline{r\,\delta r}\right)\right]\right\}\right] \quad (4.17)$$

and

$$\frac{\partial}{\partial t}\left(\frac{\delta H_z}{H}\right) = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial t}\delta r\right) + \frac{c}{\sigma H^2\mu^2}\frac{1}{r}\frac{\partial}{\partial r}\left[r\varrho\left(\frac{4Gm}{r^2}\delta r - \frac{\partial^2}{\partial t^2}\delta r\right) + r\gamma\frac{\partial}{\partial r}\left\{\frac{p}{r}\frac{\partial}{\partial r}(r\,\delta r)\right\}\right]. \quad (4.18)$$

For uniform density

$$m = \pi r^2\varrho \quad (4.19)$$

and hence integrating the equation of equilibrium

$$\frac{dp}{dr} = -2\pi G\varrho^2 r \quad (4.20)$$

we get

$$p = \pi G\varrho^2(R^2 - r^2), \quad (4.21)$$

since

$$p = 0 \text{ when } r = R.$$

We shall now write

$$\frac{r}{R} = x, \quad \frac{\delta r}{R} = \psi \quad (4.22)$$

and assume ψ to vary as e^{iwt}; then using (4.21) in (4.17) we have

$$A\,\psi + \frac{d}{dx}\left[(1 - x^2 + f)\frac{1}{x}\frac{d}{dx}(x\,\psi)\right] + \\ + \frac{i\,B\,c}{\sigma}\frac{d}{dx}\left[\frac{1}{x}\frac{d}{dx}\left\{A\,x\,\psi + x\frac{d}{dx}\left[\frac{1-x^2}{x}\frac{d}{dx}(x\,\psi)\right]\right\}\right] = 0 \tag{4.23}$$

where

$$A = \frac{1}{\gamma}\left(4 + \frac{w^2}{\pi\,G\,\varrho}\right) \tag{4.24}$$

$$B = \frac{1}{4\pi\,\mu\,R^2 w} \tag{4.25}$$

and

$$f = \frac{\mu\,H^2}{4\pi^2\,G\,\gamma\,\varrho^2\,R^2}\,. \tag{4.26}$$

Similarly, from (4.18) we have the amplitude δH_z of variation of the magnetic field:

$$\frac{\delta H_z}{H} = -\frac{1}{x}\frac{d}{dx}(x\,\psi) + \frac{B\,c}{\sigma\,f\,i}\frac{1}{x}\frac{d}{dx}\left[A\,x\,\psi + x\frac{d}{dx}\left\{\frac{1-x^2}{x}\frac{d}{dx}(x\,\psi)\right\}\right]. \;.\;.\;. \tag{4.27}$$

We have to solve the equation (4.23) under the boundary conditions:

(i) $\psi = 0 \quad \text{at } x = 0$ (4.28)

and

(ii) $\delta P = 0 \quad \text{at } x = 1$ (4.29)

i. e.,

$$\delta H_z = 0 \quad \text{at } x = 1\,. \tag{4.29'}$$

Let us now take

$$\psi = \frac{d\,\Phi}{d\,x} \tag{4.30}$$

in (4.23) and integrate it; we have

$$A\,\Phi + (1 - x^2 + f)\frac{1}{x}\left\{\frac{d}{dx}\left(x\frac{d\,\Phi}{d\,x}\right)\right\} + \\ + \frac{i\,B\,c}{\sigma}\frac{1}{x}\frac{d}{dx}\left[A\,x\frac{d\,\Phi}{d\,x} + x\frac{d}{dx}\left\{\frac{1-x^2}{x}\frac{d}{dx}\left(x\frac{d\,\Phi}{d\,x}\right)\right\}\right] = 0 \tag{4.31}$$

In terms of Φ, (4.27) becomes

$$\frac{\delta H_z}{H} = -\frac{1}{x}\frac{d}{dx}\left(x\frac{d\,\Phi}{d\,x}\right) + \\ + \frac{B\,c}{\sigma\,f\,i}\frac{1}{x}\frac{d}{dx}\left[A\,x\frac{d\,\Phi}{d\,x} + x\frac{d}{dx}\left\{\frac{1-x^2}{x}\frac{d}{dx}\left(x\frac{d\,\Phi}{d\,x}\right)\right\}\right] \\ = \frac{1}{f}\left[A\,\Phi + \frac{(1-x^2)}{x}\frac{d}{dx}\left(x\frac{d\,\Phi}{d\,x}\right)\right]. \tag{4.32}$$

Hence the boundary condition (4.29′) gives us

$$\Phi = 0 \text{ at } x = 1\,. \tag{4.33}$$

Let

$$B_0 = \frac{1}{4\pi\,\mu\,R^2 w_0} \text{ and } \frac{B_0 C}{\sigma} = \tau\,, \tag{4.34}$$

where we shall regard τ as a small dimensionales parameter of order one. We retain quantities of the order τ only.

$$\left.\begin{aligned} w &= w_0 + \tau\, w_1 \\ \psi &= \psi_0 + \tau\, \Phi_1 \\ A &= A_0 + \tau\, A_1 \\ B &= B_0 + \tau\, B_1\,, \end{aligned}\right] \tag{4.35}$$

where

$$\begin{gathered} A_0 = \frac{1}{\gamma}\left(4 + \frac{u_0^2}{\pi\, G\,\varrho}\right), \quad A_1 = \frac{2\, w_0}{\pi\, G\,\varrho\,\gamma}\, w_1 \\ B_1 = -\frac{w_1}{w_0}\, B_0\,, \end{gathered} \tag{4.36}$$

Substituting (4.35) in (4.31) and separating the various order terms we have

$$A_0 \Phi_0 + (1 - x^2 + f)\,\frac{1}{x}\,\frac{d}{d\,x}\left(x\,\frac{d\,\Phi_0}{d\,x}\right) = 0$$

with

$$\frac{d\,\Phi_0}{d\,x} = 0 \text{ at } x = 0$$

$$\Phi_0 = 0 \text{ at } x = 1 \tag{4.37}$$

and

$$A_0 \Phi_1 + (1 - x^2 + f)\,\frac{1}{x}\,\frac{d}{d\,x}\left(x\,\frac{d\,\Phi_1}{d\,x}\right) + \\ + A_1 \Phi_0 - i\,f\,\frac{1}{x}\,\frac{d}{d\,x}\left[x\,\frac{d}{d\,x}\left\{\frac{1}{x}\,\frac{d}{d\,x}\left(x\,\frac{d\,\Phi_0}{d\,x}\right)\right\}\right] = 0$$

with

$$\frac{d\,\Phi_1}{d\,x} = 0 \text{ at } x = 0$$

and

$$\Phi_1 = 0 \text{ at } x = 1 \tag{4.38}$$

Due to the presence of i in the equation (4.38), we conclude that Φ_1 and A_1 are complex. Let us substitute

$$\begin{aligned} \Phi_1 &= \xi + i\,\eta \\ A_1 &= \alpha + i\,\beta \end{aligned} \tag{4.39}$$

in (4.38) and separate the real and imaginary parts. We have

$$A_0 \xi + (1 + x^2 + f)\,\frac{1}{x}\,\frac{d}{d\,x}\left(x\,\frac{d\,\xi}{d\,x}\right) + \alpha\,\Phi_0 = 0\,, \tag{4.40}$$

$$\begin{gathered} A_0\,\eta + (1 - x^2 + f)\,\frac{1}{x}\,\frac{d}{d\,x}\left(x\,\frac{d\eta}{d\,x}\right) + \beta\,\Phi_0 - \\ - f\,\frac{1}{x}\,\frac{d}{d\,x}\left[x\,\frac{d}{d\,x}\left\{\frac{1}{x}\,\frac{d}{d\,x}\left(x\,\frac{d\,\Phi_0}{d\,x}\right)\right\}\right] = 0\,. \end{gathered} \tag{4.41}$$

In (4.40) the parameter α has to be determined from the boundary condition that $\xi = 0$ at $x = 1$. Similarly the parameter β in (4.41) has to be determined by the condition that $\eta = 0$ at $x = 1$. The boundary conditions at $x = 0$ are $\frac{d\xi}{dx} = 0$ and $\frac{d\eta}{dx} = 0$.

Taking into consideration the boundary condition at $x = 0$, we assume

$$\xi = \sum_{n=0}^{\infty} b_n x^n$$

substituting it in (4.40) we have the following recurrence relation

$$b_{n+2} = \frac{(n^2 - A_0)\, b_n - \alpha\, a_n}{(n+2)^2\,(1+f)}$$

where

$$\Phi_0 = \sum_{n=0}^{\infty} a_n x^n$$

with

$$a_{2n+1} = 0$$

and

$$a_{2n+2} = \frac{n^2 - A_0/4}{(n+1)^2\,(1+f)}\, a_{2n}$$

as found out in Part I.

Since $b_1 = 0$ on account of the boundary condition at $x = 0$, we shall have only even powers of x in ξ i. e.,

$$\xi = \sum_{n=0}^{\infty} b_{2n} x^{2n}, \tag{4.42}$$

where

$$b_{2n+2} = \frac{\left(n^2 - \frac{A_0}{4}\right) b_{2n} - \frac{\alpha}{4}\, a_{2n}}{(n+1)^2\,(1+f)}. \tag{4.43}$$

In view of the recurrence relation (4.43)

$$\xi = \frac{b_0}{a_0}\, \Phi_0 + \alpha \sum_{n=0}^{\infty} \mu_{2n}\, x^{2n}$$

where

$$\mu_{2n+2} = \frac{\left(n^2 - \frac{A_0}{4}\right) \mu_{2n} - \frac{1}{4}\, a_{2n}}{(n+2)^2\,(1+f)}. \tag{4.44}$$

At $x = 1$, $\Phi_0 = 0$, and $\xi = 0$,
and hence

$$\alpha = 0\,. \tag{4.45}$$

From (4.39) and (4.36) we conclude that the change in w due to finite conductivity is purely imaginary. Thus to our approximation the period of oscillation will be unaffected by the assumption of finite conductivity.

We shall now consider (4.41). Let us substitute

$$\eta = \sum_{n=0}^{\infty} c_n x^n \tag{4.46}$$

in it. The coefficients c_n obey the following recurrence relation:

$$C_{n+2} = \frac{(n^2 - A_0)\, C_n - \beta\, a_n + f\,(n+4)^2\,(n+2)^2\, a_{n+4}}{(n+2)^2\,(1+f)}.$$

Since

$$a_{2n+1} = 0 \text{ and } C_1 = 0 \text{ as } \frac{d\eta}{dx} = 0$$

at $x = 0$, we have

$$C_{2n+1} = 0\,. \tag{4.47}$$

and

$$C_{2n+2} = \frac{\left(n^2 - \frac{A_0}{4}\right) C_{2n} - \frac{\beta}{4}\, a_{2n} + 4\, f\,(n+2)^2\,(n+1)^2\, a_{2n+4}}{(n+1)^2\,(1+f)}\,. \tag{4.47'}$$

Let us now assume that

$$C_{2n} = C_0 \lambda_{2n} + \beta\, \mu_{2n} + \nu_{2n}\,, \tag{4.48}$$

where

$$\lambda_{2n},\ \mu_{2n},\ \nu_{2n} \text{ are independent of } C_0 \text{ and } \beta\,.$$

Therefore

$$C_{2n+2} = C_0 \lambda_{2n+2} + \beta\, \mu_{2n+2} + \nu_{2n+2}$$

$$= \frac{\left(n^2 - \frac{A_0}{4}\right)(C_0 \lambda_{2n} + \beta\, \mu_{2n} + \nu_{2n}) - \frac{\beta}{4}\, a_{2n} + 4\, f\,(n+2)^2\,(n+1)^2\, a_{2n+4}}{(n+1)^2\,(1+f)}\,.$$

Equating the coefficients of C_0, β we have the following recurrence relations for λ, μ and ν

$$\lambda_{2n+2} = \frac{n^2 - \frac{A_0}{4}}{(n+1)^2\,(1+f)}\, \lambda_{2n} \tag{4.49}$$

$$\mu_{2n+2} = \frac{\left(n^2 - \frac{A_0}{4}\right) \mu_{2n} - \frac{1}{4}\, a_{2n}}{(n+1)^2\,(1+f)} \tag{4.50}$$

and

$$\nu_{2n+2} = \frac{\left(n^2 - \frac{A_0}{4}\right) \nu_{2n} + 4\, f\,(n+2)^2\,(n+1)^2\, a_{2n+4}}{(n+1)^2\,(1+f)}\,. \tag{4.51}$$

Hence

$$\eta = C_0 \sum_{n=0}^{\infty} \lambda_{2n} x^{2n} + \beta \sum_{n=0}^{\infty} \mu_{2n} x^{2n} + \sum_{n=0}^{\infty} \nu_{2n} x^{2n} \tag{4.52}$$

In view of (4.49) the first term on the right hand side is $\frac{C_0}{a_0}\, \Phi_0$, which vanishes at $x = 1$.

Therefore at $x = 1$

$$\beta = -\frac{\sum_{n=0}^{\infty} \nu_{2n}}{\sum_{n=0}^{\infty} \mu_{2n}}, \tag{4.53}$$

as $\eta = 0$ at $x = 1$.

(4.53) determins value of β for a given value of f.

We get the starting values for λ_0, μ_0, ν_0 from (4.48) by taking $n = 0$:

$$c_0 = c_0 \lambda_0 + \beta \mu_0 + \nu_0$$

so that

$$\lambda_0 = 1, \quad \mu_0 = \nu_0 = 0. \tag{4.54}$$

5. *The change in the frequency due to the finiteness of conductivity*, to our approximation, is given by

$$\tau w_1 = i \beta \frac{\pi G \varrho \gamma}{2 w_0} \tau, \tag{5.1}$$

which is purely imaginary. Hence there is no change in the periods of oscillation in any mode.

If we define damping time t_0 as the time interval in which the amplitude falls by the factor $\frac{1}{e}$, then

$$t_0 = \frac{8\pi \mu R^2 \sigma}{c} \left\{ \frac{1}{\beta} \left(A_0 - \frac{4}{\gamma} \right) \right\}. \tag{5.2}$$

The characteristic function Φ is given by

$$\Phi = \Phi_0 + i \tau \eta \tag{5.3}$$

so that the displacement function is

$$\psi = F(x)\, e^{i\chi(x)}, \tag{5.4}$$

where the amplitude $F(x)$ and variation in phase $\chi(x)$ are given by

$$F(x) = \left[\{\psi_0(x)\}^2 + \tau^2 \left(\frac{d\eta}{dx} \right)^2 \right]^{1/2} \tag{5.5}$$

$$\sim \psi_0(x), \text{ to our approximation,}$$

and

$$\tan \chi = \frac{\tau \frac{d\eta}{dx}}{\psi_0}$$

$$= \tau \left[\frac{c_0}{a_0} + \frac{\beta \sum_{n=1}^{\infty} 2n\, \mu_{2n}\, x^{2n-1} + \sum_{n=1}^{\infty} 2n\, \nu_{2n}\, x^{2n-1}}{\sum_{n=1}^{\infty} 2n\, a_{2n}\, x^{2n-1}} \right]. \tag{5.6}$$

The damping time for the amplitude of the variation of the magnetic field is the same as t_0 but the variation in its phase is given by

$$\tan \chi_H = \tau \left[\frac{c_0}{a_0} + \frac{\sum_{n=1}^{\infty} (2n)^2 \mu_{2n} x^{2n-1} + \sum_{n=1}^{\infty} (2n)^2 \nu_{2n} x^{2n-1} - \sum_{n=1}^{\infty} (2n)^2 (2n-1)^2 a_{2n} x^{2n-3}}{\sum_{n=1}^{\infty} (2n)^2 a_{2n} x^{2n-1}} \right] \dots \quad (5.7)$$

Thus we see that finite conductivity, besides damping the mechanical and magnetic pulsations, produces a variation in phase in both of them. Comparing (5.6) and (5.7) we find that the variations in phase of ψ and $\frac{\delta H}{H}$ are not the same.

Table 5

	First Mode		Second Mode		Third Mode	
f	β	$\frac{t_0}{\left(\frac{8\pi R^2 \sigma \mu}{c}\right)}$	β	$\frac{t_0}{\left(\frac{8\pi R^2 \sigma \mu}{c}\right)}$	β	$\frac{t_0}{\left(\frac{8\pi R^2 \sigma \mu}{c}\right)}$
1	37.867	0.198	1044.0	0.0456	6257.1	0.0191
2	69.818	0.191	—	—	—	—
$3+\sqrt{6}$	183.62	0.182	—	—	—	—
9	301.882	0.178	8372.8	0.0349	51157.8	0.0141
99	3311.14	0.173	—	—	—	—

Table 5 gives the values of β and $\frac{t_0}{\left(\frac{8\pi \mu R^2 \sigma}{c}\right)}$ for various values of f discussed in Part I.

We find that as H increase the damping time decreases; the damping time also decreases for the higher modes. For $f = 1$, the ratios of damping times for second and third mode to that of the first mode are 0.230 and 0.096. Similarly for $f = 9$ the values of these ratios are 0.196 and 0.079.

References

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