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Two Loop Dilaton Tadpole Induced by Fayet-Iliopoulos D-Terms In Compactified Heterotic String Theories*

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ABSTRACT

We calculate the two loop dilaton tadpole induced by Fayet-Iliopoulos D-terms in heterotic string theories compactified on arbitrary supersymmetry preserving backgrounds. The result turns out to be a total derivative in the moduli and hence receives contribution solely from the boundary of the moduli space. This contribution is shown to be proportional to the square of the coefficient of the one loop Fayet-Iliopoulos D-term in agreement with what is expected from effective lagrangian considerations.

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1. Introduction

It has recently been demonstrated, both through effective lagrangian considerations [1] as well as through explicit string calculations [2, 3], that Fayet-Iliopoulos D-terms [4] can be generated at one loop level in string perturbation although they are absent at tree level.‡ These terms arise if the unbroken gauge group of the theory contains one or more $U(1)$ factors whose generators have a nonzero trace over the tree level massless chiral fermions. One consequence of the presence of these terms is that massless scalars charged under the $U(1)$ develop a mass at one loop. In fact this is how the Fayet-Iliopoulos D-terms were calculated in ref. [2,3] in the first place, *i.e.* via the one loop scalar masses they generate.

The Fayet-Iliopoulos D-terms are also expected to destabilize the vacuum by inducing a dilaton tadpole at two loops. This may be seen by examining the effective lagrangian involving the auxiliary D fields. This takes the following form

$$L_D = \frac{1}{2} D^{(a)} D^{(a)} + e^{-3\phi} c^{(a)} D^{(a)} + e^{-\phi} D^{(a)} \sum_i q_i^{(a)} \chi_i^* \chi_i, \tag{1.1}$$

where $D^{(a)}$ is the suitably normalized auxiliary field associated with the a'th abelian factor $U^{(a)}(1), \chi_i$ are the charged scalars and ϕ is the dilaton. The coefficients $c^{(a)}$ in the one loop term $c^{(a)} D^{(a)}$ are those computed in ref. [2] and they only depend on properties of the massless spectrum. It is now easy to see that the above structure for the auxiliary field lagrangian yields, among other things, a two loop dilaton tadpole proportional to $\sum_a (c^a)^2$ in the effective potential.

It is important to verify the validity of the above picture through explicit two loop string calculation. This would confirm our understanding of Fayet-Iliopoulos D-terms in string theories and would give new insights into string loop

‡ An indirect calculation of the D-term coefficient was also given in ref. [5]

amplitudes. In fact this is the main objective of this work. More specifically, in this paper we shall carry out a two loop calculation of the dilaton tadpole induced by the Fayet-Iliopoulos D-terms in heterotic string theories compactified on arbitrary backgrounds that preserve tree level supersymmetry. As we shall see below, our results are in full agreement with what is expected from the effective lagrangian considerations given above.

In carrying out this work, we were also motivated by the need for a better understanding of the structure of fermionic string perturbation beyond one loop. As is well known, loop calculations beyond the torus in fermionic string theories involve several new complexities over and above those encountered in bosonic string loop amplitudes. The most prominent of these, of course, is the presence of supermoduli [6] and the associated problem of non-zero background charge for the superconformal ghosts. Recently, there has been several attempts at handling these subtleties [7-15]. The main approach adopted in almost all of these attempts has been to integrate over the supermoduli in advance. This has the effect of introducing insertions of certain operators involving matter as well as ghost fields, in any correlator in addition to the vertex operators whose correlation is to be computed. This operation has been rendered well defined through a recent work of Verlinde and Verlinde [13], who have succeeded in globally defining correlators involving the superconformal ghosts. In the present calculation we shall adopt the above approach for handling the supermoduli and shall use some of the results of ref.[13] pertaining to the superconformal ghost system. A brief review of all relevant facts needed is given in the next section.

One of the interesting aspects of this calculation is the fact that the dilaton tadpole at two loops in an arbitrary supersymmetry preserving background turns out to be a total derivative in the moduli space. Thus the only contribution to the dilaton tadpole comes from the boundary of the moduli space. More specifically it comes from the region in moduli space where $\Omega_{12} \to 0$, where Ω_{12} is the off-diagonal component of the genus two period matrix. In such a limit the two loop diagram degenerates into two separate handles (tori) connected by an infinitely long neck (Fig. 1a). We explicitly verify that this boundary term is proportional to $\sum_a (c^{(a)})^2$, where $c^{(a)}$ are the coefficients of the auxiliary D field in the one-loop Fayet-Iliopoulos D-terms calculated in ref. [2]. Needless to say this means that the contribution to the dilaton tadpole could be interpreted as coming from a diagram with two seperate tori and with an auxiliary D field propagating in between as in (Fig. 1b).

Our work also sheds light on why the general arguments of ref. [16] break down at the two loop level. In [16] the dilaton vertex operator was written as the contour integral of the space-time supersymmetry current around a fermionic vertex oparator (dilatino). It was then argued that the integration contour may be deformed away from the fermionic vertex operator and shrunk to zero, since the GSO projection ensures that the supersymmetry current is periodic on the Riemann surface. The more detailed analysis of ref. [13] showed that the supersymmetry current develops unphysical poles on the Riemann surface which prevents the contour from being deformed and shrunk to zero. It was also shown the residue at these poles may be expressed as total derivatives in the moduli space. What our analysis shows is that these total derivative terms are not quite harmless, since they may give non-zero contribution from the boundary of the moduli space.

At this point it is also appropriate to ask if our analysis throws any light on higher loop (> 2) dilaton tadpole or cosmological constant in uncompactified heterotic string theory. The general analysis of ref. [13] shows that the amplitudes under consideration may in general be expressed as a total derivative in the moduli space. But as we have seen, this is not enough to show the vanishing of the corresponding amplitude. One may try to study the behaviour of the total derivative terms and try to show they vanish. Another alternative is to try to use general factorization properties which must be satisfied by the amplitude at the boundary of the moduli space, and try to see how this constrains the amplitude under consideration. This approach was pursued in ref. [10] (see also [17]) to show the vanishing of the cosmological constant in heterotic string theory. But this approach presupposes a possible choice of basis of the Beltrami differentials which makes the amplitude manifestly modular invariant, and factorize only on the physical states of the theory. It is not quite clear whether such a choice is possible in actual practice. One might also try to bypass the integration over the supermoduli by postulating a suitable ansatz for screening the background ghost charge, which gives sensible answers for all the amplitudes, consistent with the symmetry properties of the theory. In ref. [12] we proposed an ansatz for screening the background ghost charge which ensures the vanishing of all amplitudes with three or less external legs, but a derivation of the ansatz from first principles by integrating over the supermoduli is still lacking. Similar ansatz has also been discussed in refs.[11,15] Finally there are general arguments for the non-renormalization of the superpotential based on effective four dimensional field theory [18]. These arguments are based on the decoupling of the axion at zero momentum, which was proved by showing the decoupling of the zero momentum component of the vertex operator. However as has become clear from the calculation of refs. [2, 3], the decoupling of the zero momentum component of a scalar vertex operator does not always imply the decoupling of the corresponding scalar field at zero momentun, since the integration over the world sheet coordinates can produce inverse powers of momentum, thus allowing the part of the vertex operator linear in momentum to cointribute in the zero momentum limit. Whether such a phenomenon actually occurs for the axion vertex operator is not known at this stage.

This paper is organized as follows: In the next section we shall discuss the background material needed for later developements in the paper. In section three we present the details of the calculation of the two loop dilaton tadpole. We also verify that this calculation yields the expected result. Section four contains our conclusions and some more discussion of the implication of the results of section three to string multiloop calculations in general. Appendix A contains some useful formulae describing the behaviour of various functions on a genus two surface in the degeneration limit $\Omega_{12} \to 0$.

2. Background Material

This section is intended as a review of some of the background material needed for later developments in the paper. We shall mainly outline the scheme that we adopt following ref. [7,13] for handling the supermoduli and review some of the results of ref. [2] that we shall need here concerning the one loop Fayet-Iliopoulos D-terms.

A. Integration over the supermoduli

The supermoduli are zero modes of the gravitino that cannot be gauged away, (*i.e.* they provide an obstruction to the superconformal gauge choice) much like the ordinary moduli or the zero modes of the 2d metric. As such, any string amplitude will involve an integration over the moduli space as well as the space of supermoduli. The latter can be shown to be a complex space of dimension $2g-2$ for $g>1$. As mentioned earlier the practice so far has been to carry out the integration over the supermoduli in advance *i.e.* before the calculation of correlation functions. Ultimately, it would be of great interest to have a formalism where the supermoduli would be treated more on equal footing to the moduli in some super-Riemann [19] surface representation. However the theory of super Riemann surfaces is just being developed [20] and although interesting progress has been achieved, at this point in time we do not yet possess a practical scheme along these directions within which calculations can be carried out. Therefore we shall resort here to the more common practice of integrating out the supermoduli in advance as we explain next.

The gravitino (χ) couples to the world-sheet string degrees of freedom through the term,

$$\int d^2 z \chi T_F \tag{2.1}$$

in the action, where T_F is the total fermionic stress tensor. Integration over the supermoduli in string amplitudes has the effect of bringing down factors of T_F folded with appropriate zero mode wavefunctions, *i.e.* brings down $\langle \chi^{(a)}(z)|T_F\rangle =$

$\int d^2z\chi^{(a)}(z)T_F(z)$ where $\{\chi^{(a)}(z) : a = 1, \cdots 2g-2\}$ is an appropriate basis for the 3/2 differentials. A convenient choice for this basis could be $\chi^{(a)} = \delta^{(2)}(z-z_a)$ where $\{z_a : a = 1, \cdots 2g-2\}$ is some apriori arbitrary set of points on the Riemann surface. In this basis the effect of the supermoduli is to introduce insertions of the stress tensor T_F at some set of points $\{z_a : a = 1, \cdots 2g-2\}$.

An associated complexity is posed by the zero modes of the superconformal (β, γ) ghost system. Using an index theorem one can see that on genera $g \geq 2$ β develops $2g-2$ zero modes. The counting here being exactly the same as that for the supermoduli. To render correlation functions well defined integration over the zero modes of β has to be restricted in an appropriate way. Alternatively the presence of these zero modes could be thought of as signalling the presence of a background ghost charge [7], $q^{back.} = 2(1-g)$. Correlation functions of charge neutral combination of operators can be shown to vanish. Only operators which soak up the background charge survive (*i.e.* operators with a ghost charge that adds up to $2g-2 = -q^{back.}$). In the bosonized representation of ref. [7] where the superconfromal ghosts β, γ are represented by :

$$\gamma(z) = e^{\phi(z)}\eta(z), \qquad \beta(z) = e^{-\phi(z)}\partial\xi(z), \tag{2.2}$$

one could insert $2g-2$ factors of the background charge operator e^{ϕ} to soak up the ghost charge. However these operators have to be joined with $\{T_F(z_a) : a = 1, \cdots, 2g-2\}$ produced by the supermoduli integration in a $BRST$ invariant fashion. As was first pointed out in ref. [7], the unique combination of $e^{\phi(z_a)}$ and $T_F(z_a)$ that does this is given by:

$$\begin{aligned} Y(z_a) &\equiv :e^{\phi(z_a)}T_F(z_a): \\ &= \{Q_{BRST}, \xi(z_a)\} \\ &= e^{\phi(z_a)}T_F^{matter} - \frac{1}{4}\partial\eta(z_a)e^{2\phi(z_a)}b(z_a) \\ &\quad - \frac{1}{4}\partial(\eta(z_a)e^{2\phi(z_a)}b(z_a)) + c(z_a)\partial\xi(z_a), \end{aligned} \tag{2.3}$$

which is nothing but the picture changing operator. In this equation b, c stand for the reparametrization ghosts.

To sum up, the combined effect of the supermoduli and the background ghost charge is to introduce a factor of $\prod_{a=1}^{2g-2} Y(z_a)$ in any correlation function. The above prescription has been rederived from a path integral approach in ref. [13]. Also as pointed out in [13] modular invariance gives some restrictions on the apriori arbitrary positions z_a. The point here is that the space of moduli in general contains orbifold points which correspond to Riemann surfaces that possess a discrete group of automorphisms. Modular invariance would require the positions of $Y(z_a)$ on these surfaces to be left fixed or permuted among themselves under the action of the automorphism. More specific discussion of this will be given in the next section.

At this point we can write down the form of the correlator for any set of vertex operators with no net ghost charge. In heterotic string theories this is given by:

$$\langle V(x_1)\cdots V(x_N)\rangle$$

$$= \int \prod_{i=1}^{3g-3} [dm_i d\bar{m}_i] \int [DXD\psi D\varphi D\beta D\gamma DbD\bar{b}DcD\bar{c}] e^{-S} \prod_{i=1}^{3g-3} \{\langle \eta_j|b\rangle\langle\bar{\eta}_j|\bar{b}\rangle\}\xi(z_0)$$
$$\prod_{a=1}^{2g-2} Y(z_a) \prod_{i=1}^{N} V(x_i), \tag{2.4}$$

where S is the string action with χ set to zero. Here X^μ and ψ^μ $(1 \le \mu \le 4)$ denote the free bosonic fields and their right-handed fermionic partners associated with the four flat directions. $\{\varphi^j\}$ denote the set of all the other fields associated with the internal degrees of freedom. For example, for the heterotic string compactified on Calabi-Yau manifolds or orbifolds,$\{\varphi^j\}$ include the six bosonic and the right-handed fermionic fields associated with the compact dimensions, as well as the thirty two left-handed fermions responsible for generating gauge group. If we use a more general superconformal field theory to replace the internal dimensions, the set $\{\varphi^j\}$ would represent the variables of the conformal field theory. $m_i, i = 1, \cdots 3g-3$ stand for an appropriate set of moduli of the Riemann surface and $\{\eta_j : j = 1, \cdots, 3g-3\}$ are the Beltrami differentials dual to $\{dm_j\}$, satisfying

$$\frac{\partial \eta_j}{\partial m_i} - \frac{\partial \eta_i}{\partial m_j} = 0. \tag{2.5}$$

$\langle \eta_j | b \rangle = \int d^2 w \eta_j(w) b(w)$ are inserted to absorb the zero modes of the b, c ghost system. Similarily the operator $\xi(z_0)$ has been inserted to soak up the zero mode of the ξ-field.

It is clear from eqs (2.3) and (2.4) that to effectively calculate correlation functions we have to deal with correlators involving superconformal ghosts of the form $\langle \prod_{i=1}^{n+1} \xi(x_i) \prod_{j=1}^{n} \eta(y_j) \prod_k e^{q_k \phi(z_k)} \rangle_\delta$ with $\sum_k q_k = 2g-2$, where δ denotes the spin structure. An expression for this correlator was derived in ref. [13]. Here we shall only quote the answer refering the reader to [13] for its derivation:

$$\begin{aligned}
&\langle \prod_{i=1}^{n+1} \xi(x_i) \prod_{j=1}^{n} \eta(y_j) \prod_k e^{q_k \phi(z_k)} \rangle_\delta \\
&\qquad = Z_1^{\frac{1}{2}} \frac{\prod_{j=1}^{n} \vartheta[\delta](-\vec{y}_j + \sum_i \vec{x}_i - \sum_j \vec{y}_j + \sum_k q_k \vec{z}_k - 2\vec{\Delta})}{\prod_{i=1}^{n+1} \vartheta[\delta](-\vec{x}_i + \sum_i \vec{x}_i - \sum_j \vec{y}_j + \sum_k q_k \vec{z}_k - 2\vec{\Delta})} \\
&\qquad\quad \frac{\prod_{i<i'} E(x_i, x_{i'}) \prod_{j<j'} E(y_j, y_{j'})}{\prod_{i,j} E(x_i, y_j) \prod_{k<l} E(z_k, z_l)^{q_k q_l} \prod_k (\sigma(z_k))^{2q_k}}
\end{aligned} \tag{2.6}$$

In the above $E(x, y)$ is the prime form defined by [21]

$$E(x, y) = \frac{\vartheta[\alpha](\int_y^x \vec{\omega})}{h[\alpha](x) h[\alpha](y)}, \tag{2.7}$$

where $\omega_i, i = 1, \cdots g$ are the canonical abelian differentials on the Riemann surface. α denotes an odd spin structure and $h[\alpha]$ is a holomorphic half differential associated with the spin structure α. $\vec{\Delta}$ is the Riemann class characterizing the divisor of zeroes of the ϑ-function. Furthermore, for each point x on the Riemann surface we have defined a vector $\vec{x}$, $\vec{x} = \int_{P_0}^{x} \vec{\omega}$, where P_0 is a fixed base point. $\sigma(z)$ is a $g/2$ differential representing the background charge in the theory and carries the conformal anomaly. Since ultimately we will be working with multiplets of matter and ghost fields with no net conformal anomaly, only ratios of σ will appear in any of our final expressions. For those one could use,

$$\frac{\sigma(z)}{\sigma(w)} = \frac{\vartheta(\vec{z} - \sum_i \vec{p}_i + \vec{\Delta})}{\vartheta(\vec{w} - \sum_i \vec{p}_i + \vec{\Delta})} \prod_{i=1}^{g} \frac{E(w, p_i)}{E(z, p_i)} \tag{2.8}$$

where $\{p_i\}$ is an arbitrary set of points. It is not difficult to verify that (2.8) is independent of the p_i's. Finally $Z_1^{\frac{1}{2}}$ is an overall normalization factor given by,

$$Z_1 = \left[\vartheta(\sum_{i=1}^{g} \vec{u}_i - \vec{v} - \vec{\Delta}) \frac{\prod_{i<j} E(u_i, u_j)}{\prod_i E(u_i, v)} \frac{\prod_i \sigma(u_i)}{\sigma(v)} \frac{1}{det\omega_i(u_j)} \right]^{\frac{2}{3}} \tag{2.9}$$

where u_i and v are arbitrary points on the Riemann surface.

Also useful for our analysis will be a general correlation function involving the reparametrization ghost fields [22-25]:

$$\int DbDce^{-S(b,c)} \prod_{i=1}^{3g-3} b(z_i) = Z_1^{-\frac{1}{2}} \vartheta \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} (\sum_i \vec{z}_i - 3\vec{\Delta}) \prod_{i<j} E(z_i, z_j) \prod_i (\sigma(z_i))^{3(g-1)} \tag{2.10}$$

In our actual calculations below it turns out that we only need the above formulae only on genus two Riemann surfaces in the limit of degeneration* into two genus one surfaces as shown in Fig.1. In this limit all of the above formulae become much more explicit. For convenience and later reference we list all the expressions describing the degeneration of various quantities in an appendix.

* The problem of degeneration of Riemann surfaces in general in connection with string theory has been analysed by several authors [26–29, 23, 30]. For a mathematical treatment see [21].

At this point let us note that (2.6) has 'unphysical poles' at the zeroes of $\prod_j \vartheta[\delta](\sum_{i\neq j}\vec{x}_i - \sum_i \vec{y}_j + \sum_k q_k\vec{z}_k - 2\vec{\Delta})$. Since these poles play a major role in our calculation, we shall give a physical interpretation of these 'unphysical' poles: In the presence of the operators $\xi(x_i)$, $\eta(y_j)$ and $e^{q_k\phi(z_k)}$ the field γ may develop singularities of the form:

$$\begin{aligned}\gamma(z) &\sim (z-x_i)^{-1} \qquad as\ z\to x_i\\ &\sim (z-y_j) \qquad as\ z\to y_j\\ &\sim (z-z_k)^{-q_k} \qquad as\ z\to z_k\end{aligned}\tag{2.11}$$

We shall now show that $\gamma(z)$ develops a zero mode in the presence of these singularities whenever $\prod_\ell \vartheta[\delta](\sum_{i\neq\ell}\vec{x}_i - \sum_j \vec{y}_j + \sum_k q_k\vec{z}_k - 2\vec{\Delta})$ vanishes, and hence the path integral diverges. A zero mode of $\gamma(z)$ corresponds to an antianalytic $-\frac{1}{2}$ differential with the singularities given in (2.11). There is however one subtlety: Since one of the $\xi(x_i)$'s must be used to absorb the zero mode of ξ, $\gamma(z)$ can develop poles only near n of the $n+1$ x_i's. Let us for the time being use $\xi(x_{n+1})$ to absorb the ξ zero mode. We then can write down the following function as a zero mode of $\gamma(z)$:

$$\begin{aligned}\gamma_0(z) \sim &\frac{\prod_{l=1}^{2g-2}E(z,P_l)}{\prod_k(E(z,z_k))^{q_k}}\frac{\prod_{i=1}^{n}E(z,y_i)}{\prod_{i=1}^{n-1}E(z,x_i)}\\ &\frac{\vartheta[\delta](-\vec{z}+\vec{Q}_0+\sum_k q_k\vec{z}_k+\sum_{i=1}^n\vec{x}_i-\sum_{i=1}^n\vec{y}_i-2\vec{\Delta})}{\vartheta(-\vec{z}+\sum_{l=1}^{g}\vec{P}_l-\vec{\Delta})\vartheta(-\vec{z}+\sum_{l=g+1}^{2g-2}\vec{P}_l+\vec{x}_n+\vec{Q}_0-\vec{\Delta})}\end{aligned}\tag{2.12}$$

where $P_1,\cdots P_{2g-2}$ and Q_0 are some arbitrary points on the Riemann surface. (2.12) has the right singularities except for an extra pole at Q_0 coming from the zero of $\vartheta(-\vec{z}+\sum_{l=g+1}^{2g-2}\vec{P}_l+\vec{x}_n+\vec{Q}_0-\vec{\Delta})$. Thus in order for (2.12) to represent a zero mode of γ the numerator must vanish at $z=Q_0$, *i.e.* $\vartheta[\delta](\sum_k q_k\vec{z}_k + \sum_{i=1}^n\vec{x}_i-\sum_{i=1}^n\vec{y}_i-2\vec{\Delta})$ must vanish. Using the other $\xi(x_i)$'s to absorb the ξ zero mode we see that γ has a zero mode whenever $\prod_j \vartheta[\delta](\sum_k q_k \vec{z}_k + \sum_{i\neq j,\ i=1}^{n+1} \vec{x}_i - \sum_{i=1}^{n} \vec{y}_i - 2\vec{\Delta})$ vanishes, as advertised above.

We should remind the reader at this point that these unphysical poles also appear in the correlator of ghost spin fields on genus one Riemann surface. For example $\langle S_g^+(z_1) S_g^+(z_2) S_g^-(w_1) S_g^-(w_2)\rangle_\delta$ is proportional to $(\vartheta[\delta](\frac{z_1+z_2-w_1-w_2}{2}))^{-1}$ [31,32]. However as we showed in ref. [32] these poles are absent in the physical amplitudes. We should also note that in any correlator involving $\beta(= e^{-\phi}\partial\xi)$ and $\gamma(= e^{\phi}\eta)$, the arguments of β and γ drop out of the product of ϑ functions determining the positions of the unphysical poles. Thus the fields β and γ do not have any unphysical poles. This, in turn, implies that the $BRST$ current also does not have any unphysical poles, since it can be constructed entirely in terms of β and γ without explicit reference to the fields ξ, η, and ϕ.

B. *D*-Tadpole at One Loop

We next turn our attention to some of the results of ref. [2] that we shall need here, pertaining to the one loop string calculation of Fayet-Iliopoulos D-terms. The basic observation that we will utilize is the fact that the coefficients $c^{(a)}$ of the one loop Fayet-Iliopoulos D-terms in any arbitrary compactification which preserves $(2,0)$ world-sheet superconformal invariance and tree level space-time supersymmetry are given by[⋆]

$$c^{(a)} = (-\frac{ig}{6\pi}) \int d^2\tau \langle J(z) U^{(a)}(\bar{z}) \rangle_e \tag{2.13}$$

In the above equation $J(z)$ is the $U(1)$ current of the $N=2$ superconformal algebra [33], while $U^{(a)}(\bar{z})$ is the gauge $U(1)$ current associated with the a'th abelian factor $U^{(a)}(1)$ in the gauge group. For the $Spin(32)/Z_2$ heterotic string theory [34] compactified on Calabi-Yau [35] manifolds for instance, $U^{(a)}(\bar{z})$ generates the $U(1)$ factor of the unbroken $SO(26)\times U(1)$ gauge group. The subscript e in $\langle\ \ \rangle_e$ above denotes a sum over even spin structures in the right-handed sector. The contribution of the periodic-periodic sector vanishes due to the zero modes of the free right-handed fermions $\psi^{\mu}, \psi^{\bar{\mu}}, \mu = 4,5$. It is worth mentioning at this point that $c^{(a)}$ in (2.13) above may be interpreted as the expectation value of the auxiliary D field, where the vertex operator for the auxiliary $D^{(a)}$ field is simply $-\frac{i}{3}J(z)U^{(a)}(\bar{z})$.

A particularly useful representation for the correlator in (2.13) is given by:

$$\langle J(z) U^{(a)}(\bar{z}) \rangle_e = \frac{3}{2} \langle P^+(z) P^-(w) U^a(\bar{z}) \rangle \tag{2.14}$$

as demonstrated in [2]. In (2.14) P^+ and P^- are conformal fields of dimension $(1,0)$ and $(0,0)$ respectively constructed out of the various spin fields in the

⋆ In writing down (2.13) we have removed the integration over z to take into account the translational invariance on the torus. Thus in calculating the correlator we should not divide by the volume of the group of translations on the torus. This notation difers from that of ref.[2].

theory as follows:

$$
\begin{aligned}
P^+ &= S_g^- \hat{S}^+ S_4^+ S_5^+ \\
P^- &= S_g^+ \hat{S}^- S_4^- S_5^-
\end{aligned}
\tag{2.15}
$$

where $S_g^{\pm} = e^{\pm\phi/2}$ are the ghost spin fields, $S_{4,5}^{\pm}$ are the four dimensional spin fields while the operators $\hat{S}^{\pm}$ are precisely those fields that appear in the supersymmetry charges. In the free case (uncompactified internal space) $\hat{S}^{\pm}$ reduces to $e^{\pm\frac{1}{2}(\phi^1+\phi^2+\phi^3)}$ where ϕ^i are related to the internal fermions through standard bosonization by $\psi^i \sim e^{i\phi^i}$. Here we shall only need to use the operator product of P^+ and P^- given by

$$
P^+(z)P^-(w) \sim \frac{1}{(z-w)} \tag{2.16}
$$

without the need for any explicit representation of $\hat{S}^{\pm}$.

The correlator in (2.14) can be computed in terms of ϑ-functions on any arbitrary background. In a given spin strucutre $[\delta]$ the answer is given by

$$
\begin{aligned}
&\langle P^+(z)P^-(w)U^{(a)}(\bar{z})\rangle_\delta \\
&= \tilde{K}\epsilon[\delta]\frac{\vartheta[\delta](\frac{z}{2}-\frac{w}{2}-A^{(a)})\vartheta[\delta](\frac{z}{2}-\frac{w}{2}-B^{(a)})\vartheta[\delta](\frac{z}{2}-\frac{w}{2}-C^{(a)})\vartheta[\delta](\frac{z}{2}-\frac{w}{2})}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](z-w)}
\end{aligned}
\tag{2.17}
$$

where $[\delta] = \left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right], \left[\begin{smallmatrix}\frac{1}{2}\\ 0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\ \frac{1}{2}\end{smallmatrix}\right]$ denote spin structures $(P,P),(P,A),(A,A)$ and (A,P) with the corresponding $\epsilon\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right] = exp(4\pi iab)$. In the above $\tilde{K}$, $A^{(a)}, B^{(a)}, C^{(a)}$ are some parameters (functions of τ) characterizing the details of the model. We shall only need the fact that

$$
A^{(a)} + B^{(a)} + C^{(a)} = 0 \tag{2.18}
$$

Summing over spin structures on the torus and using a Riemann theta function identity and (2.18) we finally get:

$$
\begin{aligned}
c^{(a)} &= -\frac{ig}{4\pi}\int d^2\tau \langle P^+(z)P^-(w)U^{(a)}(\bar{z})\rangle \\
&= -\frac{ig}{2\pi}\int d^2\tau \tilde{K}\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(-A^{(a)})\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(B^{(a)})\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(C^{(a)})
\end{aligned}
\tag{2.19}
$$

which was shown in ref. [2] to be calculable solely in terms of the massless spectrum of the theory.

Also useful for our analysis will be the correlator,

$$
\begin{aligned}
&\int [DXD\psi D\varphi]_\delta e^{-S(X,\psi,\varphi)}\hat{S}^+(z)S_4^+(z)S_5^+(z)\hat{S}^-(w)S_4^-(w)S_5^-(w)U^{(a)}(\bar{z}) \\
&\sim(\eta(\tau))^{-3}(\overline{\eta(\tau)})^{-2}(\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0))^{1/4} \\
&\quad \tilde{K}\frac{\vartheta[\delta](\frac{z}{2}-\frac{w}{2}-A^{(a)})\vartheta[\delta](\frac{z}{2}-\frac{w}{2}-B^{(a)})\vartheta[\delta](\frac{z}{2}-\frac{w}{2}-C^{(a)})(\vartheta[\delta](\frac{z}{2}-\frac{w}{2}))^2}{(\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](z-w))^{5/4}}
\end{aligned}
\tag{2.20}
$$

where we have ignored an overall numerical factor. Eq. (2.20) is derived from eq. (2.17) by dividing the latter by the known expressions for [31]

$$
\begin{aligned}
&\int [d\beta d\gamma]_\delta S_g^-(z)S_g^+(w)e^{-S(\beta,\gamma)} \\
&\qquad = \frac{(\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](z-w))^{\frac{1}{4}}}{\vartheta[\delta](\frac{z}{2}-\frac{w}{2})}\frac{\eta(\tau)}{(\vartheta'\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](0))^{\frac{1}{4}}}
\end{aligned}
\tag{2.21}
$$

and [36]

$$
\int [DbDc]e^{-S(b,c)}b(z_0)c(z_0) = (\eta(\tau))^2 \tag{2.22}
$$

and its complex conjugate.

3. Two Loop Dilaton Tadpole

In this section we shall calculate the two loop dilaton tadpole in models where the auxiliary D-fields develop a vacuum expectation value at one loop in string perturbation. The analysis may be divided into two parts. First, using manipulations similar to those used in ref.[13], we show that the dilaton tadpole may be written as a total derivative in the moduli space. Next we calculate the dilaton tadpole by examining the integrand near the boundary of the moduli space. In carrying out this analysis we shall not be careful about the overall numerical factor.

We start by writing down the zero momentum vertex operator for the superpartner of the dilaton, i.e. the dilatino, in the $-\frac{1}{2}$ picture:

$$V^{\alpha}_{-\frac{1}{2}} = \bar{\partial} X^{\mu} (\gamma_{\mu})^{\alpha}{}_{\dot{\beta}} \hat{S}^{-} S^{\dot{\beta}} e^{-\frac{\phi}{2}} \tag{3.1}$$

where $\hat{S}^{-}$ has been defined in sec. 2. From this we can derive an expression for the dilatino vertex in the $\frac{1}{2}$ picture,

$$\begin{aligned} V^{\alpha}_{\frac{1}{2}} \equiv & [Q_{BRST}, \xi V^{\alpha}_{-\frac{1}{2}}] \\ = & \bar{\partial} X^{\mu} (\gamma_{\mu})^{\alpha\dot{\beta}} [e^{\frac{\phi}{2}} \partial X^{\nu} (\gamma_{\nu})_{\dot{\beta}\gamma} \hat{S}^{-} S^{\gamma} + \partial(c\xi e^{-\frac{\phi}{2}} \hat{S}^{-} S_{\dot{\beta}}) + \frac{1}{2} e^{\frac{3}{2}\phi} \eta b \hat{S}^{-} S_{\dot{\beta}} \\ & + e^{\frac{\phi}{2}} S_{\dot{\beta}} \lim_{w \to z} (w - z)^{\frac{1}{2}} T_F^{int}(w) \hat{S}^{-}(z)] \end{aligned} \tag{3.2}$$

In the above, S_{α}, $S_{\dot{\alpha}}$ denote four dimensional spin operators of positive and negative chiralities. Note that this definition of $V^{\alpha}_{\frac{1}{2}}$ differs from that of ref.[7] by a total derivative term. We choose not to drop the total derivative term so as to ensure that $V^{\alpha}_{\frac{1}{2}}$ is $BRST$ invariant point by point on the Riemann surface before integrating over the location of the vertex:

$$[Q_{BRST}, V_{\frac{1}{2}}(z)] \equiv \oint \frac{dw}{2\pi i} J_{BRST}(w) V_{\frac{1}{2}}(z) = 0 \tag{3.3}$$

The above definition of $V_{\frac{1}{2}}$ will prove to be more convenient for our manipulations below.

Now we could write down the vertex operator V_0 for the dilaton field as a supersymmetry commutator with the vertex operator for the dilatino in (3.2). More precisely,

$$\langle V_0(y)\rangle \delta^\alpha_\beta = \oint \frac{dx}{2\pi i}\langle J_\beta(x) V^\alpha_{\frac{1}{2}}(y)\rangle \tag{3.4}$$

where $J_\beta(x)$ is the four dimensional space-time supersymmetry current in the $-\frac{1}{2}$ picture given by,

$$J_\beta = e^{-\frac{\phi}{2}}\hat{S}^+ S_\beta \tag{3.5}$$

For definiteness we shall from now on take $\beta = (+,+)$ and $\alpha = (-,-)$ so that $\delta^\alpha_\beta = 1$. Using eqs.(3.4) and (2.4) the two loop dilaton tadpole can be written as,

$$\Lambda_D = \sum_\delta \epsilon[\delta] \int d^2y \oint_y \frac{dx}{2\pi i} \int [\prod_{i=1}^{3} dm_i d\bar{m}_i][DXD\psi D\varphi DbD\bar{b}DcD\bar{c}D\beta D\gamma]_\delta$$
$$e^{-S}\prod_{j=1}^{3}\{\langle\eta_j \mid b\rangle\langle\bar{\eta}_j \mid \bar{b}\rangle\}\xi(z_0)\{\prod_{a=1}^{2} Y(z_a)\} J_\beta(x) V^\alpha_{\frac{1}{2}}(y) \tag{3.6}$$

As a function of x the above correlator is periodic after the sum over spin structures $[\delta] = \left[\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2\end{smallmatrix}\right]$ is performed. Thus we may deform the x contour on the Riemann surface and shrink it to a point if (3.6) has no other poles as a function of x. However, from eq.(2.6) we can see that the superconformal ghost correlator in (3.6) in a spin structure $[\delta]$ contributes, among other things, to an excess factor of $\vartheta[\delta](-\frac{1}{2}\vec{x}+\frac{1}{2}\vec{y}+\sum_{a=1}^{2}\vec{z}_a - 2\vec{\Delta})$ in the denominator. Therefore, as a function of x the correlator in (3.6) in spin structure δ has unphysical poles at the zeroes of the function $\vartheta[\delta](-\frac{1}{2}\vec{x}+\frac{1}{2}\vec{y}+\sum_{a=1}^{2}\vec{z}_a - 2\vec{\Delta})$. Let r_ℓ denote the set of zeroes of the function $\prod_\delta \vartheta[\delta](-\frac{1}{2}\vec{x}+\frac{1}{2}\vec{y}+\sum_{a=1}^{2}\vec{z}_a - 2\vec{\Delta})$. Then after deforming the x contour through these points and shrinking it to a point the expression in (3.6) may be written as,

$$\Lambda_D = -\sum_\delta \sum_\ell \epsilon[\delta] \int d^2y \oint_{r_\ell} \frac{dx}{2\pi i} \int [\prod_{i=1}^{3} dm_i d\bar{m}_i][DXD\psi D\varphi DbD\bar{b}DcD\bar{c}D\beta D\gamma]_\delta$$
$$e^{-S} \prod_{j=1}^{3} \{\langle \eta_j \mid b\rangle \langle \bar{\eta}_j \mid \bar{b}\rangle\} \xi(z_0) \{\prod_{a=1}^{2} Y(z_a)\} J_\beta(x) V^\alpha_{\frac{1}{2}}(y) \tag{3.7}$$

Let $\tilde{z}_1$ be an arbitrary point. If we replace $Y(z_1)$ by $Y(\tilde{z}_1)$ in eq.(3.7) then the correlator $\langle J_\beta(x) V^\alpha_{\frac{1}{2}}(y) \xi(z_0) Y(\tilde{z}_1) Y(z_2)\rangle$ has unphysical poles at the zeroes of $\prod_\delta \vartheta[\delta](-\frac{1}{2}\vec{x}+\frac{1}{2}\vec{y}+\vec{\tilde{z}}_1+\vec{z}_2-2\vec{\Delta})$. Choosing $\tilde{z}_1$ properly we may ensure that none of the zeroes of this function coincide with any of the points r_ℓ. As a consequence the contribution to (3.7) with $Y(z_1)$ replaced by $Y(\tilde{z}_1)$ vanishes if the x contour is taken around the original points r_ℓ, i.e. the zeroes of $\prod_\delta \vartheta[\delta](-\frac{1}{2}\vec{x} + \frac{1}{2}\vec{y} + \sum_{a=1}^{2} \vec{z}_a - 2\vec{\Delta})$. Thus we may express (3.7) as,

$$\Lambda_D = -\sum_\delta \sum_\ell \epsilon[\delta] \int d^2y \oint_{r_\ell} \frac{dx}{2\pi i} \int [\prod_{i=1}^{3} dm_i d\bar{m}_i][DXD\psi D\varphi DbD\bar{b}DcD\bar{c}D\beta D\gamma]_\delta$$
$$e^{-S} \prod_{j=1}^{3} \{\langle \eta_j \mid b\rangle \langle \bar{\eta}_j \mid \bar{b}\rangle\} \xi(z_0)(Y(z_1) - Y(\tilde{z}_1)) Y(z_2) J_\beta(x) V^\alpha_{\frac{1}{2}}(y) \tag{3.8}$$

We can now use eq.(2.3) to express $Y(z_1) - Y(\tilde{z}_1)$ as a contour integral of the $BRST$ current, namely,

$$Y(z_1) - Y(\tilde{z}_1) = \oint_{z_1, \tilde{z}_1} \frac{dw}{2\pi i} J_{BRST}(w)(\xi(z_1) - \xi(\tilde{z}_1)) \tag{3.9}$$

But since the correlator involving $J_{BRST}(w)$ is periodic on the Riemann surface, the w contour may be deformed on the Riemann surface and shrunk to zero, picking up the residues at various poles. The pole at z_0 does not contribute, since the resulting correlator involving $[Q_{BRST}, \xi(z_0)](\xi(z_1) - \xi(\tilde{z}_1))Y(z_2)$ does not involve any ξ zero mode, and hence would vanish. Had we not subtracted the term involving $Y(\tilde{z}_1)$ as in (3.8), we would have had to worry about the contribution of the residue at the pole at z_0. Furthermore, $BRST$ invariance of $J_\beta(x)$, $V^\alpha_{\frac{1}{2}}(y)$ and $Y(z_2)$ ensures that there are no poles at $w = x, y$ or z_2. Thus the only poles are at the arguments of b in $\langle \eta_j \mid b \rangle$. Using [19, 7, 8]

$$[Q_{BRST}, b(z)] = T(z) \tag{3.10}$$

and the fact that,

$$\frac{\partial}{\partial m_i} S = \int d^2 w \eta_i(w) T(w) \tag{3.11}$$

for m_i a set of complex moduli and η_i a basis of Beltrami differentials satisfying eq.(2.5), we may express eq.(3.8) as,

$$\Lambda_D = \int [\prod_{i=1}^{3} dm_i d\bar{m}_i] \sum_{i=1}^{3} \frac{\partial}{\partial m_i} (\sqrt{det\ G} B_i) \tag{3.12}$$

where,

$$\sqrt{det\ G} B_i = -\sum_\delta \sum_\ell \epsilon[\delta] \int d^2 y \oint_{r_\ell} \frac{dx}{2\pi i} \Big[\int [DXD\psi D\varphi DbD\bar{b}DcD\bar{c}D\beta D\gamma]_\delta e^{-S}$$
$$\{ \prod_{j \neq i, j=1}^{3} \langle \eta_j \mid b \rangle \prod_{j=1}^{3} \langle \bar{\eta}_j \mid \bar{b} \rangle \} \xi(\tilde{z}_1) \xi(z_1) Y(z_2) J_\beta(x) V^\alpha_{\frac{1}{2}}(y) \Big] \tag{3.13}$$

where we have set $z_0 = \tilde{z}_1$ since the expression is independent of z_0. $G_{i\bar{j}}$ is a suitable metric in the moduli space.

This shows that the dilaton one point function may be expressed as a total derivative in the moduli space. If the field B_i defined in eq.(3.13) is a globally defined vector field in the moduli space, (this requirement constrains the choice of points z_1 and z_2, as we shall see later) the contribution to the dilaton tadpole comes solely from the boundary of the moduli space. In particular, we shall parametrize the space of moduli by the period matrix Ω_{ij} $(1 \leq i \leq j \leq 2)$ and consider the boundary $\Omega_{12} = 0$. In this limit the genus two surface degenerates into two genus one surfaces as depicted in Fig.1(a). Near the boundary, the space of moduli may be characterized by three complex parameters $\Omega_{11} \equiv \tau_1$, $\Omega_{22} \equiv \tau_2$ and $\Omega_{12} \equiv t$. τ_1 and τ_2 may be interpreted as the Teichmuller parameters of the tori T_1 and T_2 respectively. As we shall see below, at the boundary $\Omega_{12} = 0$ the contribution from each torus turns out to be proportional to the D-tadpole calculated at the one loop order [2]. Thus the net contribution to the two loop dilaton tadpole is given by the square of the D-tadpole, as expected from the low energy effective lagrangian considerations.

To see how this works in detail, we have to analyse the behaviour of $\sqrt{det\ G}B_t$, defined in eq.(3.13), near the boundary of $\Omega_{12} \equiv t \rightarrow 0$. Since Ω_{12} is a complex parameter, the boundary $\Omega_{12} = 0$ is a manifold of real codimension two, and hence we would not get any contribution from this boundary unless the integrand becomes singular at the boundary. We can determine what sort of singularity is needed for a non-zero contribution by noting that the part of the measure involving t near the boundary is just $dtd\bar{t}$. Let us define the real variables (r, θ) through,

$$t = re^{i\theta} \tag{3.14}$$

Then Λ_D is given by an integral of the form,*

$$\Lambda_D = \lim_{a \rightarrow 0} \int\limits_{|t| \geq a} dtd\bar{t} \frac{\partial}{\partial t}(F(t, \bar{t})) \tag{3.15}$$

where F is obtained from B_t by integrating over τ_1 and τ_2. Eq.(3.15) may be

* In actual practice, when a genus two surface degenerates into two tori, there is a symmetry $t \rightarrow -t$, and so we should only integrate over the upper half t plane. Alternatively, we can integrate over the whole t plane and divide the final result by two.

rewritten by doing an integration by parts as,

$$\Lambda_D = \lim_{a\to 0} \int_0^{2\pi} d\theta (\bar{t}F(t,\bar{t}))|_{r=a} \tag{3.16}$$

So in order to get a non-zero contribution in the $a \to 0$ limit, $F(t,\bar{t})$ should diverge as $\frac{1}{t}$ as $t \to 0$. As we shall see shortly, $F(t,\bar{t})$ calculated from B_t defined in eq.(3.13) does indeed diverge in this way.

Let η_1, η_2 and η_t be the Beltrami differentials associated with the moduli τ_1, τ_2 and t respectively. From eqs.(3.12) and (3.13) we see that the final contribution to the dilaton tadpole in (3.12) after doing the t integration is given by eq.(3.16) with,

$$F(t,\bar{t}) = \lim_{z_2'\to z_2} (-\frac{1}{4}) \int d^2y d^2\tau_1 d^2\tau_2 (2\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_2'}) G(z_2, z_2') + H(t,\bar{t}) \tag{3.17}$$

where,

$$\begin{aligned} H(t,\bar{t}) = -\int d^2y d^2\tau_1 d^2\tau_2 \sum_\delta \sum_\ell \epsilon[\delta] \oint_{r_\ell} \frac{dx}{2\pi i} \int [DXD\psi D\varphi DbD\bar{b}DcD\bar{c}D\beta D\gamma]_\delta \\ e^{-S} \langle \eta_1 \mid b\rangle \langle \eta_2 \mid b\rangle \langle \bar{\eta}_1 \mid \bar{b}\rangle \langle \bar{\eta}_2 \mid \bar{b}\rangle \langle \bar{\eta}_t \mid \bar{b}\rangle \\ \xi(\tilde{z}_1)\xi(z_1) e^{\phi(z_2)} T_F^{matter}(z_2) J_\beta(x) V^\alpha_{\frac{1}{2}}(y) \end{aligned} \tag{3.18}$$

and,

$$\begin{aligned} G(z_2, z_2') = -\sum_\delta \sum_\ell \epsilon[\delta] \oint_{r_\ell} \frac{dx}{2\pi i} \int [DXD\psi D\varphi DbD\bar{b}DcD\bar{c}D\beta D\gamma]_\delta \\ e^{-S} \langle \eta_1 \mid b\rangle \langle \eta_2 \mid b\rangle \langle \bar{\eta}_1 \mid \bar{b}\rangle \langle \bar{\eta}_2 \mid \bar{b}\rangle \langle \bar{\eta}_t \mid \bar{b}\rangle \\ \xi(\tilde{z}_1)\xi(z_1)\eta(z_2) e^{2\phi(z_2')} b(z_2') J_\beta(x) V^\alpha_{\frac{1}{2}}(y) \end{aligned} \tag{3.19}$$

where we have used eq.(2.3). Notice that we have thrown away terms for which the total power of e^ϕ does not add up to 2, since these terms will vanish by ghost number conservation. By the same token, we may, at this stage, ignore the $\partial(c\xi e^{\frac{-\phi}{2}}\hat{S}^{-}S_{\beta})$ term from $V^{\alpha}_{\frac{1}{2}}$ in eq.(3.2). However, it was crucial to include such a term in $V^{\alpha}_{\frac{1}{2}}$ to start with in order to ensure that one could go from eq.(3.6) to eq.(3.12) before integrating over y. The necessity for doing this will become clear later.

We shall show later that the contribution to H at the boundary $t = 0$ vanishes, so let us concentrate on the contribution from $G(z_2, z_2')$. The first step in calculating F from eq.(3.17-3.19). is to find the number of unphysical poles and their positions r_ℓ. As mentioned earlier, these poles occur at the zeroes of the function,

$$f(x) = \prod_{\delta} \vartheta[\delta](-\frac{1}{2}\vec{x} + \frac{1}{2}\vec{y} + \sum_{a=1}^{2} \vec{z}_a - 2\vec{\Delta}) \tag{3.20}$$

in the x plane. To find their number, note that if we translate x along any of the homology cycles, $\vartheta[\delta](-\frac{1}{2}\vec{x} + \frac{1}{2}\vec{y} + \sum_{a=1}^{2} \vec{z}_a - 2\vec{\Delta})$ gets transformed up to a multiplicative factor into a theta function of the same argument but of a different spin structure δ. Since, however, $\prod_\delta$ in (3.20) involves product over all spin structures, $f(x)$ just picks up a multilicative factor under translation. More specifically,

$$\begin{aligned} f(x) &\rightarrow f(x) \quad \text{under translation along } \mathrm{A_k} \text{ cycle}, \\ &\rightarrow exp[2^{2g-1}\{-i\frac{\pi}{2}\Omega_{kk} + 2\pi i(\frac{1}{2}(\vec{y} - \vec{x}) + \vec{z}_1 - \vec{z}_2 + 2\vec{z}_2' - 2\vec{\Delta})_k\}]f(x) \\ &\qquad \text{under translations along } \mathrm{B_k} \text{ cycle} \end{aligned} \tag{3.21}$$

where $g(= 2)$ is the genus of the Riemann surface. From this it is straightforward to calculate the number of zeroes of $f(x)$ using Green's theorem [21]. This number turns out to be,

$$2^{2g-2}g = 8 \qquad \text{for g} = 2 \tag{3.22}$$

The next task is to locate the positions of these zeroes. Since we shall be interested in the behavior of (3.19) near the boundary of the moduli space ($\Omega_{12} =$

0), it is sufficient to study the positions of the unphysical poles in this limit. As mentioned before, in this limit, the genus two surface breaks up into two tori T_1 and T_2 connected by a thin tube, the axis of the tube meeting the torus T_1 at a point p_1 and T_2 at p_2. The radius of the tube and the twist angle may be identified with the variables r and θ introduced in eq.(3.14). We shall choose the points z_1 and z_2 to lie on the tori T_1 and T_2 respectively in this limit. The point $\tilde{z}_1$ will be taken to lie on the torus T_2. Also, for the time being, we shall take the point y on the torus T_1. Ultimately we are to integrate over y over the whole Riemann surface, and hence must also include regions of integration where y is on T_2. We shall see later that the contribution to $G(z_2, z_2')$ from the region where y lies on the torus T_2 may be brought into the same form as the contribution when y lies on the torus T_1 by suitable manipulation.

We are now ready to evaluate the dilaton tadpole through eqs.(3.16)-(3.19). We start with the correlator of the superconformal ghosts appearing in eq.(3.19). In a given spin structure δ the latter can be written down using eq.(2.6). The answer is given by,

$$\langle e^{-\frac{\phi}{2}(x)} e^{\frac{\phi}{2}(y)} \xi(\tilde{z}_1)\xi(z_1)\eta(z_2)e^{2\phi(z_2')}\rangle_\delta$$

$$= (Z_1)^{\frac{1}{2}} \frac{\vartheta[\delta](\frac{1}{2}\vec{y} - \frac{1}{2}\vec{x} + \vec{\tilde{z}}_1 + \vec{z}_1 - 2\vec{z}_2 + 2\vec{z}_2' - 2\vec{\Delta})}{\vartheta[\delta](\frac{1}{2}\vec{y} - \frac{1}{2}\vec{x} + \vec{z}_1 - \vec{z}_2 + 2\vec{z}_2' - 2\vec{\Delta})\vartheta[\delta](\frac{1}{2}\vec{y} - \frac{1}{2}\vec{x} + \vec{\tilde{z}}_1 - \vec{z}_2 + 2\vec{z}_2' - 2\vec{\Delta})}$$

$$\frac{E(z_1,\tilde{z}_1)E(x,z_2')(E(y,x))^{\frac{1}{4}}}{E(z_1,z_2)E(\tilde{z}_1,z_2)E(y,z_2')} \frac{\sigma(x)}{\sigma(y)(\sigma(z_2'))^4} \tag{3.23}$$

The unphysical poles come from the zeroes of the first theta-function in the denominator. If x lies on the torus T_1 in the $t \to 0$ limit, this theta function decomposes as,

$$\vartheta\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}(z_1 + \frac{1}{2}y - \frac{1}{2}x - p_1 \mid \tau_1)\vartheta\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}(2z_2' - z_2 - p_2 \mid \tau_2) \tag{3.24}$$

where $\left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} a_2 \\ b_2 \end{smallmatrix}\right]$ denote spin structures on the tori T_1 and T_2 respectively.

Note that the position of the zero of (3.24) as a function of x does not depend on the spin structure $\left[\begin{smallmatrix}a_2\\b_2\end{smallmatrix}\right]$ in this limit. Depending on the precise location of the position of the points z_1 and y relative to the point p_1 only one of the four functions $\vartheta\left[\begin{smallmatrix}a_1\\b_1\end{smallmatrix}\right](z_1+\frac{1}{2}y-\frac{1}{2}x-p_1\mid\tau_1)$ (for four different spin structures) has a zero in the fundamental region of integration in the x plane. Since the zero is repeated four times in $\prod_\delta \vartheta[\delta](\frac{1}{2}\vec{y}-\frac{1}{2}\vec{x}-\vec{z}_1-\vec{z}_2+2\vec{z}_2'-\vec{\Delta})$ due to four different spin structures $\left[\begin{smallmatrix}a_2\\b_2\end{smallmatrix}\right]$ on T_2, this accounts for four of the eight zeroes of the function predicted from Green's theorem.

It is easy to see, however, that the residue of (3.23) at each of these poles vanishes identically. To see this let us note that the ϑ-function in the numerator decomposes in the $t\to 0$ limit as,

$$\vartheta\begin{bmatrix}a_1\\b_1\end{bmatrix}(z_1+\frac{1}{2}y-\frac{1}{2}x-p_1\mid\tau_1)\vartheta\begin{bmatrix}a_2\\b_2\end{bmatrix}(2z_2'-2z_2+\tilde{z}_1-p_2\mid\tau_2) \tag{3.25}$$

This function has zeroes precisely at the same points as those of the function (3.24). Thus the residues of the poles of (3.23) at these points vanish identically.

Next we look for the remaining four zeroes of the function $\prod_\delta \vartheta[\delta](\frac{1}{2}\vec{y}-\frac{1}{2}\vec{x}-\vec{z}_1-\vec{z}_2+2\vec{z}_2'-\vec{\Delta})$. They happen to lie on the torus T_2. To see this let us take x to lie on the torus T_2 and note that in the $t\to 0$ limit the function $\vartheta[\delta](\frac{1}{2}\vec{y}-\frac{1}{2}\vec{x}-\vec{z}_1-\vec{z}_2+2\vec{z}_2'-\vec{\Delta})$ decomposes as,

$$\vartheta\begin{bmatrix}a_1\\b_1\end{bmatrix}(z_1+\frac{1}{2}y-\frac{3}{2}p_1\mid\tau_1)\vartheta\begin{bmatrix}a_2\\b_2\end{bmatrix}(-\frac{1}{2}x+2z_2'-z_2-\frac{1}{2}p_2\mid\tau_2) \tag{3.26}$$

Now the position of the zero is independent of the spin structure $\left[\begin{smallmatrix}a_1\\b_1\end{smallmatrix}\right]$. Again, depending on the positions of z_2 and z_2' relative to p_2, only one of the four functions $\vartheta\left[\begin{smallmatrix}a_2\\b_2\end{smallmatrix}\right](-\frac{1}{2}x+2z_2'-z_2-\frac{1}{2}p_2\mid\tau_2)$ will have a zero. This contributes four zeroes in $\prod_\delta \vartheta[\delta](\frac{1}{2}\vec{y}-\frac{1}{2}\vec{x}-\vec{z}_1-\vec{z}_2+2\vec{z}_2'-\vec{\Delta})$, due to four different spin structures $\left[\begin{smallmatrix}a_1\\b_1\end{smallmatrix}\right]$. If $\left[\begin{smallmatrix}\hat{a}_2\\\hat{b}_2\end{smallmatrix}\right]$ denotes the spin structure for which $\vartheta\left[\begin{smallmatrix}\hat{a}_2\\\hat{b}_2\end{smallmatrix}\right](-\frac{1}{2}x+2z_2'-z_2-\frac{3}{2}p_2\mid\tau_2)$

has a zero, then the position of x at this zero is given by,

$$-\frac{1}{2}x = -2z_2' + z_2 - \hat{a}_2\tau_2 - \hat{b}_2 + \frac{1}{2}p_2 \tag{3.27}$$

where τ_2 is the Teichmuller parameter for the torus T_2. The residue of this pole in the $t \to 0$ limit takes the form,

$$\eta(\tau_1)\eta(\tau_2)\frac{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](3z_2' - 2z_2 - p_2 \mid \tau_2)}{\vartheta\left[\begin{smallmatrix}a_1\\ b_1\end{smallmatrix}\right](\frac{1}{2}y - \frac{1}{2}p_1 \mid \tau_1)\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](4z_2' - 2z_2 - 2p_2 \mid \tau_2)}$$
$$\frac{\left(\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](4z_2' - 2z_2 - 2p_2 \mid \tau_2)\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](y - p_1 \mid \tau_1)\right)^{\frac{1}{4}}}{\left(\vartheta'\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](0 \mid \tau_1)\right)^{\frac{1}{4}}\left(\vartheta'\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](0 \mid \tau_2)\right)^{\frac{13}{4}}}\frac{\left(\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](z_2' - p_2 \mid \tau_2)\right)^3}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](z_2 - p_2 \mid \tau_2)}t^{\frac{3}{8}} \tag{3.28}$$

where we have also used the fact that $(Z_1)^{\frac{1}{2}}$ factorizes to $\eta(\tau_1)\eta(\tau_2)$ in the $t \to 0$ limit.

At this point we see that all dependence on the spurious points z_1 and $\tilde{z}_1$ has dropped out from eq.(3.28). As can be seen from eq.(3.19), the correlator involving the other fields X, ψ, φ, b, c, $\bar{b}$ and $\bar{c}$ do not have any dependence on z_1 and $\tilde{z}_1$, and hence the final expression for the dilaton tadpole is completely independent of these points. The story, however, is different for the point z_2 (and z_2', which needs to be set equal to z_2 at the end of the calculation). (3.28) certainly depends on $z_2(z_2')$. There is a dependence on z_2' from the correlator involving the b fields, since (3.19) involves an explicit factor of $b(z_2')$. There is also a dependence on $z_2(z_2')$ coming from the correlator involving $J_\beta(x)$, since the correlator has to be evaluated at a value of x given in eq.(3.27), which depends on z_2 and z_2'. As we shall see, even after combining these results together, and setting $z_2' = z_2$, the final expression has explicit dependence on z_2. Hence we must use some guideline to determine the position of the point z_2. These have already been discussed in ref. [13], and are as follows,

i) The positions of the points z_i must be independent of the moduli $\{m_i\}$. This was shown to be a necessary condition for the validity of eq.(2.4). The implementation of this condition of course requires assigning a definite metric on the surface for a fixed point in the Teichmuller space,i.e. a choice of gauge. (Otherwise we can always shift the position of the points z_i on the surface by a reparametrization of the surface, changing the metric in this process). The choice of metric is restricted by the second condition:

ii) In order that (3.12) contributes only from the boundary of the moduli space, B^j defined in eq.(3.13) must be a globally defined vector field in the moduli space. Since the expression involves correlators of the fields at the points z_i, a necessary condition is that the points z_i are either left fixed, or permuted among themselves under the global diffeomorphism which generates the modular transformation for the specific choice of metric. More specifically, if we consider an orbifold point in the moduli space (a point in the Teichmuller space left fixed by a subgroup of modular transformations), the metric associated with this point is invariant under a global diffeomorphism. This diffeomorphism must leave the positions of the points z_1 and z_2 fixed.

Since the above condition has to be satisfied for all possible modular transformations, it is not clear if there is a global obstruction to such a choice. What we shall show is that if such a choice is possible at all, the nodes p_1 and p_2 must approach the points z_1 and z_2 respectively in the degeneration limit. For definiteness, let us discuss the location of z_2, the location of z_1 may be found in the same way. We choose the reference metric in such a way that it reduces to the standard form $e^{\rho(u,v)} \mid du + \tau_2 dv \mid^2$ on the torus T_2 in the $t \to 0$ limit, where the torus is parametrized by $0 \le u \le 1$, $0 \le v \le 1$, and ρ is a conformal factor. In the $t \to 0$ limit, a subgroup of the full modular group on the genus two surface is the modular group of the torus T_2 with a marked point p_2. Taking the origin as the point p_2, these transformations are generated by $u \to u + v$, $v \to v$, and $u \to v$, $v \to -u$. The only point on the torus which is left invariant under these operations is the point p_2. Thus for consistency, we must set the point $z_2(z_2')$

at p_2. Since individual terms are not well-defined in the $z_2(z_2') \to p_2$ limit, we shall carry out the computation keeping $z_2(z_2')$ away from p_2. At the end of the calculation we shall first set $z_2 = z_2'$, and then take the limit $z_2 \to p_2$. As we shall see, the limit is finite and well defined. Also, at the intermediate stages of calculation we may set z_2 and z_2' to be equal to p_2 in terms which are finite in this limit.

We now can put together all other factors that appear in expression (3.19) in the $t \to 0$ limit. The integration over the ghost fields $\bar{b}$, $\bar{c}$ produces the anti-holomorphic ghost determinant, which decomposes into two ghost determinants on T_1 and T_2, together with a factor of $\bar{t}^{-2}$[26-29,23], i.e.

$$\begin{aligned}&\lim_{t\to 0}\int D\bar{b}D\bar{c}e^{-S(\bar{b},\bar{c})}\prod_{i=1}^{3}\langle\bar{\eta}_i \mid \bar{b}\rangle\\ &\sim \bar{t}^{-2}\overline{(\eta(\tau_1))}^2\overline{(\eta(\tau_2))}^2\end{aligned} \tag{3.29}$$

where $\eta(\tau)$ is the deDekind η function. The integration over the fields b and c gives a correlator of the form[23],

$$\begin{aligned}&\lim_{t\to 0}\int DbDce^{-S(b,c)}\langle\eta_1 \mid b\rangle\langle\eta_2 \mid b\rangle b(z_2')\\ &\sim (\eta(\tau_1))^2(\eta(\tau_2))^2\frac{(\vartheta'\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](0 \mid \tau_2))^2}{(\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](z_2'-p_2 \mid \tau_2))^2}t^{-1}\end{aligned} \tag{3.30}$$

In deriving (3.30) we have set $z_2' = p_2$ whenever the limit $z_2' \to p_2$ is not zero or singular.

Let us now turn our attention to the integral over the bosons, the Lorentz fermions, and the gauge fermions. The relevant correlator appearing in (3.19) is,

$$\int DXD\psi D\varphi e^{-S}\hat{S}^+(x)S_\beta(x)\hat{S}^-(y)S^\alpha(y)\bar{\partial}X^\mu(y)\partial X^\nu(y) \tag{3.31}$$

The correlator $\langle\bar{\partial}X^\mu\partial X^\nu\rangle$ on the torus T_1 gives a factor proportional to $\frac{1}{\text{Im }\tau_1}$. The leading t behaviour of the rest of the correlator may be obtained by using the factorization theorem,[29]

$$\langle A_1(z_1)A_2(z_2)\rangle \sim \sum_\phi \langle A_1(z_1)\phi(p_1)\rangle_{T_1}\langle \phi^\dagger(p_2)A_2(z_2)\rangle_{T_2} t^{h_\phi}\bar{t}^{\bar{h}_\phi} \tag{3.32}$$

where $A_1(z_1)$ and $A_2(z_2)$ are any two local operators on the tori T_1 and T_2 respectively. $\langle\ \rangle_{T_i}$ denotes correlator calculated on the torus T_i. The sum over ϕ runs over all the conformal fields in the theory with $(h_\phi, \bar{h}_\phi)$ being the conformal dimension of the field ϕ. In our problem, the relevant operator on the torus T_1 is $\hat{S}^-(y)S^\alpha(y)$ and the operator $A_2(z_2)$ on the torus T_2 is $\hat{S}^+(x)S_\beta(x)$, where $\alpha = (--)$ and $\beta = (++)$ in the four dimensional helicity basis. Then the field $\phi(p_1)$ which contributes to the most singular part in (3.32) in the $t \to 0$ limit is $\hat{S}^+(p_1)S_4^+(p_1)S_5^+(p_1)$ of dimension $(\frac{5}{8}, 0)$. The corresponding net contribution from the torus T_1 may be identified to $\langle P^-(y)P^+(p_1)\rangle_{T_1}$,(with $P^\pm$ as defined in sec.2) if we note that the τ_1 dependent contribution from (3.28) may be identified to $\langle S_g^+(y)S_g^-(p_1)\rangle_{T_1}$. Using the same manipulations as in ref.[2] which led to eq.(2.14), $\langle P^-(y)P^+(p_1)\rangle$ may be shown to be equal to $\langle J(y)\rangle_e$. Since $J(y)$ is an operator of conformal dimension (1,0), $(\text{Im }\tau)^2\langle J(y)\rangle_e \equiv f(\tau,\bar{\tau})$ transforms to $\bar{\tau}^{-1}f(\tau,\bar{\tau})$ under the modular transformation $\tau \to -\frac{1}{\tau}$, and remains invariant under $\tau \to \tau + 1$.* On the other hand, using manipulations similar to those in ref. [2], we may relate $\langle P^-(y)P^+(p_1)\rangle$ to $\langle\langle I\rangle\rangle_{PP}$, where $\langle\langle I\rangle\rangle_{PP}$ denotes the contribution to the partition function from the interacting fields φ^j with periodic boundary condition on the right-handed fermions along both cycles of the torus. We do not integrate over the free fields X^μ, ψ^μ or the ghost fields in calculating $\langle\langle I\rangle\rangle_{PP}$. This, in turn, may be used to show that $f(\tau,\bar{\tau})$ receives contribution only from the $L_0^{int} = \frac{3}{8}$ ($L_0^{tot} = 0$) states, and hence is independent of τ. In order to determine the $\bar{\tau}$ dependence of $f(\tau,\bar{\tau})$, we may compute the contribution to $\langle\langle I\rangle\rangle_{PP}$ from the $L_0^{int} = \frac{3}{8}, \bar{L}_0^{int} = 0$, and $L_0^{int} = \frac{3}{8}, \bar{L}_0^{int} = 1$ states explicitly. There are two states at the $\bar{L}_0^{int} = 0$ level, those created by the operators $\hat{S}^+$ and

---

* The fact that $\langle J(z)\rangle_e \sim (\overline{\eta(\tau)})^{-2}\langle\langle I\rangle\rangle_{PP}$ transforms as a modular form of weight -1 has been shown by Schellekens and Warner[37].

$\hat{S}^-$. They give equal and opposite contribution to $\langle\langle I\rangle\rangle_{PP}$. The contribution from the $\bar{L}_0^{int} = 1$ states, on the other hand, may be shown to be equal to $\sum_i n_i h_i$, where n_i is the number of massless fermions carrying helicity h_i in the four dimensional theory at tree level. Since a fermion and its CPT conjugate state always carry opposite helicities, this sum vanishes identically. Thus the leading contribution to $f(\tau,\bar{\tau})$ in the Im $\tau \to \infty$ limit comes from the $\bar{L}_0^{int} = 2$ states, which, in turn, implies that $f(\tau,\bar{\tau})$ is bounded by $e^{-2\pi \text{Im } \tau}$ as Im $\tau \to \infty$.

Putting all the facts together, we see that f is an anti-holomorphic function of τ in the upper half plane, vanishes in the limit Im $\tau \to \infty$, and is a modular form of weight -1. This, in turn, shows that f vanishes on the real axis [21]. Such a function can easily be seen to vanish identically using complex function theory.

The next to leading contribution comes from operators ϕ of dimension $(\frac{5}{8}, 1)$ of the form,

$$\hat{S}^+(p_1)S_4^+(p_1)S_5^+(p_1)U^{(a)}(p_1) \tag{3.33}$$

where $U^{(a)}(\bar{z})$ is the dimension (0,1) conformal field associated with the a'th $U(1)$ gauge group.† Thus the net contribution from the internal bosonic and the fermionic determinant in the spin structure $[\delta] = [\delta_1, \delta_2]$ is given by,

$$\begin{aligned} I_1 =& t^{\frac{5}{8}}\bar{t}\frac{1}{\text{Im } \tau_1}\langle\hat{S}^+(p_1)S_4^+(p_1)S_5^+(p_1)U^{(a)}(p_1)\hat{S}^-(y)S_4^-(y)S_5^-(y)\rangle_{\delta_1} \\ & \langle\hat{S}^-(p_2)S_4^-(p_2)S_5^-(p_2)U^{(a)}(p_2)\hat{S}^+(x)S_4^+(x)S_5^+(x)\rangle_{\delta_2} \end{aligned} \tag{3.34}$$

† One could try to construct other dimension $(\frac{5}{8}$,1) operators by combining the product $S_4^+S_5^+$ with some internal field $\hat{\phi}$ of dimension $(\frac{3}{8}$,1), which is not of the form $\hat{S}^+U^{(a)}$. It will turn out that when we take the $z_2 \to p_2$ limit at the end, only operators $\hat{\phi}^\dagger(z)(\hat{\phi}(z))$ which have leading singularity of the form $(z-w)^{-\frac{3}{4}}M(w)$ near $\hat{S}^+(w)$ $(\hat{S}^-(w))$ have non-vanishing contribution to $G(z_2, z_2')$ (See eq.(3.36), (3.39) and (3.40)). From dimensional counting we see that $M(w)$ above is an operator of dimension (0,1), hence it must be a linear combination of the gauge currents $U^{(a)}(\bar{z})$. It follows from this that any operator $\hat{\phi}$ of dimension $(\frac{3}{8}, 1)$ may be written as a linear combination of $\hat{S}^+U^{(a)}$, plus an operator which does not have $(z-w)^{-\frac{3}{4}}$ singularity near $\hat{S}^-(w)$. Thus the final contribution to the dilaton tadpole comes only from the operators displayed in eq.(3.33).

where $\langle\ \rangle_{\delta_i}$ is defined as,

$$\langle\prod_k V_k\rangle_{\delta_i} = \int [DXD\psi D\varphi]_{\delta_i} e^{-S(X,\psi,\varphi)} \prod_k V_k \tag{3.35}$$

on the i'th torus. Using the results of sec.2 we may write down the answer for this correlator in the form,

$$\begin{aligned} I_1 =& t^{\frac{5}{8}}\bar{t}\tilde{K}(\tau_1)\eta(\tau_1)^{-3}\overline{\eta(\tau_1)}^{-2}(\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0\mid\tau_1))^{\frac{1}{4}}\tilde{K}(\tau_2)\eta(\tau_2)\overline{\eta(\tau_2)}^{2}(\vartheta'\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0\mid\tau_2))^{\frac{1}{4}} \\ & \vartheta[\delta_1](\frac{1}{2}y-\frac{1}{2}p_1-A^{(a)}(\tau_1)\mid\tau_1)\vartheta[\delta_1](\frac{1}{2}y-\frac{1}{2}p_1-B^{(a)}(\tau_1)\mid\tau_1) \\ & \vartheta[\delta_1](\frac{1}{2}y-\frac{1}{2}p_1-C^{(a)}(\tau_1)\mid\tau_1)(\vartheta[\delta_1](\frac{1}{2}y-\frac{1}{2}p_1\mid\tau_1))^2(\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(y-p_1\mid\tau_1))^{-\frac{5}{4}} \\ & \vartheta[\delta_2](\frac{1}{2}p_2-\frac{1}{2}x-A^{(a)}(\tau_2)\mid\tau_2)\vartheta[\delta_2](\frac{1}{2}p_2-\frac{1}{2}x-B^{(a)}(\tau_2)\mid\tau_2) \\ & \vartheta[\delta_2](\frac{1}{2}p_2-\frac{1}{2}x-C^{(a)}(\tau_2)\mid\tau_2)(\vartheta[\delta_2](\frac{1}{2}p_2-\frac{1}{2}x\mid\tau_2))^2(\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(p_2-x\mid\tau_2))^{-\frac{5}{4}} \end{aligned} \tag{3.36}$$

We are now ready to put together all the factors given in (3.28-30) and (3.36). Let us first examine the contributions from the torus T_1. For a given spin structure $[\delta_1]$ this contribution is proportional to,

$$\begin{aligned} &(\text{Im}\ \tau_1)^{-1}\tilde{K}(\tau_1)\epsilon[\delta_1]\vartheta[\delta_1](\frac{1}{2}y-\frac{1}{2}p_1-A^{(a)}(\tau_1)\mid\tau_1)\vartheta[\delta_1](\frac{1}{2}y-\frac{1}{2}p_1-B^{(a)}(\tau_1)\mid\tau_1) \\ &\vartheta[\delta_1](\frac{1}{2}y-\frac{1}{2}p_1-C^{(a)}(\tau_1)\mid\tau_1)\vartheta[\delta_1](\frac{1}{2}y-\frac{1}{2}p_1\mid\tau_1)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(y-p_1\mid\tau_1) \end{aligned} \tag{3.37}$$

The sum over spin structures may be performed with the help of a Riemann theta identity to yield,

$$-2(\text{Im}\ \tau_1)^{-1}\tilde{K}(\tau_1)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(A^{(a)}(\tau_1)\mid\tau_1)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(B^{(a)}(\tau_1)\mid\tau_1)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(C^{(a)}(\tau_1)\mid\tau_1) \tag{3.38}$$

using eq.(2.18). The contribution from the torus T_2, on the other hand, comes from one specific spin structure $\left[\begin{smallmatrix}\hat{a}_2\\ \hat{b}_2\end{smallmatrix}\right]$ as the residue of the pole at $-\frac{1}{2}x = -2z_2' + z_2 - \hat{a}_2\tau_2 - \hat{b}_2 + \frac{1}{2}p_2$. The final contribution, however, is independent of $\hat{a}_2$ and $\hat{b}_2$, and is,

$$\tilde{K}(\tau_2)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(A^{(a)}(\tau_2)\mid\tau_2)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(B^{(a)}(\tau_2)\mid\tau_2)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(C^{(a)}(\tau_2)\mid\tau_2)$$
$$\frac{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](z_2'-p_2\mid\tau_2)\left(\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](2z_2'-z_2-p_2\mid\tau_2)\right)^2\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](3z_2'-2z_2-p_2\mid\tau_2)}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](z_2-p_2\mid\tau_2)\left(\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](4z_2'-2z_2-2p_2\mid\tau_2)\right)^2\vartheta'\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](0\mid\tau_2)} \tag{3.39}$$

where we have set $z_2 = z_2' = p_2$ wherever permitted.

This gives,

$$\lim_{z_2\to p_2}\lim_{z_2'\to z_2}\left(2\frac{\partial}{\partial z_2}+\frac{\partial}{\partial z_2'}\right)G(z_2,z_2')$$
$$\sim(\mathrm{Im}\,\tau_1)^{-1}\tilde{K}(\tau_1)\tilde{K}(\tau_2)\bar{t}^{-1}$$
$$\sum_a\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(A^{(a)}(\tau_1)\mid\tau_1)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(B^{(a)}(\tau_1)\mid\tau_1)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(C^{(a)}(\tau_1)\mid\tau_1) \tag{3.40}$$
$$\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(A^{(a)}(\tau_2)\mid\tau_2)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(B^{(a)}(\tau_2)\mid\tau_2)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(C^{(a)}(\tau_2)\mid\tau_2)$$

The contribution from $H(t,\bar{t})$ defined in eq.(3.18) may be analyzed in the same way. It turns out that after summing over spin structures using Riemann theta identity, and the result $\langle\langle I\rangle\rangle_{PP}=0$, and taking the limit $z_1\to p_1$, $z_2\to p_2$, $H(t,\bar{t})$ does not have the necessary singularity in the $t\to 0$ limit so as to contribute to (3.16). The origin of this may be traced to the fact that the relevant part of $V_{\frac{1}{2}}(y)$ in the calculation has a factor of $\bar{\partial}X^\mu(y)$, which must be contracted with a $\partial X(z_2)$ coming from $T_F^{matter}(z_2)$. Since y and z_2 lie on different tori, the intermediate operator ϕ in eq.(3.32) must carry a factor of ∂X or $\bar{\partial}X$, which is accompanied by a factor of t or $\bar{t}$. Thus (3.17) gets contribution only from the term involving $G(z_2, z_2')$. The y integral in (3.17) produces a factor of Im τ_1. Substitution of (3.40) into (3.17) and (3.16) gives the contribution to the dilaton tadpole Λ_D from the region of y integration on the torus T_1:

$$\sum_a c^{(a)} c^{(a)} \tag{3.41}$$

up to an overall numerical factor. $c^{(a)}$ has been defined in eq.(2.19).

Let us now turn to the contribution to Λ_D from the region where y lies on the torus T_2. We start from the expression of $G(z_2, z_2')$ defined in eq.(3.19), and take z_1 on the torus T_1 and z_2 on the torus T_2 as before. The first result to notice is that for any value of t, $G(z_2, z_2')$ is independent of the position of the point $\tilde{z}_1$[13]. This may be proved by first noting that the positions of the poles r_ℓ are independent of $\tilde{z}_1$, and the residue at any of these points r_ℓ, considered as a function of $\tilde{z}_1$ is periodic, and has at most $g-1$ poles. As a result it must be independent of $\tilde{z}_1$. Thus we can shift the positions of $\tilde{z}_1$ without affecting the value of $G(z_2, z_2')$. We shall use this freedom to choose $\tilde{z}_1$ in such a way that if in the $t \to 0$ limit y goes over to the torus T_2, then the point $\tilde{z}_1$ goes over to the torus T_1.

Next we write

$$\begin{aligned}
&\lim_{z_2' \to z_2} \left(2\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_2'}\right) G(z_2, z_2') \\
&= -\sum_\delta \sum_\ell \epsilon[\delta] \oint_{r_\ell} \frac{dx}{2\pi i} \int [DXD\psi D\varphi DbD\bar{b}DcD\bar{c}D\beta D\gamma]_\delta \\
&\quad e^{-S} \langle \eta_1 \mid b \rangle \langle \eta_2 \mid b \rangle \langle \bar{\eta}_1 \mid \bar{b} \rangle \langle \bar{\eta}_2 \mid \bar{b} \rangle \langle \bar{\eta}_t \mid \bar{b} \rangle \\
&\quad \xi(\tilde{z}_1) \xi(z_1) \left(\oint \frac{dw}{2\pi i} J_{BRST}(w) \xi(z_2) \right) J_\beta(x) V_{\frac{1}{2}}^\alpha(y)
\end{aligned} \tag{3.42}$$

where we have used eq.(2.3) to express $Y(z_2)$ as a $BRST$ contour integral around $\xi(z_2)$. We may now deform the $BRST$ contour integral on the Riemann surface, picking up residues at various poles. As before, there is no contribution from the points x or y. the contribution from the arguments in b in $\langle\eta_i \mid b\rangle$ $(i = 1, 2)$ may be expressed as,

$$
\begin{aligned}
&-\sum_{i=1}^{2} \frac{\partial}{\partial m_i} \sum_{\delta} \sum_{\ell} \epsilon[\delta] \oint_{r_\ell} \frac{dx}{2\pi i} \int [DXD\psi D\varphi DbD\bar{b}DcD\bar{c}D\beta D\gamma]_\delta \\
&e^{-S} \Big(\prod_{j\neq i, j=1}^{2} \langle\eta_j \mid b\rangle \Big) \langle\bar{\eta}_1 \mid \bar{b}\rangle \langle\bar{\eta}_2 \mid \bar{b}\rangle \langle\bar{\eta}_t \mid \bar{b}\rangle \\
&\xi(\tilde{z}_1)\xi(z_1)\xi(z_2) J_\beta(x) V^{\alpha}_{\frac{1}{2}}(y)
\end{aligned}
\tag{3.43}
$$

This contribution vanishes identically, since the number of ξ's minus the number of η's in a correlator must be equal to unity in order to get a non-zero answer. There are two more terms, coming from the residues of the poles at z_1 and $\tilde{z}_1$ respectively. The contribution from the residue of the pole at z_1 is given by,

$$
\begin{aligned}
&-\sum_{\delta} \sum_{\ell} \epsilon[\delta] \oint_{r_\ell} \frac{dx}{2\pi i} \int [DXD\psi D\varphi DbD\bar{b}DcD\bar{c}D\beta D\gamma]_\delta \\
&e^{-S} \langle\eta_1 \mid b\rangle \langle\eta_2 \mid b\rangle \langle\bar{\eta}_1 \mid \bar{b}\rangle \langle\bar{\eta}_2 \mid \bar{b}\rangle \langle\bar{\eta}_t \mid \bar{b}\rangle \\
&\xi(\tilde{z}_1)\xi(z_2) Y(z_1) J_\beta(x) V^{\alpha}_{\frac{1}{2}}(y)
\end{aligned}
\tag{3.44}
$$

The residue at the pole at $\tilde{z}_1$ is given by eq.(3.44) with z_1 and $\tilde{z}_1$ interchanged. But by explicit calculation we can see that this term does not have any pole in the x plane at the points r_ℓ, and hence vanishes after the x contour integral. Thus we are left with the contribution (3.44). But this expression now has the same structure as the original contribution (3.17-19), when y and z_1 lie on the torus T_1 and $\tilde{z}_1$ and z_2 lie on the torus T_2; except that the roles of the tori T_1 and T_2, as well as the points z_1 and z_2 have been interchanged here. In this form the contribution to the dilaton tadpole from the region of integration where y lies on the torus T_2 is also given by eq.(3.41). Note that since during the manipulations described above, the deformation of the $BRST$ contour is carried out for a fixed value of y, it is important that $V_{\frac{1}{2}}(y)$ is $BRST$ invariant before integrating over y.

In carrying out the above analysis we have investigated the integration region where y lies on either the torus T_1 or the torus T_2. The formulae we have used in our analysis may need to be modified when y is within a distance of order $|t|^{\frac{1}{2}}$ from the nodes. One might then ask if the contribution from this region of integration should be investigated separately. Since (3.40) does not have any singularity in the $y \to p_1$ limit, one might expect the contribution from the region $|y - p_1| \sim |t|^{\frac{1}{2}}$ to be suppressed by powers of t due to the smallness of the integration volume. A more careful study of F_t in the region $|y - p_1| \sim |t|^{\frac{1}{2}}$ using the degeneration formulae of ref.[23] verifies this result.

To sum up, the final answer for the dilaton tadpole at two loops is given by eq. (3.41), where $c^{(a)}$ are the coefficients of the Fayet-Iliopoulos D-terms generated at one loop.

4. Conclusion

In this paper we have calculated the two loop dilaton tadpole in compactified heterotic string theories with unbroken tree level space-time supersymmetry. In some of these theories, one loop radiative corrections may generate a Fayet-Iliopoulos D term. Precisely in these theories we find a non-vanishing contribution to the dilaton tadpole at the two loop order. Furthermore, the contribution is shown to be proportional to the square of the coefficient of the D term generated at the one loop order, as expected from the analysis in the low energy effective field theory [1].

Besides providing an explicit verification of the effective lagrangian arguments, our analysis also throws some light on the structure of the general fermionic string perturbation theory. One of the major obstacles in developing the fermionic string perturbation theory is the integration over the supermoduli. In a recent paper Verlinde and Verlinde [13] have given a general prescription for carrying out integration over the supermoduli. Using their prescription we have shown that the two loop dilaton tadpole is a total derivative in the moduli space and hence receives contribution only from the boundary terms. The final contribution to the dilaton tadpole induced by the Fayet-Iliopoulos D term comes from the particular boundary where the genus two surface degenerates into two tori. This analysis shows that total derivative terms may not always be ignored. Since gauge transformations in string theories generate total derivative terms in the moduli space [7,29], this may provide a mechanism for breakdown of some gauge symmetries by higher loop corrections in string theory.

Finally, we should point out that in our analysis we have ignored a global issue; which is that there may be an obstruction to choosing the super-Beltrami differentials in a way such that they are independent of the moduli, and at the same time are either invariant or get transformed into each other under modular transformations. In our analysis we proceed by assuming that such a choice is possible, and make use of the restrictions imposed by these criteria near the boundary of the moduli space. A global obstruction to such a choice of basis for the super-Beltrami differentials may generate new contributions to the dilaton tadpole, and provide a new source of breakdown of space-time supersymmetry.

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Appendix

Degeneration Formulae

In this appendix we list the formulae describing the behaviour of various functions in the limit of degeneration of the genus two Riemann surface to two genus one surfaces (Fig. 1a). We parametrize the moduli space of genus two by the period matrix $\Omega_{ij}, 1 \leq i < j \leq 2$. The degeneration in question is then described by $t = \Omega_{12} \to 0$. Degeneration formulae for arbitrary genera appear in several references, see for instance [21,23,30].

Let $x \in T_1$ and $y \in T_2$ be points on the first and second tori respectively. Then the degeneration formulae that we need are:

$$E(x,y) \to t^{-\frac{1}{2}} \left(\frac{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](x - p_1|\tau_1)}{\vartheta'\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](0|\tau_1)} \right) \left(\frac{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](p_2 - y|\tau_2)}{\vartheta'\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](0|\tau_2)} \right) \tag{A.1}$$

$$E(x,x') \to \left(\frac{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](x - x'|\tau_1)}{\vartheta'\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](0|\tau_1)} \right) \tag{A.2}$$

$$[\delta] = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \to \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \tag{A.3}$$

$$\begin{aligned} &\vartheta[\delta]\Big(\sum_{i=1}^{m} \vec{x}_i - \sum_{j=1}^{n} \vec{y}_j - (m-n)\vec{\Delta}\Big) \\ &\to \vartheta\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}\Big(\sum_{i=1}^{m} x_i - mp_1 - \frac{1}{2}(m-n)(1+\tau)\Big)\vartheta\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\Big(np_2 - \sum_{j=1}^{n} y_j - \frac{1}{2}(m-n)(1+\tau)\Big) \end{aligned} \tag{A.4}$$

$$\sigma(x) \to \frac{\vartheta'\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](0|\tau_1)}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](x-p_1)} \tag{A.5}$$

$$Z_1^{\frac{1}{2}} \to \eta(\tau_1)\ \eta(\tau_2) \tag{A.6}$$

$$\omega^1(z|t) \to \begin{cases} 1+O(t) & z \ \epsilon \ T_1 \\ O(t) & z \ \epsilon \ T_2 \end{cases} \tag{A.7}$$

$$\omega^2(z|t) \to \begin{cases} O(t) & z \ \epsilon \ T_1 \\ 1+O(t) & z \ \epsilon \ T_2 \end{cases} \tag{A.8}$$

where p_1, p_2 are nodes on T_1, T_2 ω^i are the abelian differentials and $\eta(\tau)$ is the deDekind η function. We should also mention that there is an ambiguity in the degeneration of the ϑ-function given in (A.4); the arguments of the ϑ-functions on the right hand side of this equation may be shifted by integral multiples of 1 and τ_1 (or 1 and τ_2). This corresponds to an ambiguity in defining the arguments $\vec{x}_i$ and $\vec{y}$ of the original ϑ-function. This is the ambiguity in the choice of path from the base point P_0 to the point x in defining $\vec{x} = \int_{P_0}^{P} \vec{\omega}$. In physical correlation functions, however this ambiguity may be resolved by demanding the correct periodicity properties.

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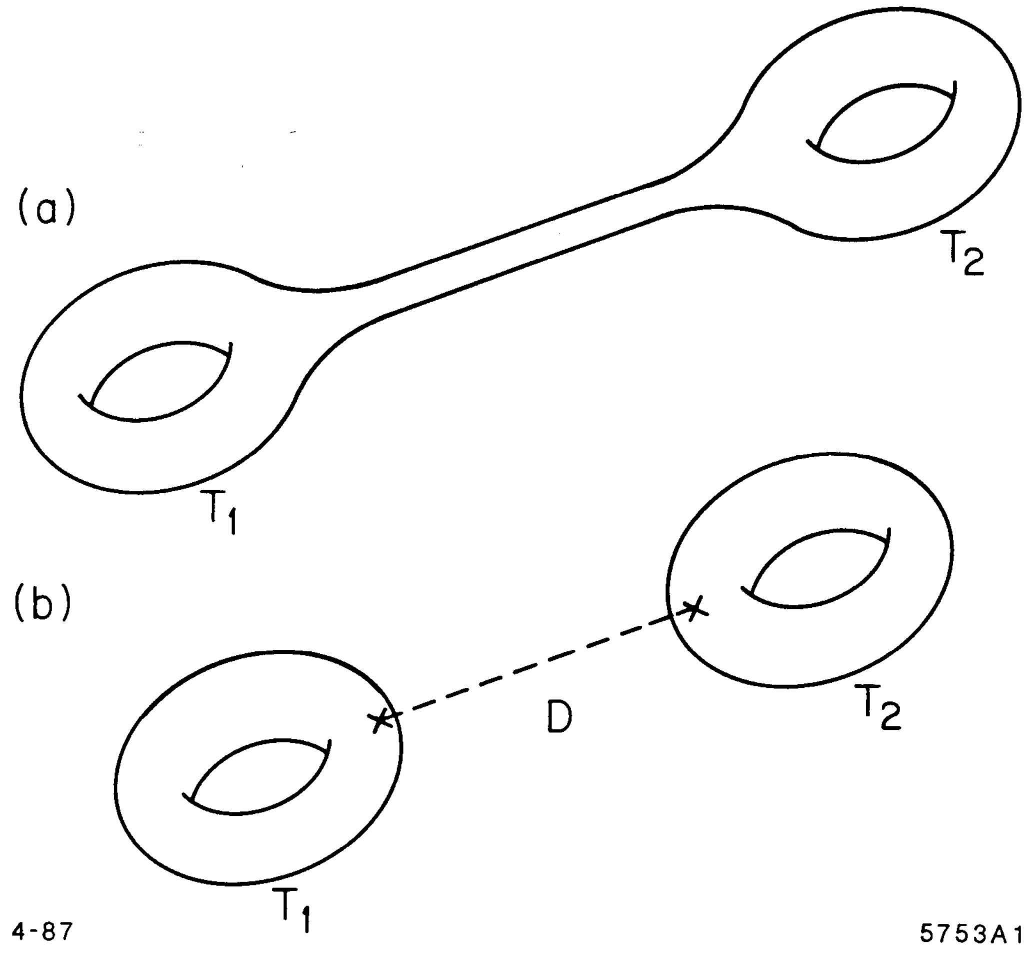

Fig. 1