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GROTHENDIECK'S THEOREM ON NON-ABELIAN H^2 AND LOCAL-GLOBAL PRINCIPLES

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0. Introduction

Let k be a field with separable closure k_s and Galois group $\Gamma_k = \mathrm{Gal}(k_s/k)$. We assume that k is perfect throughout this introduction; then k_s is an algebraic closure of k. Let G be an algebraic group over k. (By this we always mean a smooth group scheme of finite type over $\mathrm{Spec}\, k$.) Recall that a homogeneous G-space is a non-empty smooth algebraic scheme X together with a (right) action by G on X, both X and the action being defined over k, such that the action of $G(k_s)$ on $X(k_s)$ is transitive. The G-space X is called a principal homogeneous G-space (or a G-torsor) if in addition this last action is free, i.e. has trivial stabilizer subgroups. The non-abelian Galois cohomology set $H^1(k, G) = H^1(\Gamma_k, G(k_s))$ is the set of isomorphism classes of principal homogeneous G-spaces over k. A well-known theorem of Steinberg ("Conjecture I" of Serre; see [S2], III, §2.3, Thm. 1′) asserts that when k has cohomological dimension at most one ($\mathrm{cd}(k) \leq 1$) and G is connected and linear, the cohomology set $H^1(k, G)$ is trivial. In other words, each principal homogeneous G-space is trivial, which means, has a k-point.

There is a general definition of the second non-abelian cohomology set in terms of gerbes (by Grothendieck, Dedecker, Giraud [G]; see also Deligne-Milne [DM], Breen [Br]). Springer [Sp] constructed a non-abelian H^2-set in terms of group extensions; this set has an equivalent description in terms of 2-cocycles. His approach was recently taken up again by Borovoi [B1]. Here is a brief review of this setup: Given an algebraic group $\overline{G}$ over k_s, let $\mathrm{SOut}(\overline{G}/k)$ be the quotient of the group $\mathrm{SAut}(\overline{G}/k)$ of k-semilinear automorphisms of $\overline{G}$ by the subgroup $\mathrm{Int}(\overline{G})$ of inner automorphisms of $\overline{G}$. A k-kernel in $\overline{G}$ is a homomorphic section κ of the natural homomorphism $\mathrm{SOut}(\overline{G}/k) \to \Gamma_k$ which satisfies a certain continuity condition. (The notion of continuity is delicate.) The set $H^2(k, \overline{G}, \kappa)$ classifies group extensions of $\overline{G}(k_s)$ by Γ_k compatible with the kernel κ. It may be empty. If it is not, it is a principal homogeneous set under the abelian group $H^2(k, Z)$, where Z is the center of $\overline{G}$, considered as an algebraic group over k by means of κ. The elements of $H^2(k, \overline{G}, \kappa)$ which correspond to split extensions are called neutral. The set of neutral elements is denoted by $N^2(k, \overline{G}, \kappa)$, and the kernel κ is said to be trivial if there exists a neutral element. Theorem 3.5 of [Sp] — attributed to Grothendieck — asserts that if k is perfect with $\mathrm{cd}(k) \leq 1$ and $\overline{G}$ is any algebraic group over k_s, then any k-kernel κ in $\overline{G}$ is trivial, and $H^2(k, \overline{G}, \kappa)$ consists of a single element which is (therefore) neutral.

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The interest in non-abelian H^2-cohomology stems at least in part from its relation to homogeneous spaces. Let G be an algebraic group over k and X a homogeneous G-space, defined over k. To X one associates in a natural way a k-kernel κ_X in $\overline{H}$, where $\overline{H}$ is the stabilizer of a k_s-point of X, and a class α_X in $H^2(k, \overline{H}, \kappa_X)$. The class α_X is neutral if and only if X is dominated (over k) by some *principal* homogeneous G-space. This fact can be regarded as part of an "exact sequence" relating the H^i-sets of G and $\overline{H}$ to relative H^i-sets of G mod $\overline{H}$, $i = 0, 1, 2$; see [Sp], Prop. 1.27.

Grothendieck's theorem therefore implies Springer's theorem, which says: Over a (perfect) field k with $\mathrm{cd}(k) = 1$, each homogeneous space under an algebraic group G is dominated by a principal homogeneous space under G. Another proof of this fact, also due to Springer, which does not use non-abelian H^2, can be found in [S2], III, §2.4. If the group G is connected and linear, one can therefore use Steinberg's theorem to conclude that each homogeneous G-space has a k-point.

In this paper we prove a formally real analogue of Grothendieck's theorem. That is, we assume that the ground field k is perfect and has *virtual* cohomological dimension one, $\mathrm{vcd}(k) \le 1$. This means that k has some finite extension K with $\mathrm{cd}(K) \le 1$ (one can take $K = k(\sqrt{-1})$). Our main result specializes to Grothendieck's theorem if $\mathrm{cd}(k) \le 1$, which is equivalent to k being not formally real, i.e. having no orderings. However, even in this case our proof is independent of the proof given in [Sp]. Typical examples of formally real fields with $\mathrm{vcd}(k) = 1$ are function fields of curves over $\mathbb{R}$, or the power series field $\mathbb{R}((T))$; one can replace $\mathbb{R}$ by any real closed field here ([S2], II, §3.3). Before we describe our results in more detail, let us briefly review previous work on formally real analogues of the theorems by Steinberg and Springer.

It was Colliot-Thélène [CT] who proposed to study analogues of the classical Hasse principle for the function field $k = \mathbb{R}(Y)$ of a smooth projective curve Y over $\mathbb{R}$, in which the role of the local places would be played by the completions k_P of k at the $\mathbb{R}$-points P of Y. As observed by the second author in [Sch], one can consider as local objects the real closures k_ξ of k with respect to its orderings ξ. Apart from having technical advantages, this point of view leads to stronger results. The orderings of k are the points of a compact, totally disconnected topological space $\Omega_k = \mathrm{Sper}\ k$, the real spectrum of k. The following Hasse principle was proved in [Sch]: If k is a perfect field with $\mathrm{vcd}(k) \le 1$, G is a connected linear group over k and X is a G-torsor with a k_ξ-point for each ξ in a dense subset of Ω_k, then X has a k-point. An equivalent way of expressing this is that the natural map

$$(*) \qquad H^1(k, G) \longrightarrow \prod_\xi H^1(k_\xi, G)$$

is injective, where in the product ξ ranges over any dense subset of the real spectrum Ω_k. Note that this reduces to Steinberg's theorem if k has no orderings. An important point in the proof is the notion of locally constant families of local cohomology classes, technically realized through the construction of a sheaf $\mathcal{H}^1(G)$ on Ω_k, and the description of the precise image of $(*)$ in these terms.

The classical framework for Hasse principles is the context of local and global fields, and in particular, of number fields. Here one has exactly the same sort of Hasse principle for torsors if the algebraic group G is semisimple and simply connected. This is due to work by Kneser and Harder in the 60s, and to more recent work of Chernousov on E_8. Note that global or (non-archimedean) local fields k have $\mathrm{vcd}(k) = 2$. For torsors under semisimple, simply connected classical groups (as well as G_2 and F_4), these results have recently been generalized to all fields with $\mathrm{vcd}(k) = 2$, by Bayer-Fluckiger and Parimala [BP1], [BP2].

Assume again that (k is perfect and) $\mathrm{vcd}(k) = 1$. The paper [Sch] contains also the analogue of Springer's theorem (Thm. 6.5). Namely, if G is an algebraic group over k and X is a homogeneous G-space such that over each real closure k_ξ there exists a G-torsor which dominates X (over k_ξ), then there exists a G-torsor over k which dominates X (over k). The proof used ideas of Springer's second proof in [S2]. Oddly, it needed the Feit-Thompson theorem on groups of odd order. The question of whether the assumption (of the existence of local dominating torsors) can be relaxed to a dense subset of Ω_k, instead of all of Ω_k, remained open and was raised in [Sch].

We shall now describe our main result. Assume that k is perfect with $\mathrm{vcd}(k) \leq 1$, let $\overline{G}$ be an algebraic group over k_s and κ a k-kernel in $\overline{G}$. Our extension of Grothendieck's theorem takes the form of local-global principles for $H^2(k, \overline{G}, \kappa)$. Its main features are:

a) The set $H^2(k, \overline{G}, \kappa)$ contains a neutral element if and only if $H^2(k_\xi, \overline{G}, \kappa)$ contains one for all ξ in a dense subset of Ω_k. In other words, κ is trivial if and only if κ_ξ is trivial for all ξ in a dense subset of Ω_k.
b) For α, $\beta \in H^2(k, \overline{G}, \kappa)$, we have that $\alpha = \beta$ if and only if $\alpha_\xi = \beta_\xi$ for all ξ in a dense subset of Ω_k.
c) The element α is neutral if and only if α_ξ is neutral for all ξ in a dense subset of Ω_k.

In more technical terms, we construct a sheaf of sets $\mathcal{H}^2(\overline{G}, \kappa)$ on Ω_k which is locally constant and whose stalk at ξ is the finite set $H^2(k_\xi, \overline{G}, \kappa)$. Further we construct a subsheaf $\mathcal{N}^2(\overline{G}, \kappa)$ of $\mathcal{H}^2(\overline{G}, \kappa)$ which is again locally constant and whose stalk at ξ is the subset $N^2(k_\xi, \overline{G}, \kappa)$ of $H^2(k_\xi, \overline{G}, \kappa)$. Finally we show that the natural map $H^2(k, \overline{G}, \kappa) \to \Gamma(\Omega_k, \mathcal{H}^2(\overline{G}, \kappa))$ is bijective, and its restriction to $N^2(k, \overline{G}, \kappa)$ bijects to $\Gamma(\Omega_k, \mathcal{N}^2(\overline{G}, \kappa))$.

The most difficult step is the proof of c). We also present an alternative approach, which is technically easier but applies only when $\overline{G}$ is connected and linear. It is based on Borovoi's elegant technique of hypercohomological abelianization of the non-commutative H^2 ([B1], [B2]). Both approaches use the technique of sheafified H^1 from [Sch].

As an application of our extension of Grothendieck's theorem, we give a new and simpler proof to the formally real analogue of Springer's theorem ([Sch], Thm. 6.5). It is the fact that we know the sheaves $\mathcal{H}^2$ and $\mathcal{N}^2$ to be locally constant, which actually allows us to work with just a dense subset of Ω_k. In particular, we obtain an affirmative answer to the question mentioned above, of whether domination of a homogeneous space locally over a dense subset of Ω_k suffices to conclude the existence of a dominating k-torsor.

The paper is organized as follows. In Section 1 we introduce kernels and non-abelian H^2 for algebraic groups, as well as the subset N^2 of neutral elements. Section 2 constructs the locally constant sheaves $\mathcal{H}^2$ and $\mathcal{N}^2$ and studies their properties. The formally real analogues of Grothendieck's theorem are proven in Section 3. Section 4 presents the alternative approach based on Borovoi's abelianization technique. Finally, applications to local-global principles for homogeneous spaces are discussed in Section 5.

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Notations and conventions. Let k be a field with separable closure k_s and absolute Galois group $\Gamma_k = \mathrm{Gal}(k_s/k)$. The cohomological dimension $\mathrm{cd}(k)$ of k is the largest integer n for which there is a finite discrete Γ_k-module A with $H^n(\Gamma_k, A) \neq 0$ (resp. is ∞ if no largest such n exists). The virtual cohomological dimension $\mathrm{vcd}(k)$ is the common cohomological dimension of all sufficiently large finite separable extensions of k; equivalently, $\mathrm{vcd}(k) = \mathrm{cd}\, k(\sqrt{-1})$. Note that $\mathrm{vcd}(k) < \mathrm{cd}(k)$ can happen only if k has an ordering, in particular, only if $\mathrm{char}(k) = 0$ ([S2], II, Prop. 4.1). Generally we do not assume the base field k to be perfect, but our main results will need this hypothesis.

The real spectrum of k, which we denote here by Ω_k and which is often written $\mathrm{Sper}\, k$, is the topological space of all orderings ξ of k. Its topology is generated by the subsets $\{\xi \in \Omega_k\colon\ a > 0 \text{ at } \xi\}$ for $a \in k$. This is a boolean (= compact and totally disconnected) topological space. Given $\xi \in \Omega_k$, one denotes the real closure of k at ξ by k_ξ. Note that Ω_k is naturally homeomorphic to the topological quotient space of $\mathrm{Inv}(\Gamma_k)$ (the space of elements of Γ_k of order two) modulo conjugation by Γ_k. See e.g. [Scha], ch. 3, §5, for some basic information on Ω_k.

Throughout the paper, by an algebraic group over k we mean a *smooth* group scheme of finite type over $\mathrm{Spec}\, k$. If $\mathrm{char}(k) = 0$, the smoothness assumption holds automatically ([DG], II, §6, no. 1). An algebraic group over k which is absolutely reduced (reduced over an algebraic closure $\overline{k}$ of k) is smooth over k ([DG], II, §5, 2.1(v)).

1. Non-commutative H^2 for algebraic groups

In this section we define kernels and non-commutative H^2 for (not necessarily linear) algebraic groups. We will follow mainly Borovoi [B1], who rewrote part of Springer [Sp], but only for linear groups and fields of characteristic zero. For a general account in terms of gerbes see Giraud [G], Deligne-Milne [DM], Breen [Br].

(1.1) Let k be a field and k_s a fixed separable closure of k. Denote the profinite group $\mathrm{Gal}(k_s/k)$ by Γ_k, or just by Γ. Given $s \in \Gamma_k$, let s^* denote the morphism $\mathrm{Spec}\, k_s \to \mathrm{Spec}\, k_s$ induced by s. Note that $(st)^* = t^*s^*$.

(1.2) Let $\overline{G}$ be an algebraic group over k_s. Denote by $\mathrm{Aut}(\overline{G})$ the group of automorphisms of $\overline{G}$ (where $\overline{G}$ is considered as a group scheme over k_s). Given $s \in \Gamma_k$, let $s_*\overline{G}$ denote the base change of $p\colon \overline{G} \to \mathrm{Spec}\, k_s$ by s^*. Then $s_*\overline{G}$ is another group scheme over $\mathrm{Spec}\, k_s$, which as a k_s-scheme is isomorphic to $(s^*)^{-1} \circ p\colon\ \overline{G} \to \mathrm{Spec}\, k_s$.

An *s-semilinear automorphism* of $\overline{G}$ is by definition an isomorphism of algebraic groups over k_s from $s_*\overline{G}$ to $\overline{G}$. A *k-semilinear automorphism* φ of $\overline{G}$ will mean an s-semilinear automorphism of $\overline{G}$ for some $s \in \Gamma_k$. Note that s is uniquely determined by φ, since $s^* = p \circ \varphi \circ e$, where $e\colon\ \mathrm{Spec}\, k_s \to \overline{G}$ is the identity point.

The set of k-semilinear automorphisms of $\overline{G}$ forms a group which we denote by $\mathrm{SAut}(\overline{G}/k)$. If $\varphi\colon\ s_*\overline{G} \to \overline{G}$ (resp. $\psi\colon\ t_*\overline{G} \to \overline{G}$) is an s-semilinear (resp.

t-semilinear) automorphism of $\overline{G}$, then the product $\psi \cdot \varphi$ is by definition the $t \circ s$-semilinear automorphism $\psi \circ t_*\varphi\colon\ t_*s_*\overline{G} \to \overline{G}$ of $\overline{G}$. Sending an s-semilinear automorphism to s defines an exact sequence

$$1 \to \operatorname{Aut}(\overline{G}) \to \operatorname{SAut}(\overline{G}/k) \to \Gamma_k. \tag{1}$$

The last map need not be surjective in general.

(1.3) There is a natural action of the group $\operatorname{SAut}(\overline{G}/k)$ on the group $\overline{G}(k_s)$, given as follows. For $s \in \Gamma_k$ let $\beta_s\colon\ \overline{G}(k_s) \to (s_*\overline{G})(k_s)$ be defined by $\beta_s(x) := x \circ s^*$. Then β_s is an isomorphism of groups. If $\varphi\colon s_*\overline{G} \to \overline{G}$ is an s-semilinear automorphism, define $\varphi_*\colon\ \overline{G}(k_s) \to \overline{G}(k_s)$ by composing φ with β_s, i.e. $\varphi_*(x) := \varphi \circ x \circ s^*$ for $x \in \overline{G}(k_s)$. Then φ_* is an automorphism of the group $\overline{G}(k_s)$; moreover $(\psi\varphi)_* = \psi_* \circ \varphi_*$ if ψ is another semilinear automorphism of $\overline{G}$.

Thus we have defined a group homomorphism $\operatorname{SAut}(\overline{G}/k) \to \operatorname{Aut}\overline{G}(k_s)$, $\varphi \mapsto \varphi_*$. In general this homomorphism need not be injective, e.g. if $\overline{G}$ is a finite constant group scheme. Usually we will simply write $\varphi(x)$ instead of $\varphi_*(x)$.

Observe that $\operatorname{SAut}(\overline{G}/k)$ acts also on $k_s[\overline{G}] := \Gamma(\overline{G}, \mathcal{O}_{\overline{G}})$. In this case the action is given by $\varphi^*(a) := a \circ \varphi$ ($\varphi \in \operatorname{SAut}(\overline{G}/k)$, $a \in k_s[\overline{G}]$; here we regard φ as a morphism of schemes $\overline{G} \to \overline{G}$ which satisfies $p \circ \varphi = (s^*)^{-1} \circ p$).

(1.4) A *k-form* of $\overline{G}$ is an algebraic group G over k together with an isomorphism $\overline{G} \cong G \times_k k_s$ of algebraic groups over k_s. If a k-form of $\overline{G}$ is fixed, the group $\overline{G}$ is also said to be defined over k. Any k-form of $\overline{G}$ defines a splitting $\Gamma_k \to \operatorname{SAut}(\overline{G}/k)$ of (1), by $s \mapsto \mathrm{id} \times_k (s^{-1})^*$. For a converse see (1.15) below.

(1.5) Given $x \in \overline{G}(k_s)$ we write $\operatorname{int}(x)$ for the inner automorphism $y \mapsto xyx^{-1}$ of $\overline{G}$. Let $\operatorname{Int}(\overline{G})$ be the subgroup of $\operatorname{Aut}(\overline{G})$ consisting of the $\operatorname{int}(x)$, $x \in \overline{G}(k_s)$. Thus $\operatorname{Int}(\overline{G}) = \overline{G}(k_s)/\overline{Z}(k_s)$ where $\overline{Z}$ is the center of $\overline{G}$. The subgroup $\operatorname{Int}(\overline{G})$ is normal in $\operatorname{SAut}(\overline{G}/k)$. Let

$$\operatorname{Out}(\overline{G}) \;:=\; \operatorname{Aut}(\overline{G})/\operatorname{Int}(\overline{G})$$

and

$$\operatorname{SOut}(\overline{G}/k) \;:=\; \operatorname{SAut}(\overline{G}/k)\big/\operatorname{Int}(\overline{G}).$$

Taking (1) modulo $\operatorname{Int}(\overline{G})$ we get the exact sequence

$$1 \to \operatorname{Out}(\overline{G}) \to \operatorname{SOut}(\overline{G}/k) \to \Gamma_k. \tag{2}$$

(1.6) Definition. We equip $\operatorname{SAut}(\overline{G}/k)$ with the weak topology with respect to the family of evaluation maps $\mathrm{ev}_x\colon\ \operatorname{SAut}(\overline{G}/k) \to \overline{G}(k_s)$, $\varphi \mapsto \varphi(x)$ ($x \in \overline{G}(k_s)$, see (1.3)), where $\overline{G}(k_s)$ is given the discrete topology. A map $t \mapsto f_t$ from a topological space T to $\operatorname{SAut}(\overline{G}/k)$ will be called *weakly continuous* if it is continuous with respect to this topology. This is the case if and only if for every $x \in \overline{G}(k_s)$ the map $T \to \overline{G}(k_s)$, $t \mapsto f_t(x)$ is continuous (= locally constant), if and only if the natural map $T \times \overline{G}(k_s) \to \overline{G}(k_s)$ is continuous.

(1.7) Proposition. *Let $\overline{G}$ be an algebraic group over k_s and $f\colon\ \Gamma_k \to \operatorname{SAut}(\overline{G}/k)$ a set-theoretic section of* (1). *Let $K \supset k$ be a finite Galois extension such that there is a K-form $\widetilde{G}$ of $\overline{G}$. Let $\sigma\colon\ \Gamma_K \to \operatorname{SAut}(\overline{G}/K) \subset \operatorname{SAut}(\overline{G}/k)$ be the splitting of* (1) *associated with $\widetilde{G}$. Consider the following conditions:*

(i) *f is weakly continuous (cf. (1.6));*

(ii) *for every* $s \in \Gamma_k$ *the map*

$$\Gamma_K \to \mathrm{Aut}(\overline{G}), \quad t \mapsto \sigma_t^{-1} f_s^{-1} f_{st} \tag{3}$$

is locally constant.

Then (ii) *implies* (i)*, and the converse holds if* $\mathrm{char}(k) = 0$.

Moreover, if $\overline{G}$ *is linear and* $\mathrm{char}(k)$ *is arbitrary,* (ii) *is equivalent to*

(iii) *for every* $a \in k_s[\overline{G}]$ *the map*

$$\Gamma_k \to k_s[\overline{G}], \quad s \mapsto f_s^*(a) \tag{4}$$

is locally constant.

Proof. Let K, $\widetilde{G}$ and σ be as in the proposition. Fix $s \in \Gamma_k$ and write $\varphi_t := \sigma_t^{-1} f_s^{-1} f_{st}$ for $t \in \Gamma_K$, so that $f_{st} = f_s \sigma_t \varphi_t$. If the map $\Gamma_K \to \mathrm{Aut}(\overline{G})$, $t \mapsto \varphi_t$ is locally constant, then obviously the map $\Gamma_K \to \overline{G}(k_s)$, $t \mapsto f_{st}(x)$ is locally constant for every $x \in \overline{G}(k_s)$. From this one concludes that (ii) implies (i).

Assume that $\overline{G}$ is linear. Since $k_s[\overline{G}]$ is a finitely generated k_s-algebra, the map (3) is locally constant if and only if for every $a \in k_s[\overline{G}]$ the map $\Gamma_K \to k_s[\overline{G}]$, $t \mapsto \varphi_t^*(a)$ is locally constant. On the other hand, (iii) can be reformulated as saying that for every $s \in \Gamma_k$ and every $a \in k_s[\overline{G}]$ the map $\Gamma_K \to k_s[\overline{G}]$, $t \mapsto f_{st}^*(a)$ is locally constant. Since $f_{st} = f_s \sigma_t \varphi_t$, it is clear that (ii) and (iii) are equivalent.

Finally assume $\mathrm{char}(k) = 0$. Then the implication (i) $\Rightarrow$ (ii) follows from the next lemma, from which it follows that in order to verify that $t \mapsto \varphi(t)$ is locally constant, we only need to check that $t \mapsto \varphi_t(x)$ is, for finitely many $x \in \overline{G}(k_s)$:

(1.8) Lemma. *Let* $\overline{k}$ *be an algebraically closed field of characteristic zero. Then for every algebraic group* $\overline{G}$ *over* $\overline{k}$ *there exists a finitely generated subgroup* S *of* $\overline{G}(\overline{k})$ *which is Zariski dense in* $\overline{G}$.

Proof. We can assume that $\overline{G}$ is connected. Let $\overline{H}$ be the (unique) maximal element among the identity connected components of Zariski closures of finitely generated subgroups of $\overline{G}(k_s)$. If $\overline{H} \neq \overline{G}$, then $\overline{G}/\overline{H}$ is a non-trivial connected algebraic group for which the group $(\overline{G}/\overline{H})(\overline{k})$ is locally finite (i.e. every finitely generated subgroup is finite). But every non-trivial connected algebraic group over $\overline{k}$ contains an element of infinite order. This is obvious for $\mathbb{G}_a$ and $\mathbb{G}_m$, and is known for abelian varieties (e.g. [FJ], Theorem 10.1), from which the general case follows. Therefore we must have $\overline{H} = \overline{G}$. $\square$

(1.9) *Remarks.* 1. Lemma (1.8) is also valid if $\overline{k}$ is a separably closed field of positive characteristic which is not the algebraic closure of a finite field, and if $\overline{G}$ has no non-trivial unipotent quotient groups (same proof). Therefore the implication (i) $\Rightarrow$ (ii) of Proposition (1.7) holds in this case as well.

2. In characteristic zero, Lemma (1.8) implies that the weak topology on $\mathrm{Aut}(\overline{G})$ (cf. (1.6)) is discrete. In positive characteristics this is in general not true. For example, the group $\overline{G} = \mathbb{G}_a \times \mathbb{G}_a$ has automorphisms of the form $(x, y) \mapsto \big(x + P(y),\, y\big)$ where P is any additive polynomial. These automorphisms form a subgroup of $\mathrm{Aut}(\overline{G})$ which is not discrete.

From this example it follows easily that in (1.7), (i) does not always imply (ii) and (iii) if $\mathrm{char}(k) = p > 0$.

(1.10) Definition. A section f of (1) will be called *continuous* if it satisfies condition (ii) of Proposition (1.7). If f is continuous, then it is weakly continuous, the converse being true if $\mathrm{char}(k) = 0$ (1.7).

(1.11) Definition. A *k-kernel in $\overline{G}$* is a group homomorphism $\kappa\colon \Gamma_k \to \mathrm{SOut}(\overline{G}/k)$ which splits (2) and lifts to a continuous *map* $f\colon \Gamma_k \to \mathrm{SAut}(\overline{G}/k)$. A pair $L = (\overline{G}, \kappa)$ consisting of an algebraic group $\overline{G}$ over k_s and a k-kernel κ in $\overline{G}$ will simply be called a *k-kernel*. Other terms would be k-lien or k-band [G], [DM], [Br].

(1.12) Lemma. *The weak topology on* $\mathrm{Int}(\overline{G})$ *is discrete.*

Proof. We have to show that there is a finitely generated subgroup S of $\overline{G}(k_s)$ whose centralizer is the center of $\overline{G}(k_s)$. Since the Zariski topology is noetherian, there is a minimal group among all centralizers of finitely generated subgroups of $\overline{G}(k_s)$. This group is necessarily the center of $\overline{G}(k_s)$. □

(1.13) Corollary. *Let κ be a splitting of* (2)*, and let f, f' be two weakly continuous maps from Γ_k to* $\mathrm{SAut}(\overline{G}/k)$*, each of which lift κ. If f is continuous, then so is f'.*

Proof. This follows from Lemma (1.12). □

(1.14) *Remarks.* We add a few comments here on other definitions of kernels of algebraic groups found in the literature. Let $\overline{G}$ be an algebraic group over k_s.

Springer ([Sp], p. 176, bottom) defines more generally K/k-kernels for any Galois extension K/k. We consider his definition only in the case $K = k_s$. In [Sp], a k_s/k-kernel in $\overline{G}$ is a group homomorphism $\Gamma_k \to \mathrm{Out}\,\overline{G}(k_s)$, $s \mapsto \kappa(s)$, such that $\kappa(s)$ is induced by an s-semilinear automorphism of $\overline{G}$ ($s \in \Gamma_k$) and such that κ is a locally constant map. Assume — as is done in [Sp], end of p. 176 — that the canonical map $\mathrm{SAut}(\overline{G}/k) \to \mathrm{Aut}\,\overline{G}(k_s)$ is injective, cf. (1.3). Then $\mathrm{SOut}(\overline{G}/k) \to \mathrm{Out}\,\overline{G}(k_s)$ is also injective, so κ can be viewed as a locally constant homomorphism $\Gamma_k \to \mathrm{SOut}(\overline{G}/k)$. But then *a fortiori* its composition with the canonical projection $\mathrm{SOut}(\overline{G}/k) \to \Gamma_k$ is locally constant. By definition, this composition is the identity of Γ_k, which implies that Γ_k is discrete.

Borovoi takes the problem of continuity into account. He only considers the case where $\mathrm{char}(k) = 0$ and the group $\overline{G}$ is linear ([B1], 1.3(b)). His definition of a k-kernel κ in $\overline{G}$ is the same as our Definition (1.11) (in fact, it inspired our definition), except that his notion of continuity, for maps $f\colon \Gamma_k \to \mathrm{SAut}(\overline{G}/k)$ which are sections of (1), is different. His condition requires that for any $a \in k_s[\overline{G}]$ the stabilizer of a in Γ_k be open. Since f is not assumed to be a homomorphism, this stabilizer is only a *subset*, not a subgroup, in general. Examples show that his condition does not imply condition (iii) in Proposition (1.7), which (under his hypotheses) is equivalent to our notion of continuity (and, under his hypotheses, also to weak continuity). Therefore Borovoi's definition seems too weak in general.

(1.15) *Remark.* The splitting of (1) defined by a k-form G of $\overline{G}$ is continuous. Hence reading this splitting modulo $\mathrm{Int}(\overline{G})$ gives a k-kernel in $\overline{G}$. We will denote it by κ_G. Conversely, a continuous splitting of (1) defines a unique k-form of $\overline{G}$. (See [BS], Lemme 2.12; condition (b) holds by our definition of continuity, while (c) holds by [S1], ch. V no. 20, Cor. 2.) If $\mathrm{char}(k) = 0$, therefore, every weakly continuous splitting of (1) defines a unique k-form of $\overline{G}$.

(1.16) If $L = (\overline{G}, \kappa)$ is a k-kernel and $\overline{Z}$ is the center of $\overline{G}$, then κ induces a k-kernel in $\overline{Z}$, that is, κ defines a k-form Z of $\overline{Z}$. We call Z (which is a commutative algebraic group over k) the *center* of the k-kernel L.

Now let $L = (\overline{G}, \kappa)$ be a k-kernel. Recall the definition of the cohomology set $H^2(k, L)$:

(1.17) Definition. A 2-*cocycle with coefficients in* L is a pair (f, g) of maps

$$f\colon\ \Gamma \to \mathrm{SAut}(\overline{G}/k),\ s \mapsto f_s, \quad \text{and} \quad g\colon\ \Gamma \times \Gamma \to \overline{G}(k_s),\ (s,t) \mapsto g_{s,t},$$

such that

(1) f is continuous (1.10), and $f \bmod \mathrm{Int}(\overline{G}) = \kappa$;
(2) $g\colon\ (s,t) \mapsto g_{s,t}$ is continuous (= locally constant), and for $s, t, u \in \Gamma$ one has

$$f_s \circ f_t \;=\; \mathrm{int}(g_{s,t}) \circ f_{st} \quad \text{and} \quad f_s(g_{t,u}) \cdot g_{s,tu} \;=\; g_{s,t} \cdot g_{st,u}. \tag{5}$$

Let $Z^2(k, L)$ denote the set of these 2-cocycles. Two cocycles (f, g) and (f', g') are called *equivalent* if there is a continuous (= locally constant) map $h\colon\ \Gamma \to \overline{G}(k_s)$ such that

$$f'_s \;=\; \mathrm{int}(h_s) \circ f_s \quad \text{and} \quad g'_{s,t} \;=\; h_s \cdot f_s(h_t) \cdot g_{s,t} \cdot h_{st}^{-1} \tag{6}$$

for all $s, t \in \Gamma$. The cohomology set $H^2(k, L)$ is defined to be the set of equivalence classes in $Z^2(k, L)$. If $\overline{G}$ is commutative, then $H^2(k, L)$ is the usual second (Galois) cohomology group $H^2(k, G)$, where G is the k-form of $\overline{G}$ defined by κ.

(1.18) The description of cocycles above follows [Sp], rather than [B1] who takes (f, g^{-1}) instead of (f, g). An alternative and useful (see (3.4) and (3.5) below) description of $H^2(k, L)$ in terms of group extensions is recalled next (compare [Sp]). Consider extensions

$$1 \to \overline{G}(k_s) \xrightarrow{i} E \xrightarrow{\pi} \Gamma \to 1 \tag{7}$$

of topological groups, where $\overline{G}(k_s)$ (resp. $\Gamma = \Gamma_k$) carries the discrete (resp. the natural profinite) topology, and i and π are open onto their respective images. Two such extensions E and E' are called *equivalent* if there is an isomorphism $E \to E'$ of topological groups which induces the identity on $\overline{G}(k_s)$ and on Γ.

(1.19) Lemma. *The set* $H^2(k, L)$ *is in natural bijection with the set of equivalence classes of extensions* (7) *for which the induced homomorphism from* Γ *to* $\mathrm{Out}\,\overline{G}(k_s)$ *coincides with the composite homomorphism*

$$\Gamma \xrightarrow{\kappa} \mathrm{SOut}(\overline{G}/k) \to \mathrm{Out}\,\overline{G}(k_s). \tag{8}$$

The second map in (8) *is induced by the canonical map* $\mathrm{SAut}(\overline{G}/k) \to \mathrm{Aut}\,\overline{G}(k_s)$ *(1.3).*

Proof. Given such an extension (7), choose a continuous section $\Gamma \to E$, $s \mapsto z_s$. Such sections exist since $\overline{G}(k_s)$ is a discrete subgroup of E. For any $s \in \Gamma$ there is a unique s-semilinear automorphism f_s of $\overline{G}$ with $f_s(x) = z_s x z_s^{-1}$ for every $x \in \overline{G}(k_s)$. Put $g_{s,t} := z_s z_t z_{st}^{-1}$ for $s, t \in \Gamma$. The pair (f, g) is a 2-cocycle with values in L. Indeed, it is obvious that f is weakly continuous. Since κ has a continuous lift by hypothesis, f is in fact continuous by Corollary (1.13). Choosing a different continuous section $s \mapsto z'_s$ yields an equivalent cocycle.

Conversely, to a cocycle $(f,g) \in Z^2(k,L)$ one associates an extension (7) by putting

$$E := \overline{G}(k_s) \times \Gamma \quad \text{with multiplication rule} \quad (x,s)\cdot(y,t) := \big(x\, f_s(y)\, g_{s,t},\ st\big).$$

The group E is given the product topology, and i, π are defined in the obvious way. The equivalence class of this extension depends only on the cohomology class of (f,g), and the two processes are inverses of each other. □

(1.20) Given a k-kernel $L = (\overline{G}, \kappa)$, the set $H^2(k,L)$ may be empty. One has the following well-known criterion ([M], IV, §8; [G], VI, §2). Let Z be the center of $\overline{G}$, considered as an algebraic group over k via κ. With L one associates a class $\mathrm{obs}(L)$ in the cohomology group $H^3(k,Z)$ as follows:

Let $s \mapsto f_s$ be a continuous map $\Gamma \to \mathrm{SAut}(\overline{G}/k)$ which lifts κ; cf. (1.11). The map $(s,t) \mapsto f_s f_t f_{st}^{-1}$ from $\Gamma \times \Gamma$ to $\mathrm{Int}(\overline{G})$ is locally constant by Lemma (1.12), hence there exists a locally constant map $\Gamma\times\Gamma \to \overline{G}(k_s)$, $(s,t)\mapsto g_{s,t}$ such that

$$f_s \circ f_t \;=\; \mathrm{int}(g_{s,t}) \circ f_{st}.$$

Let $z\colon\ \Gamma\times\Gamma\times\Gamma \to \overline{Z}(k_s)$ be the locally constant map determined by

$$f_s(g_{t,u}) \cdot g_{s,tu} \;=\; z_{s,t,u} \cdot g_{s,t} \cdot g_{st,u}.$$

Then $z \in Z^3(k,Z)$, and the class of z in $H^3(k,Z)$ is independent of the choices made. Denote it by $\mathrm{obs}(L)$. One has:

(1.21) Proposition. *The set $H^2(k,L)$ is non-empty if and only if* $\mathrm{obs}(L) = 0$ *in* $H^3(k,Z)$.

Proof. See [M], IV, Thm. 8.7. □

This yields the following local-global principle for non-emptiness of H^2:

(1.22) Corollary. *Assume* $\mathrm{vcd}(k) \le 2$, *and let $L = (\overline{G},\kappa)$ be a k-kernel. Then $H^2(k,L) \neq \emptyset$ if and only if $H^2(k_\xi, L) \neq \emptyset$ for all ξ in a dense subset of Ω_k.*

For the proof we need a straightforward (partial) generalization of [Sch], Thm. 3.1, which we formulate in a greater generality than actually required here. The proof is analogous to *loc.cit.* Recall that sheaves $\mathcal{H}^i(A)$ on Ω_k are attached in [Sch], (2.8), to any commutative algebraic group A over k and $i \ge 1$. They are shown to be locally constant in [Sch], (2.13(a)), and to have stalks $H^i(k_\xi, A)$ in [Sch], (2.9(b)).

(1.23) Lemma. *If k is perfect with* $\mathrm{vcd}(k) \le d$ *and A is a commutative algebraic group over k, then the map $H^n(k,A) \to \Gamma\big(\Omega_k,\, \mathcal{H}^n(A)\big)$ is bijective for $n > d$ and surjective for $n = d$.*

Proof of Corollary (1.22). Assume $H^2(k_\xi, L) \neq \emptyset$ for all ξ in a dense subset of Ω_k. Let Z be the center of L. Writing L_ξ for the restriction of L to k_ξ, $\mathrm{obs}(L)$ maps to $\mathrm{obs}(L_\xi)$ under the restriction map $H^3(k,Z) \to H^3(k_\xi, Z)$. By (1.21), and by the assumption, $\mathrm{obs}(L)$ lies in the kernel of $H^3(k,Z) \to \Gamma\big(\Omega_k,\, \mathcal{H}^3(Z)\big)$. By Lemma (1.23) this kernel is zero. □

(1.24) Let L be a k-kernel with center Z. There is a natural action of the abelian group $H^2(k,Z)$ on the set $H^2(k,L)$, which is free and transitive provided $H^2(k,L) \neq \emptyset$. In fact, the abelian group $Z^2(k,Z)$ acts on the set $Z^2(k,L)$ by $z \cdot (f,g) = (f, zg)$, and this action descends to an action of $H^2(k,Z)$ on $H^2(k,L)$.

For completeness we also mention the interpretation of the action in terms of group extensions: Given an extension (7) whose class in $H^2(k,L)$ is α, and an extension

$$1 \to \overline{Z}(k_s) \to B \to \Gamma \to 1$$

whose class in $H^2(k,Z)$ is ζ, form the extension

$$1 \to \overline{G}(k_s) \to (B \times_\Gamma E)/D \to \Gamma \to 1 \tag{9}$$

where D is the subgroup $\{(z, z^{-1})\colon\ z \in \overline{Z}(k_s)\}$ of the fiber product. Then (9) has a class in $H^2(k,L)$, and this class is $\zeta \cdot \alpha$.

Observe that the action is compatible with extension of the base field, i.e. with restriction in cohomology.

(1.25) Definition. A 2-cocycle $(f,g) \in Z^2(k,L)$ is called *neutral* if $g_{s,t} = 1$ for all s,t. An element $\alpha \in H^2(k,L)$ is called *neutral* if it can be represented by a neutral cocycle. The subset of $H^2(k,L)$ consisting of the neutral elements is denoted $N^2(k,L)$. The kernel $L = (\overline{G}, \kappa)$ is called *trivial* if $N^2(k,L)$ is nonempty.

(1.26) In terms of group extensions the neutral elements have the following description: Given an extension (7) compatible with the kernel κ, its class in $H^2(k,L)$ is neutral if and only if the extension splits by a continuous homomorphic section $\Gamma \to E$.

(1.27) The kernel $L = (\overline{G}, \kappa)$ is trivial (i.e. $N^2(k,L) \neq \emptyset$, (1.25)) if and only if the homomorphism $\kappa\colon \Gamma \to \mathrm{SOut}(\overline{G}/k)$ lifts to a continuous homomorphism $\Gamma \to \mathrm{SAut}(\overline{G}/k)$. By (1.15) it is equivalent to saying that κ belongs to some k-form G of $\overline{G}$. Observe that $N^2(k,L)$ can be empty even if $H^2(k,L)$ is non-empty (see also (3.6) below).

2. The sheaves $\mathcal{H}^2$ and $\mathcal{N}^2$

Given an algebraic group G over k, a natural sheaf $\mathcal{H}^1(G)$ of pointed sets on the space Ω_k of orderings of k was defined in [Sch]. It is locally constant and its stalk at ξ is $H^1(k_\xi, G)$. As noted in (1.22), if G is commutative, locally constant sheaves $\mathcal{H}^n(G)$ of abelian groups were defined for all $n \geq 1$; they have the analogous property of the stalks.

We now want to sheafify the non-commutative cohomology sets H^2 and N^2 in a similar way. We frequently present sheaves on boolean (i.e. compact and totally disconnected) spaces (such as Ω_k) just by giving the sections over clopen subsets. This is justified by [Sch], Appendix C.1.

(2.1) Let $L = (\bar{G}, \kappa)$ be a k-kernel, with center Z. The cohomology set $H^2(E,L)$ is defined for every finite separable extension E of k and, more generally, for every finite étale k-algebra E. In the latter case, if $E = K_1 \times \cdots \times K_r$ with K_i/k finite separable field extensions, then $H^2(E,L) = H^2(K_1,L) \times \cdots \times H^2(K_r,L)$. Therefore we can imitate the definition in 2.3 of [Sch]:

(2.2) Definition. The sheaf (of sets) $\mathcal{H}^2(L)$ on Ω_k is defined by

$$U \quad \longmapsto \quad \varinjlim_{(E,s)\in J_U} H^2(E,L),$$

for $U \subset \Omega_k$ clopen. Here J_U is the category of pairs (E,s) where E is an étale k-algebra and $s\colon U \to \Omega_E$ is a section of the restriction map $\Omega_E \to \Omega_k$ over U (see [Sch], Sect. 2.3).

Observe that one has a canonical map $H^2(k,L) \to \Gamma\big(\Omega_k,\, \mathcal{H}^2(L)\big)$.

Next we identify the stalks of the sheaves $\mathcal{H}^2(L)$. For this we need

(2.3) Lemma. *If $K \supset k$ is a separable algebraic extension, then $H^2(K,L)$ is the direct limit of the sets $H^2(F,L)$, where F ranges over the finite subextensions $F \supset k$ of $K \supset k$.*

Proof. From Proposition (1.21) it follows that the lemma holds if $H^2(F,L) = \emptyset$ for all these F. Hence we may assume $H^2(F,L) \neq \emptyset$ for some F. Fixing an element $\alpha \in H^2(F,L)$ we get a bijection $H^2(F',Z) \xrightarrow{\sim} H^2(F',L)$ for every $F' \supset F$, and these bijections are compatible with restriction (1.24). So the assertion follows from the corresponding fact for abelian cohomology of Z, which is well known. □

(2.4) Corollary. *The sheaf $\mathcal{H}^2(L)$ is locally constant with finite stalks. The stalk at $\xi \in \Omega_k$ is canonically isomorphic to $H^2(k_\xi, L)$. There is a canonical action of the sheaf $\mathcal{H}^2(Z)$ (of abelian groups) on the sheaf $\mathcal{H}^2(L)$, which is (stalkwise) free and transitive.*

Proof. The identification of the stalks follows from Lemma (2.3). Also it is clear from (1.24) that one has a canonical action of $\mathcal{H}^2(Z)$ on $\mathcal{H}^2(L)$, which is stalkwise free and transitive (wherever the stalks of the second sheaf are non-empty). Since $\mathcal{H}^2(Z)$ is known to be a locally constant sheaf with finite stalks ([Sch], Thm. 2.13a), the same will follow for $\mathcal{H}^2(L)$ once it is shown that the subset $\{\xi\colon\ H^2(k_\xi,L) \neq \emptyset\}$ is clopen in Ω_k. But this is the set of ξ where $\mathrm{obs}(L)_\xi = 0$ in $H^3(k_\xi, Z)$, and hence it is obviously clopen. □

(2.5) *Notation.* Given $\alpha \in H^2(k,L)$, we write α_ξ for the restriction of α to k_ξ, so $\alpha_\xi \in H^2(k_\xi, L)$.

(2.6) Corollary. *If* $\mathrm{vcd}(k) \le 1$*, then the map $H^2(k,L) \to \Gamma\big(\Omega_k,\, \mathcal{H}^2(L)\big)$ is bijective.*

Proof. If $H^2(k,L) = \emptyset$, this is true by (1.22). Otherwise use Corollary (2.4) together with bijectivity of $H^2(k,Z) \to \Gamma\big(\Omega_k,\, \mathcal{H}^2(Z)\big)$ ([Sch], Thm. 3.1a). □

(2.7) Definition. The sheaf (of sets) $\mathcal{N}^2(L)$ on Ω_k is defined by

$$U \quad \longmapsto \quad \varinjlim_{(E,s)\in J_U} N^2(E,L),$$

for $U \subset \Omega_k$ clopen. It is clear that this is a subsheaf of $\mathcal{H}^2(L)$.

The proof that this subsheaf $\mathcal{N}^2(L)$ of $\mathcal{H}^2(L)$ has the right stalks and that it is again locally constant requires slightly more work than for $\mathcal{H}^2(L)$. We first prove

(2.8) Lemma. *Let $\alpha \in H^2(k,L)$ be an element whose restriction to k_ξ is neutral, where $\xi \in \Omega_k$ is fixed. Then there is a finite subextension $k \subset K \subset k_\xi$ such that already the restriction of α to K is neutral.*

Proof. Let (f,g) be a cocycle which represents α. We can assume that (f,g) is normalized, i.e. that $f_1 = \mathrm{id}$ and $g_{1,s} = g_{s,1} = 1$ for $s \in \Gamma$. Fix a real closure $k_\xi \subset k_s$ with respect to ξ, and let t be the corresponding involution in Γ. That α is neutral at ξ means (see (6)) that there is $h \in \overline{G}(k_s)$ such that

$$h \cdot f_t(h) \cdot g_{t,t} = 1.$$

Choose an open normal subgroup U of Γ with $t \notin U$, and put $V := U \cup tU$. If one takes U sufficiently small, then the following properties hold:

a) If $x, y \in V$, then $g_{x,y} = 1$ if x or y is in U, and $g_{x,y} = g_{t,t}$ otherwise;
b) $f_x(h) = h$ and $f_{xt}(h) = f_t(h)$ for every $x \in U$.

Property b) follows since f is (weakly) continuous. Now the restriction of the cocycle (f,g) to the open subgroup V of Γ is neutralized by the continuous map $V \to \bar{G}(k_s)$ which is 1 on U and h on tU. $\square$

(2.9) Proposition. *The subsheaf $\mathcal{N}^2(L)$ of $\mathcal{H}^2(L)$ is locally constant. Its stalk at ξ is the subset $N^2(k_\xi, L)$ of $H^2(k_\xi, L)$.*

Proof. It is clear that the stalk of $\mathcal{N}^2(L)$ at ξ is contained in $N^2(k_\xi, L)$. The other inclusion follows directly from Lemma (2.8). A subsheaf of the locally constant sheaf $\mathcal{H}^2(L)$ is locally constant if and only if it is (not only open but also) closed in $\mathcal{H}^2(L)$, both sheaves being regarded as espaces étalés. This follows immediately from [Sch], Lemma C.3, which asserts that a sheaf on a boolean space is locally constant with finite stalks if and only if its espace étalé is compact. Therefore we have to show: Given a section c of $\mathcal{H}^2(L)$ over a clopen subset $U \subset \Omega_k$, the set of $\xi \in U$ with $c_\xi \in N^2(k_\xi, L)$ is closed. For this in turn it suffices to show for every finite extension E of k and every $\alpha \in H^2(E, L)$ that the set $\{\eta \in \Omega_E\colon\ \alpha_\eta$ is neutral$\}$ is closed. But this follows again from Lemma (2.8). $\square$

3. A formally real analogue of Grothendieck's theorem

Grothendieck's theorem, by which we mean Theorem 3.5 in [Sp], asserts: If k is a perfect field with $\mathrm{cd}(k) \le 1$, and $L = (\overline{G}, \kappa)$ is a k-kernel, then $H^2(k, L)$ consists of precisely one element, and this element is neutral. In particular, any k-kernel is trivial.

The following theorem specializes to Grothendieck's theorem if the field k has no orderings. It can be regarded as its "formally real" analogue.

(3.1) Theorem. *Let k be a perfect field with $\mathrm{vcd}(k) \le 1$, and let $L = (\overline{G}, \kappa)$ be a k-kernel.*

a) *The sheaf $\mathcal{H}^2(L)$ on Ω_k is locally constant. Its stalk at ξ is the finite set $H^2(k_\xi, L)$.*
b) *The subsheaf $\mathcal{N}^2(L)$ of $\mathcal{H}^2(L)$ is again locally constant. Its stalk at ξ is the subset $N^2(k_\xi, L)$ of $H^2(k_\xi, L)$.*
c) *The natural map $H^2(k, L) \to \Gamma\big(\Omega_k,\, \mathcal{H}^2(L)\big)$ is bijective, and its restriction to $N^2(k, L)$ is a bijection onto $\Gamma\big(\Omega_k,\, \mathcal{N}^2(L)\big)$.*

(3.2) Corollary. *Let $\alpha,\ \beta \in H^2(k, L)$.*

a) *$\alpha = \beta \Leftrightarrow \alpha_\xi = \beta_\xi$ for all ξ in a dense subset of Ω_k.*
b) *α is neutral $\Leftrightarrow \alpha_\xi$ is neutral for all ξ in a dense subset of Ω_k.*
c) *$H^2(k, L)$ contains a neutral element $\Leftrightarrow H^2(k_\xi, L)$ contains a neutral element for all ξ in a dense subset of Ω_k.*

Proof. a) and b) follow from (3.1c), using the characterization of the stalks of the sheaves $\mathcal{H}^2(L)$ and $\mathcal{N}^2(L)$. c) follows from (3.1c) together with the fact that the sheaf $\mathcal{N}^2(L)$ is locally constant. $\square$

Our proof does not assume Grothendieck's theorem, i.e. the case of (3.1) where k has no orderings; rather this case will be covered by our proof as well.

For the proof of Theorem (3.1) we will need the following two lemmas. The first is the general principle underlying the main induction step in the proof.

(3.3) Lemma. *Let k be a perfect field and let $\mathcal{P}$ be a property of algebraic groups over k. (Recall that all algebraic groups are assumed to be smooth of finite type.) Suppose that $\mathcal{P}(G)$ holds under each of the following conditions:*

(i) *G is finite;*
(ii) *G is commutative;*
(iii) *G is connected linear semisimple with trivial center;*
(iv) *there is an algebraic k-subgroup N of G which is invariant under all semilinear automorphisms of G and such that $\mathcal{P}(N)$ and $\mathcal{P}(G/N)$ hold.*

Then $\mathcal{P}(G)$ holds for every G.

Proof. Assume that the lemma is false. From the descending chain condition for closed algebraic subschemes, there exists a minimal element among all counterexamples. We may therefore assume that there is a counterexample G such that $\mathcal{P}(H)$ holds for every proper algebraic k-subgroup H of G. Applying (iv) with $N = G^0$, and using [DG], II, §5, 2.1(ii) ("G is smooth if and only if G^0 is smooth"), (i) implies that G is connected. Applying (iv) with $N =$ largest connected linear subgroup L of G, and using [R], Thm. 16, (ii) implies that G is linear. Note that in [R] all algebraic groups are reduced over $\overline{k}$, hence they are smooth over their field of definition (see [DG], II, §5, 2.1(v)); the "k-closed" subgroup L of [R], Thm. 16, is defined over k since k is perfect, hence it is smooth over k. Inductively one also sees that $\mathcal{P}(H)$ holds whenever H is solvable. Applying (iv) with $N =$ radical of G one sees that $S := G/N$ is also a counterexample. Using (iv) with $N =$ center of S one sees from (i) (or (ii)) and (iii) that $\mathcal{P}(S)$ holds, thereby giving a contradiction. $\square$

If H is any (abstract) group, let $\mathrm{Inv}(H) = \{h \in H\colon\ h^2 = 1,\ h \neq 1\}$ denote the set of involutions in H. The following lemma is used in the proof of Theorem (3.1) to verify step (iv) in Lemma (3.3), the property $\mathcal{P}$ being the local-global principle for neutral elements.

(3.4) Lemma. *Let k be a perfect field with $\mathrm{vcd}(k) \leq 1$, let $\overline{H}$ be an algebraic group over $\overline{k} = k_s$ and let*

$$1 \to \overline{H}(\overline{k}) \to F \xrightarrow{\pi} \Gamma \to 1 \tag{10}$$

be an extension of the type considered in (1.18), where $\Gamma := \Gamma_k$.

a) *Suppose (10) splits locally, i.e. $\mathrm{Inv}(F) \to \mathrm{Inv}(\Gamma)$ is surjective. Then there exists a continuous map $\tau\colon \mathrm{Inv}(\Gamma) \to \mathrm{Inv}(F)$ with $\pi \circ \tau = \mathrm{id}$ and such that for every $t \in \mathrm{Inv}(\Gamma)$ and $x \in F$, the elements $x\tau(t)x^{-1}$ and $\tau\big(\pi(x) \cdot t \cdot \pi(x)^{-1}\big)$ are conjugate under $\overline{H}(\overline{k})$. We call such τ an Inv-section of (10).*
b) *If (10) splits, then for any Inv-section τ of (10) there is a splitting $\sigma\colon \Gamma \to F$ of (10) such that for every $t \in \mathrm{Inv}(\Gamma)$, $\sigma(t)$ is conjugate to $\tau(t)$ under $\overline{H}(\overline{k})$.*

Proof. a) We repeatedly use the following obvious fact: If $p\colon X \to T$ is a surjective map of topological spaces which is a local homeomorphism, and if T is a boolean space, then p has a (continuous) section. By this device there exist continuous sections $\sigma\colon \operatorname{Inv}(\Gamma) \to \operatorname{Inv}(F)$ and $\varrho\colon \Gamma \to F$ of π. The canonical map $\operatorname{Inv}(\Gamma) \to \operatorname{Inv}(\Gamma)/\text{conj.} = \Omega_k$ is known to have a continuous section ([H], Lemma 5.3(a)). Hence there is a closed subset Z of $\operatorname{Inv}(\Gamma)$ which is a system of representatives of conjugacy classes in $\operatorname{Inv}(\Gamma)$. Let $\alpha\colon \Gamma \times \operatorname{Inv}(\Gamma) \to \operatorname{Inv}(\Gamma)$ be the conjugation action of Γ on $\operatorname{Inv}(\Gamma)$. Let ι be the involution on $\Gamma \times Z$ defined by $\iota(s,t) = (st,t)$. Then α induces a homeomorphism $(\Gamma \times Z)/\iota \xrightarrow{\sim} \operatorname{Inv}(\Gamma)$. Since ι has no fixed points, there is a closed subset Y of $\Gamma \times Z$ such that $\alpha|Y$ is a homeomorphism $Y \xrightarrow{\sim} \operatorname{Inv}(\Gamma)$. Define τ by $\tau(sts^{-1}) := \varrho(s)\sigma(t)\varrho(s)^{-1}$ for $(s,t) \in Y$.

b) Fix a splitting ϱ of (10), and write ${}^x a := \varrho(x)\, a\, \varrho(x)^{-1}$ for $x \in \Gamma$, $a \in \overline{H}(\overline{k})$. Let the map $c\colon \operatorname{Inv}(\Gamma) \to \overline{H}(\overline{k})$ be defined by $\tau(t) = c(t)\varrho(t)$ for $t \in \operatorname{Inv}(\Gamma)$. The map c is locally constant. Using that τ is an Inv-section one finds easily that c satisfies

$$c(t) \cdot {}^t c(t) \,=\, 1 \text{ for } t \in \operatorname{Inv}(\Gamma)$$

and

$$\text{for } t \in \operatorname{Inv}\Gamma \text{ and } x \in \Gamma \text{ there is } a \in \overline{H}(\overline{k}) \text{ with } {}^x c(x^{-1}tx) = a^{-1} \cdot c(t) \cdot {}^t a.$$

In other words, $c \in Z^1_{\Omega_k}(\overline{H}(\overline{k}))$, in the terminology of [Sch], Lemma 2.7. By Prop. 2.9 of *loc. cit.*, c represents a global section of the sheaf $\mathcal{H}^1(H)$, where H denotes the k-form of $\overline{H}$ defined by ϱ. By [Sch], Thm. 5.1, there is a class $\gamma \in H^1(k,H)$ which maps to this global section. Let $(h_s)_{s\in\Gamma}$ be a cocycle representing γ, and define the splitting σ of (10) by $\sigma(s) := h_s \cdot \varrho(s)$. Then σ has the desired property. $\square$

(3.5) We shall now prove Theorem (3.1). Assertions a), b) and the first part of c) have already been proved in (2.4), (2.9) and (2.6), respectively. Therefore it remains to show the following local-global principle for neutral elements:

($*$) An element $\alpha \in H^2(k,L)$ whose restriction α_ξ to k_ξ is neutral for every $\xi \in \Omega_k$ is neutral.

Our proof is based on an induction over $\overline{G}$, using Lemma (3.3). Take $K = \overline{k}$ in the lemma, and let $\mathcal{P}(\overline{G})$ be the property that ($*$) holds for all k-kernels in $\overline{G}$. Property (i) required in (3.3) is known (see [Sch], Sect. 5.12). Property (ii) is part of [Sch], Thm. 3.1. To prove property (iii) we remark that if $L = (\overline{G},\kappa)$ is a k-kernel with $\overline{G}$ connected reductive, then $N^2(k,L)$ is not empty. (Here k may be any field; cf. [D] and also [B1], Prop. 3.1.) If further $\overline{G}$ has trivial center, then by (1.24) $H^2(k,L)$ has exactly one element, which is neutral.

It remains to show that property (iv) from (3.3) holds as well. So let $L = (\overline{G},\kappa)$ and $\alpha \in H^2(k,L)$ be given with $\alpha_\xi \in N^2(k_\xi,L)$ for all ξ, and let $\overline{N}$ be an algebraic subgroup of $\overline{G}$ which is invariant under all semilinear automorphisms of $\overline{G}$ and such that ($*$) holds for $\overline{N}$ and for $\overline{G}/\overline{N}$. Let $\Gamma = \Gamma_k$, and let

$$1 \to \overline{G}(\overline{k}) \to E \xrightarrow{\pi} \Gamma \to 1 \tag{11}$$

be the extension corresponding to α; cf. (1.19). The hypothesis says that the map $\operatorname{Inv}(E) \to \operatorname{Inv}(\Gamma)$ induced by π is surjective. We have to show (1.26) that (11) splits. The subgroup $\overline{N}(\overline{k})$ of E is normal, and by hypothesis we know that (11) splits modulo $\overline{N}(\overline{k})$, i.e. that

$$1 \to \overline{G}(\overline{k})\big/\overline{N}(\overline{k}) \to E\big/\overline{N}(\overline{k}) \xrightarrow{\bar{\pi}} \Gamma \to 1 \tag{12}$$

splits. Since (11) splits locally, there is an Inv-section τ of (11), by part a) of Lemma (3.4). Since (12) splits, and since $\bar{\tau} := \tau \bmod \overline{N}(\overline{k})$ is an Inv-section of (12), part b) of that lemma shows that there is a splitting σ of (12) for which $\sigma(t)$ is conjugate to $\bar{\tau}(t)$ under $\overline{G}\big/\overline{N}(\overline{k})$, for every $t \in \mathrm{Inv}(\Gamma)$.

Fix this σ, and let S be the preimage of $\sigma(\Gamma)$ under $E \to E\big/\overline{N}(\overline{k})$. Then S is a subextension

$$1 \to \overline{N}(\overline{k}) \to S \to \Gamma \to 1 \tag{13}$$

of (11). The extension (13) corresponds to a k-kernel in $\overline{N}$. Indeed, let $s \mapsto z_s$ be a locally constant section of (13), and let f_s be the (unique) s-semilinear automorphism of $\overline{G}$ which induces $\mathrm{int}(z_s)$ on $\overline{G}(\overline{k})$. Then $s \mapsto f_s$ is weakly continuous, hence continuous by (1.13) since κ is a kernel. But this implies that also $s \mapsto f_s|\overline{N}$ is continuous. Therefore $s \mapsto$ image of $f_s|\overline{N}$ in $\mathrm{SOut}(\overline{G}/k)$ is a k-kernel to which (13) belongs.

Since for every $t \in \mathrm{Inv}(\Gamma)$, S contains a $\overline{G}(\overline{k})$-conjugate of $\tau(t)$, the extension (13) splits locally. By the induction hypothesis, therefore, (13) splits, and *a fortiori* (11) splits. □

An alternative approach to the above induction process, which applies only for connected linear groups, is introduced in Section 4. Using Borovoi's construction of abelianized non-commutative H^2, an independent proof is given there for the local-global principle for neutral elements, if the kernel is connected and linear. Using this result one may in the above proof directly proceed from general $\overline{G}$ to its connected component, and then to the maximal linear connected subgroup of $\overline{G}$.

(3.6) *Remark.* From Theorem (3.1) it is clear that it can happen (over suitable k with $\mathrm{vcd}(k) = 1$) that a k-kernel L is trivial over *some* real closure k_ξ, but not over another (and *a fortiori* not over k), even if $H^2(k, L) \neq \emptyset$. As an illustration we give the following explicit example: Let

$$1 \to G \to E \to \mathbb{Z}/2 \to 1 \tag{14}$$

be an extension of finite groups which does not split modulo the center of G. (Example: Take E to be the group of all transformations $aX + b$ over the field with five elements, and G the subgroup consisting of those with $a = \pm 1$.) Let τ be the involution in $\mathrm{Out}(G)$ defined by (14). Then τ does not lift to an involution in $\mathrm{Aut}(G)$.

Let k be a field with $\mathrm{vcd}(k) = 1$, and let K/k be a quadratic extension such that K is formally real and there is an ordering of k which does not extend to K. (Example: $k = \mathbb{R}(t)$, $K = k(\sqrt{t})$.) Let $\chi\colon \Gamma_k \to \mathbb{Z}/2$ be the homomorphism with kernel Γ_K, and let

$$1 \to G \to \tilde{E} \to \Gamma_k \to 1 \tag{15}$$

be the extension obtained by pulling back (14) via χ. This extension defines a k-kernel $L = (G, \kappa)$ (G is considered as a constant finite group scheme here) and an element in $H^2(k, L)$. For every ordering ξ of k which extends to K, (15) is trivial over k_ξ, and hence $N^2(k_\xi, L) \neq \emptyset$. On the other hand, if ξ does not extend to K, then $N^2(k_\xi, L)$ is empty since τ does not lift to an involution in $\mathrm{Aut}(G)$.

4. Hypercohomological approach

The aim of this section is to introduce another technique which provides an alternative proof of the induction step in Theorem (3.1). We consider only connected linear algebraic groups in this section. The main tool here is the abelian Galois hypercohomology group H^2 of a complex of length two, which was used by Borovoi [B1] in the context of number fields.

(4.1) Let $L = (\overline{G}, \kappa)$ be a connected reductive k-kernel (i.e. $\overline{G}$ is a connected linear reductive algebraic group over k_s). Let $\overline{G}^{\mathrm{ss}} = [\overline{G}, \overline{G}]$ be its derived group; it is semisimple. We denote by $\overline{G}^{\mathrm{sc}}$ the simply connected cover of $\overline{G}^{\mathrm{ss}}$ and by $\overline{\rho}$ the composite map

$$\overline{G}^{\mathrm{sc}} \to \overline{G}^{\mathrm{ss}} \to \overline{G}.$$

Let $\overline{Z}$ (resp. $\overline{Z}^{\mathrm{ss}}$, $\overline{Z}^{\mathrm{sc}}$) be the center of $\overline{G}$ (resp. $\overline{G}^{\mathrm{ss}}$, $\overline{G}^{\mathrm{sc}}$). Observe that κ defines k-forms Z, Z^{ss} and Z^{sc} of $\overline{Z}$, $\overline{Z}^{\mathrm{ss}}$ and $\overline{Z}^{\mathrm{sc}}$ since these groups are abelian. Further, if $L = (G \times_k k_s, \kappa_G)$ for some k-group G (cf. (1.15)), then the k-groups Z, Z^{ss} and Z^{sc} are the respective centers of G, G^{ss} and G^{sc} (cf. (1.16)). The restricted homomorphism $\rho : Z^{\mathrm{sc}} \to Z$ is then defined over k. This induces a short complex of discrete Γ_k-modules

$$1 \to Z^{\mathrm{sc}}(k_s) \xrightarrow{\rho} Z(k_s) \to 1 \tag{16}$$

placed in degrees -1 and 0. Borovoi [B1], Sect. 5, defines the abelianized cohomology groups $H^i_{\mathrm{ab}}(k, L)$, $i \geq 0$, by

$$H^i_{\mathrm{ab}}(k, L) := \mathbb{H}^i(k,\, Z^{\mathrm{sc}} \to Z)$$

where the right hand side denotes the Γ_k-hypercohomology of the complex (16). Borovoi also constructs an abelianization map [B1], Sect. 5.3,

$$\mathrm{ab}^2 : H^2(k, L) \to H^2_{\mathrm{ab}}(k, L).$$

These constructions are valid for a field k of any characteristic, although [B1] assumes that $\mathrm{char}(k) = 0$.

(4.2) To extend the definition of the abelianization map to the case of any connected linear k-kernel $L = (\overline{G}, \kappa)$ (i.e. $\overline{G}$ is a connected linear algebraic group over $\overline{k}$), we assume that k is perfect. This assumption remains in force for the rest of this section.

Let $\overline{G}^{\mathrm{red}}$ denote the connected reductive group which is the quotient of $\overline{G}$ by its unipotent radical and let $L^{\mathrm{red}} = (\overline{G}^{\mathrm{red}}, \kappa)$ be the induced kernel. There is a natural map [B1], Sect. 4, $r : H^2(k, L) \to H^2(k, L^{\mathrm{red}})$. Borovoi proved [B1], Prop. 4.1, that an element $\eta \in H^2(k, L)$ is neutral if and only if $r(\eta)$ is neutral. Note that [B1], Prop. 4.1, holds for any perfect field k. Indeed, Lemma 4.3 of [B1] holds for all such fields, since [DG], IV, §2, Cor. 3.9, asserts that a (connected smooth) unipotent k-group has a central composition series with quotients $\mathbb{G}_a$. Setting $H^2_{\mathrm{ab}}(k, L) = H^2_{\mathrm{ab}}(k, L^{\mathrm{red}})$, the map ab^2 is defined as the composite

$$H^2(k, L) \xrightarrow{r} H^2(k, L^{\mathrm{red}}) \xrightarrow{\mathrm{ab}^2} H^2_{\mathrm{ab}}(k, L^{\mathrm{red}}).$$

The main advantage of the abelianization map is that it helps detect neutral elements. More precisely, we have

(4.3) Proposition. *Let k be a perfect field with $\mathrm{vcd}(k) = 1$ and let $L = (\overline{G}, \kappa)$ be a connected linear k-kernel. An element $\eta \in H^2(k, L)$ is neutral if and only if $\mathrm{ab}^2(\eta)$ is 0.*

Proof. The proof is the same as in [B1], Sect. 5.8, once we replace [B1], Lemma 5.7, by

(4.4) Lemma. *Let G be a semisimple simply connected linear algebraic group over a perfect field k with $\mathrm{vcd}(k) \leq 1$ and put $G^{\mathrm{ad}} = G/Z$, where Z is the center of G. Then the connecting map $\delta : H^1(k, G^{\mathrm{ad}}) \to H^2(k, Z)$ is surjective.*

Proof. (See [Sch], Cor. 5.4, for another proof.) Since $\mathrm{cd}(k(i)) \leq 1$, Steinberg's theorem ([S2], III, §2.3) implies that $H^1(k(i), G)$ is trivial and G is quasi-split over $k(i)$. Therefore G has a Borel subgroup B defined over $k(i)$. Further, B can be chosen such that $B \cap \sigma B$ (where σ is the involution of $k(i)$ over k) is a torus T (necessarily maximal in G and defined over k; cf. [Sch], Prop. 4.9). The torus T has the property that $H^2(k_\xi, T) = 0$ (cf. [Sch], Prop. 1.6 and proof of Corollary 1.7) for every ordering $\xi \in \Omega_k$. Since the map $H^2(k, T) \to \prod_\xi H^2(k_\xi, T)$ is injective [Sch], Thm. 3.1, the group $H^2(k, T)$ is itself trivial. Using this in the long exact cohomology sequence associated to the short exact sequence

$$1 \to Z \to T \to T^{\mathrm{ad}} \to 1$$

which defines T^{ad}, we see that the map $H^1(k, T^{\mathrm{ad}}) \to H^2(k, Z)$ is surjective. Hence in the commutative diagram

$$\begin{array}{ccc} H^1(k, T^{\mathrm{ad}}) & \twoheadrightarrow & H^2(k, Z) \\ \downarrow & & \| \\ H^1(k, G^{\mathrm{ad}}) & \to & H^2(k, Z), \end{array}$$

the surjectivity of the top arrow implies the surjectivity of the bottom arrow. $\square$

The following proposition proves $(*)$ of Sect. 3 (cf. (3.5)) for connected linear algebraic groups.

(4.5) Proposition. *Let $L = (\overline{G}, \kappa)$ be a connected linear k-kernel, where k is a perfect field with $\mathrm{vcd}(k) \leq 1$. Then $\eta \in H^2(k, L)$ is neutral if and only if its localizations $\eta_\xi \in H^2(k_\xi, L)$ are neutral for all orderings ξ in a dense subset of Ω_k.*

Proof. We may assume without loss of generality (cf. (4.2)) that $\overline{G}$ is connected and reductive. Hence $H^2(k, L)$ has a neutral element (cf. [B1], Prop. 3.1). Denote by G the corresponding k-form of $\overline{G}$. Let T be a maximal k-torus of G, $p : G^{\mathrm{sc}} \to G^{\mathrm{ss}}$ the simply connected cover of the derived group G^{ss} of G, and put $T^{\mathrm{ss}} = T \cap G^{\mathrm{ss}}$ and $T^{\mathrm{sc}} = p^{-1}(T^{\mathrm{ss}})$.

In this set up, Borovoi [B2], Sect. 3, showed that the complexes $(T^{\mathrm{sc}} \to T)$ and $(Z^{\mathrm{sc}} \to Z)$ are quasi-isomorphic. Therefore we have

$$H^i_{\mathrm{ab}}(k, L) = \mathbb{H}^i(k, T^{\mathrm{sc}} \to T) = \mathbb{H}^i(k, Z^{\mathrm{sc}} \to Z).$$

We consider a maximal torus $T = B \cap \sigma B$ as in the proof of Lemma (4.4). Associated to the short exact sequence

$$1 \to (1 \to T) \to (T^{\mathrm{sc}} \to T) \to (T^{\mathrm{sc}} \to 1) \to 1$$

of short complexes placed in degrees -1 and 0, there is a commuting diagram of long exact hypercohomology sequences as follows:

$$\begin{array}{ccccccc} H^2(k,T^{\mathrm{sc}}) & \to & H^2(k,T) & \to & H^2_{\mathrm{ab}}(k,L) & \to & H^3(k,T^{\mathrm{sc}}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \prod_\xi H^2(k_\xi,T^{\mathrm{sc}}) & \to & \prod_\xi H^2(k_\xi,T) & \to & \prod_\xi H^2_{\mathrm{ab}}(k_\xi,L) & \to & \prod_\xi H^3(k_\xi,T^{\mathrm{sc}}). \end{array}$$

As noted in the proof of Lemma (4.4), the terms on the left are zero, hence the second horizontal arrows are injections. By [Sch], Cor. 3.2, the second and fourth vertical arrows are injective, even when ξ ranges only over a dense subset of Ω_k. Hence the third vertical map is injective. Our proposition follows from the commutative diagram

$$\begin{array}{ccc} H^2(k,L) & \stackrel{\mathrm{ab}^2}{\to} & H^2_{\mathrm{ab}}(k,L) \\ \downarrow & & \downarrow \\ \prod_\xi H^2(k_\xi,L) & \stackrel{\mathrm{ab}^2_\xi}{\to} & \prod_\xi H^2_{\mathrm{ab}}(k_\xi,L), \end{array}$$

on using Proposition (4.3) and the injectivity of the vertical arrow on the right. □

(4.6) Proposition (4.5) establishes ($*$) of (3.5), and consequently (iii) of Lemma (3.3), for a connected linear algebraic group, which is not necessarily semisimple or of trivial center. This eliminates a few steps in the proof of Theorem (3.1).

5. Applications

As an application of Grothendieck's theorem (more precisely, of the local-global principle for neutral elements) we get a new proof of Theorem 6.1 of [Sch]. We can also answer now the question raised in *loc.cit.* after 6.5.

(5.1) Let k be a field, let G be an algebraic group over k and let X be a homogeneous (right) G-space, defined over k. From X one constructs canonically a k-kernel L_X and an element $\alpha_X \in H^2(k, L_X)$. The class α_X is neutral if and only if there is a principal homogeneous G-space T which dominates X (over k).

Let us recall this construction ([Sp], Sect. 1.20; [B1], Sect. 7.7). Choose a point $x_0 \in X(k_s)$, and let $\overline{H}$ be the stabilizer of x_0. This is an algebraic subgroup of $\overline{G} := G \times_k k_s$. There is a locally constant map $\Gamma = \Gamma_k \to G(k_s)$, $s \mapsto a_s$, such that $s(x_0) = x_0 \cdot a_s$ for every $s \in \Gamma$. Then $\mathrm{int}(a_s) \circ s$ is an s-semilinear automorphism of $\overline{G}$ which leaves $\overline{H}$ invariant. Let f_s be its restriction to $\overline{H}$; the map $s \mapsto f_s$ from Γ to $\mathrm{SAut}(\overline{H}/k)$ is continuous (1.10), and $s \mapsto f_s \bmod \mathrm{Int}(\overline{H})$ is a k-kernel in $\overline{H}$. Denote it by κ_X, and write $L_X = (\overline{H}, \kappa_X)$. Up to a canonical isomorphism, this kernel does not depend on the choices. The pair (f, g) with f as above and $g_{s,t} := a_s s(a_t) a_{st}^{-1}$ is a 2-cocycle for L_X, and α_X is its class in $H^2(k, L_X)$. Again, this class does not depend on the choices. It is neutral if and only if X is dominated by a principal homogeneous G-space, as one may verify by a direct cocycle calculation.

In terms of group extensions the class α_X has a more natural appearance: Let $G(k_s) \cdot \Gamma$ be the natural semidirect product, and let $E = E_{X,x_0}$ be its subgroup consisting of all products gs with $g \in G(k_s)$, $s \in \Gamma$ such that $s(x_0) = x_0 \cdot g$. Then E is a subextension of the split extension $G(k_s) \cdot \Gamma$ as follows:

$$1 \to \overline{H}(k_s) \to E \to \Gamma \to 1. \tag{17}$$

This extension belongs to the k-kernel κ_X, and the class of (17) in $H^2(k, L_X)$ is α_X.

(5.2) Theorem. *Let k be a perfect field with* $\operatorname{vcd}(k) \le 1$*. Let X be a homogeneous space under an algebraic group G, both defined over k, and suppose that for every ξ in a dense subset of Ω_k, X is dominated by some principal homogeneous G-space over k_ξ. Then there exists a principal homogeneous G-space over k which dominates X over k.*

This was proved in [Sch], Thm. 6.5, under the stronger hypothesis that X is dominated by a principal homogeneous space over *every* k_ξ, and the question was raised there (p. 341) whether the above sharpening holds. The proof is very easy now:

Proof. Let the kernel L_X and the class $\alpha_X \in H^2(k, L_X)$ be as constructed in (5.1). By the above, the hypothesis says that the class α_X becomes neutral over k_ξ for all ξ in a dense subset. By Corollary (3.2), α_X is neutral over k. Again by (5.1), this translates into the existence of a principal homogeneous G-space as desired. □

In [Sch], the above-mentioned Thm. 6.5 was derived from the following main result:

(5.3) Theorem ([Sch], Thm. 6.1). *Let k be a perfect field with* $\operatorname{vcd}(k) \le 1$*, and let X be a homogeneous space under an algebraic group G, both defined over k. Suppose $X(k_\xi) \neq \emptyset$ for all ξ in a dense subset of Ω_k. Then there exists a principal homogeneous G-space T over k which dominates X over k and is trivial over every k_ξ.*

We show how conversely Theorem (5.3) can be derived from Theorem (5.2) (and thus from our main results in Sect. 3); in fact, the weaker version of (5.2) proved in [Sch] suffices for this.

First of all we have $X(k_\xi) \neq \emptyset$ for all orderings ξ, by general reasons (e.g. [Sch], Cor. 2.2). Therefore by [Sch], Thm. 6.5 — cf. (5.2) above — there exists a G-torsor P together with a G-equivariant map $\alpha\colon P \to X$, both defined over k. Let H be the algebraic group over k which consists of those G-equivariant automorphisms of P which commute with α. For any extension K/k the set $H^1(K, H)$ parametrizes the K-forms $\beta\colon Q \to X$ of $\alpha\colon P \to X$. So we have a commutative diagram

$$\begin{array}{ccc} H^1(k,H) & \to & H^1(k,G) \\ \downarrow & & \downarrow \\ \Gamma(\Omega_k,\, \mathcal{H}^1(H)) & \to & \Gamma(\Omega_k,\, \mathcal{H}^1(G)) \end{array} \tag{18}$$

in which the upper horizontal arrow takes $\beta\colon Q \to X$ to Q.

For every ξ let $\gamma_\xi \in H^1(k_\xi, H)$ be the class of a G-map $T_\xi \to X$ over k_ξ, where T_ξ is the trivial G-torsor over k_ξ. The family $(\gamma_\xi)_\xi$ is locally constant, i.e. is a global section of the sheaf $\mathcal{H}^1(H)$. Since the vertical maps in (18) are surjective by [Sch], Thm. 5.1, there exists a G-torsor $\beta\colon Q \to X$ over X which is trivial over each k_ξ. □

Keep the assumption that k is perfect with $\operatorname{vcd}(k) \le 1$, and suppose that G is connected and linear. Recall that any G-torsor which is trivial over k_ξ for all ξ in a dense subset of Ω_k is trivial over k ([Sch], Cor. 4.2). Combining this with Theorem (5.3) immediately implies the following Hasse principle for general homogeneous spaces under G:

(5.4) Corollary ([Sch], Cor. 6.2). *Let k be a perfect field with* $\operatorname{vcd}(k) \le 1$*, G a connected linear algebraic group and X a homogeneous G-space, both defined over k. If X has a k_ξ-point for all ξ in a dense subset of Ω_k, then X has a k-point.*

As noted after (6.2) of [Sch], the conclusion holds for any G for which the natural map $H^1(k,G) \to \prod_\xi H^1(k_\xi, G)$ has trivial kernel.

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