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Dyon Spectrum in CHL Models

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Abstract

We propose a formula for the degeneracy of quarter BPS dyons in a class of CHL models. The formula uses a modular form of a subgroup of the genus two modular group $Sp(2, \mathbb{Z})$. Our proposal is S-duality invariant and reproduces correctly the entropy of a dyonic black hole to first non-leading order for large values of the charges.

Contents

1 Introduction and Summary

Some years ago, Dijkgraaf, Verlinde and Verlinde[1] proposed a formula for the exact degeneracy of dyons in toroidally compactified heterotic string theory.[1] By now there has been substantial progress towards verifying this formula[2, 3, 4, 5] and also extending it to toroidally compactified type II string theory[6]. These formulæ are invariant under the S-duality transformations of the theory, and also reproduce the entropy of a dyonic black hole in the limit of large charges[2].

In this paper we generalize this proposal to a class of CHL models[7, 8, 9, 10, 11, 12]. The construction of these models begins with heterotic string theory compactified on a six torus $T^4 \times S^1 \times \widetilde{S}^1$, and modding out the theory by a $\mathbb{Z}_N$ transformation that involves

[1] Throughout this paper we shall use the word degeneracy to denote an appropriate helicity trace that vanishes for long supermultiplets but is non-zero for the intermediate supermultiplets which describe quarter BPS dyons. In simple terms this corresponds to counting supermultiplets with weight ± 1, the weight being $+1$ (-1) if the lowest helicity state in the supermultiplet is bosonic (fermionic).

$1/N$ unit of translation along $\widetilde{S}^1$ and a non-trivial action on the internal conformal field theory (CFT) describing heterotic string compactification on T^4. Using string-string duality[13, 14, 15, 16, 17] one can relate these models to $\mathbb{Z}_N$ orbifolds of type IIA string theory on $K3 \times S^1 \times \widetilde{S}^1$, with the $\mathbb{Z}_N$ transformation acting as $1/N$ unit of shift along $\widetilde{S}^1$ together with an action on the internal CFT describing type IIA string compactification on $K3$. Under this map the S-duality of the heterotic string theory gets related to T-duality of the type IIA string theory, which in turn can be determined by analyzing the symmetries of the corresponding conformal field theory. Using this procedure one finds that the orbifolding procedure breaks the S-duality group of the heterotic string theory from $SL(2,Z)$ to a subgroup $\Gamma_1(N)$ of $SL(2,Z)$[18]. This acts on the electric charge vector Q_e and the magnetic charge vector Q_m as

$$Q_e \to aQ_e + bQ_m, \qquad Q_m \to cQ_e + dQ_m\,,$$
$$a,b,c,d \in \mathbb{Z}, \quad ad-bc=1, \quad c=0 \text{ mod } N, \quad a,d=1 \text{ mod } N\,. \tag{1.1}$$

The spectra of these models contain 1/4 BPS dyons. We consider a class of dyon states in this theory, and propose a formula for the degeneracy of these dyons. For technical reasons we have to restrict our analysis to the case of prime values of N only, – among known models the allowed prime values of N are 1, 2, 3, 5 and 7. Since the analysis is somewhat technical, we shall summarize the proposal here.

- We first define a set of coefficients $f_n^{(k)}$ $(n \geq 1)$ through the relations:

$$f^{(k)}(\tau) \equiv \eta(\tau)^{k+2}\,\eta(N\tau)^{k+2}\,, \tag{1.2}$$

$$\sum_{n\geq 1} f_n^{(k)} e^{2\pi i\tau(n-\frac{1}{4})} = \eta(\tau)^{-6}\, f^{(k)}(\tau)\,, \tag{1.3}$$

  where $\eta(\tau)$ is the Dedekind function and

$$k = \frac{24}{N+1} - 2\,. \tag{1.4}$$

  $f^{(k)}(\tau)$ is the unique cusp form of weight $(k+2)$ of the S-duality group $\Gamma_1(N)$ described in (1.1).

- Next we define the coefficients $C(m)$ through

$$C(m) = (-1)^m \sum_{\substack{s,n\in\,\mathbb{Z}\\ n\geq 1}} f_n^{(k)} \delta_{4n+s^2-1,m}\,. \tag{1.5}$$

As will be explained in appendix A, the $C(m)$'s are related to the Fourier coefficients of a (weak) Jacobi form of the group $\Gamma_1(N)$.

- We now define

$$\Phi_k(\rho,\sigma,v) = \sum_{\substack{n,m,r\in \mathbb{Z} \\ n,m\geq 1,\, r^2<4mn}} a(n,m,r)\, e^{2\pi i(n\rho+m\sigma+rv)}\,, \tag{1.6}$$

where

$$a(n,m,r) = \sum_{\substack{\alpha\in \mathbb{Z};\alpha>0 \\ \alpha|(n,m,r),\, \text{g.c.d.}(\alpha,N)=1}} \chi(\alpha)\, \alpha^{k-1}\, C\left(\frac{4mn-r^2}{\alpha^2}\right)\,, \tag{1.7}$$

and $\chi(\alpha)$ takes values 1 or -1 depending on the values of N and of α mod N. As explained in appendix A, $\chi(a)$ is a Dirichlet character mod 2 of the modular transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, i.e. it describes a homomorphism map from $\Gamma_0(N)$ to $\mathbb{Z}_2$. If k defined in (1.4) is even (i.e. if $N = 1, 2, 3$ or 5) then $\chi(\alpha) = 1$ identically. For $N = 7$ we have

$$\begin{aligned} \chi(\alpha) &= 1 \quad \text{for} \quad \alpha = 1,2,4 \bmod 7\,, \\ &= -1 \quad \text{for} \quad \alpha = 3,5,6 \bmod 7\,. \end{aligned} \tag{1.8}$$

It has been shown in appendix B that Φ_k, defined in (1.6), transforms as a modular form of weight k under a subgroup G of $Sp(2, \mathbb{Z})$ defined in (2.11).

- In the next step we define:

$$\widetilde\Phi_k(\widetilde\rho,\widetilde\sigma,\widetilde v) = \widetilde\sigma^{-k}\Phi_k\left(\widetilde\rho - \frac{\widetilde v^2}{\widetilde\sigma}, -\frac{1}{\widetilde\sigma}, \frac{\widetilde v}{\widetilde\sigma}\right)\,. \tag{1.9}$$

$\widetilde\Phi_k$ transforms as a modular form of weight k under a *different subgroup* $\widetilde G$ of $Sp(2, \mathbb{Z})$ defined in (2.20), (2.21).

- Now consider a dyon carrying electric charge vector Q_e and magnetic charge vector Q_m. The precise conventions for defining Q_e and Q_m and their inner products have been described in eqs.(2.3), (2.4). We consider those dyons whose electric charge arises from a twisted sector of the theory, carrying fractional winding number along $\widetilde S^1$. According to our proposal the degeneracy $d(Q)$ of such a dyon is given by:

$$d(Q) = g\left(\frac{1}{2}Q_m^2, \frac{1}{2}Q_e^2, Q_e\cdot Q_m\right)\,, \tag{1.10}$$

where $g(m,n,p)$ is defined through the Fourier expansion

$$\frac{1}{\widetilde{\Phi}_k(\widetilde{\Omega})} = \frac{1}{N\,K} \sum_{\substack{m,n,p \\ m\geq -1, n\geq -1/N}} e^{2i\pi(m\widetilde{\rho}+n\widetilde{\sigma}+p\widetilde{v})} g(m,n,p)\,, \tag{1.11}$$

and K is an appropriate normalization factor given in eqs.(3.9), (3.11). We shall see that in (1.11) the sum over m and p run over integer values, whereas the sum over n runs over integer multiples of $1/N$.

In section 2 we explain in detail the construction outlined above. The proposal made in [1] is a special case ($N=1$, $k=10$) of our more general proposal.

In section 3 we subject our proposal to some consistency tests. First we prove, using the modular transformation properties of $\widetilde{\Phi}_k$, that the proposed formula for $d(Q)$ is invariant under the S-duality transformation (1.1). Next we prove, by studying the explicit form of the Fourier expansion of $\widetilde{\Phi}_k$, that the degeneracies $d(Q)$ defined through (1.10), (1.11) are integers. Finally we analyze the behaviour of the proposed formula in the limit of large Q_e^2, Q_m^2 and $Q_e\cdot Q_m$. Defining the statistical entropy S_{stat} as $\ln d(Q)$, we find that for large charges it is given by:

$$\begin{aligned} S_{stat} &= \frac{\pi}{2}\Big[\frac{a^2+S^2}{S}Q_m^2 + \frac{1}{S}Q_e^2 - 2\,\frac{a}{S}\,Q_e\cdot Q_m + 128\,\pi\,\phi(a,S)\Big] \\ &\quad +\text{constant} + \mathcal{O}(Q^{-2})\,, \end{aligned} \tag{1.12}$$

$$\phi(a,S) \equiv -\frac{1}{64\pi^2}\Big\{(k+2)\,\ln S + \ln f^{(k)}(a+iS) + \ln f^{(k)}(a+iS)^*\Big\} \tag{1.13}$$

where $f^{(k)}(\tau)$ has been defined in eq.(1.2) and the variables S and a are to be determined by extremizing (1.12). The entropy of a black hole in these theories carrying the same charges was computed in [19], generalizing earlier results of [20, 21] for toroidal compactification. (1.12) agrees with this formula for large charges not only at the leading order but also at the first non-leading order. This generalizes a similar result found in [2] for toroidal compactification.

Appendices A and B are devoted to proving that Φ_k defined in (1.6) transforms as a modular form under an appropriate subgroup of $Sp(2,\,\mathbb{Z})$. Modular transformation properties of $\widetilde{\Phi}_k$ follow from these results. Appendix C is devoted to proving that up to an overall normalization factor, $\widetilde{\Phi}_k$ has a Fourier expansion in $\widetilde{\rho}$, $\widetilde{\sigma}$ and $\widetilde{v}$ with integer coefficients, and the leading term with unit coefficient. This is necessary for proving that

$g(m,n,p)$ defined through (1.11) are integers. We also give an algorithm for computing these coefficients.

Before concluding this section we must mention a possible caveat in our proposal. According to (1.10), (1.11) the result for $d(Q)$ depends only on the combinations Q_e^2, Q_m^2 and $Q_e \cdot Q_m$ and not on the details of the charge vectors. This would be true had all the charge vectors with given Q_e^2, Q_m^2 and $Q_e \cdot Q_m$ been related by appropriate T-duality transformations. However since the same Q_e^2 value may arise from both the twisted and the untwisted sector of the orbifold model which are not related by a T-duality transformation, it is not *a priori* obvious that the formula for the degeneracy should nevertheless depend only on the combinations Q_e^2, Q_m^2 and $Q_e \cdot Q_m$. We have already pointed out that the proposed formula is suitable only for states carrying twisted sector electric charges. It is conceivable that there are further restrictions on the charge vector Q for which our proposal for $d(Q)$ holds. In the absence of a derivation of (1.10), (1.11) from first principles (*e.g.* along the lines of [3]) we are unable to address this issue in more detail.

2 Proposal for the dyon spectrum

In this section we shall first give a brief introduction to the class of CHL models which we shall study, then introduce the necessary mathematical background involving modular forms of appropriate subgroups of $Sp(2, \mathbb{Z})$, and finally write down our proposal for the degeneracy of dyons in the CHL model in terms of these modular forms.

2.1 CHL models

We consider a class of CHL string compactification[7, 8, 9, 10, 11, 12] where we begin with a heterotic string theory compactified on a six torus $T^4 \times \widetilde{S}^1 \times S^1$ and mod out the theory by a $\mathbb{Z}_N$ transformation that acts as a $1/N$ unit of shift along $\widetilde{S}^1$ together with some $\mathbb{Z}_N$ action on the internal conformal field theory associated with T^4 and the 16 left-moving chiral bosons of heterotic string theory. There is a dual description of the theory as a $\mathbb{Z}_N$ orbifold of type IIA string theory on $K3 \times \widetilde{S}^1 \times S^1$ where the $\mathbb{Z}_N$ acts as a $1/N$ unit of translation along $\widetilde{S}^1$, together with some action on the internal conformal field theory associated with the $K3$ compactification. We shall restrict our analysis to the case of prime values of N since some of the uniqueness results which we shall use hold only when N is prime. In this case the rank of the gauge group is reduced to

$$r = \frac{48}{N+1} + 4 \tag{2.1}$$

from its original value of 28 for toroidal compactification[11, 12]. We also define:

$$k = \frac{r-4}{2} - 2 = \frac{24}{N+1} - 2\,. \tag{2.2}$$

For N prime we have the following values of k: $(N,k) = (2,6),(3,4),(5,2),(7,1)$.[2] The case of toroidal compactification corresponds to $(N,k) = (1,10)$.

We shall analyze the degeneracy of dyons carrying momentum and winding charges as well as Kaluza-Klein and H-monopole charges[3] along various $\mathbb{Z}_N$ invariant compact directions. Let $\tilde{n}$, $\tilde{w}$, $\widetilde{K}$ and $\widetilde{H}$ denote the number of units of momentum, winding, Kaluza-Klein monopole and H-monopole charges associated with the circle $\widetilde{S}^1$. Then we define

$$Q_e = \begin{pmatrix} \tilde{n} \\ \tilde{w} \\ \vdots \end{pmatrix}, \qquad Q_m = \begin{pmatrix} \widetilde{H} \\ \widetilde{K} \\ \vdots \end{pmatrix}, \qquad Q = \begin{pmatrix} Q_m \\ Q_e \end{pmatrix}, \tag{2.3}$$

where $\cdots$ in the expression for Q_e denote the momentum and winding along S^1 and the other $\mathbb{Z}_N$ invariant directions associated with the internal conformal field theory and $\cdots$ in the expression for Q_m denote their magnetic counterpart. We also define

$$Q_e^2 = 2(\tilde{n}\tilde{w} + \cdots), \quad Q_m^2 = 2(\widetilde{K}\widetilde{H} + \cdots), \quad Q_e \cdot Q_m = (\tilde{n}\widetilde{K} + \tilde{w}\widetilde{H} + \cdots)\,, \tag{2.4}$$

where $\cdots$ denote the contribution from the other components of the charges. Now note that while the charge quantum numbers $\tilde{n}$ and $\widetilde{K}$ are integers, the winding charge $\tilde{w}$ associated with the circle $\widetilde{S}^1$ can take values in units of $1/N$ once we include twisted sector states in the spectrum. Also the H-monopole charge $\widetilde{H}$ associated with $\widetilde{S}^1$ is quantized in units of N, since in order to get a $\mathbb{Z}_N$ invariant configuration of five-branes

[2] Although there is no known CHL model with $N = 11$, our analysis goes through for $N = 11$ as well. This leads us to suspect that there may be a consistent CHL model with $N = 11$. In this case $k = 0$ and $r = 8$. These eight gauge fields include all the six right-handed gauge fields (as is required by $\mathcal{N} = 4$ supersymmetry) and the two left handed gauge fields associated with $\widetilde{S}^1 \times S^1$. Thus all the left-handed gauge fields associated with T^4 are projected out. In the dual type IIA construction the orifolding must project out all the 19 anti-self dual two forms on $K3$ as well as a linear combination of the zero form and the four form. The latter result suggests that the orbifold action cannot be geometric, and must form part of the mirror symmetry group of $K3$.

[3] An H-monopole associated with $\widetilde{S}^1$ corresponds to a 5-brane transverse to $\widetilde{S}^1$ and the three non-compact directions.

transverse to $\widetilde{S}^1$ we need to have N equispaced five-branes on $\widetilde{S}^1$. The contribution from $\cdots$ terms are quantized in integer units. As a result, $Q_e^2/2$ is quantized in units of $1/N$, whereas $Q_m^2/2$ and $Q_e \cdot Q_m$ are quantized in integer units.

The S-duality symmetry of the theory is generated by a set of 2×2 matrices of the form[10, 12, 18]:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1, \quad c = 0 \bmod N, \quad a, d = 1 \bmod N\,. \tag{2.5}$$

This group of matrices is known as $\Gamma_1(N)$. The S-duality transformation acts on the electric and the magnetic charge vectors as:

$$Q_e \to aQ_e + bQ_m, \qquad Q_m \to cQ_e + dQ_m\,, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)\,. \tag{2.6}$$

Our goal is to propose a formula for the degeneracy $d(Q)$ of dyons in this theory carrying charge vector Q such that it is invariant under the S-duality transformation (2.6), and also, for large charges, $\ln d(Q)$ reproduces the entropy of the BPS black hole carrying charge Q.

2.2 The modular form Φ_k

Since the proposed expression for $d(Q)$ involves an integration over the period matrices of the genus two Riemann surfaces, we need to begin by reviewing some facts about this period matrix. It is parametrized by the 2×2 complex symmetric matrix

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}, \tag{2.7}$$

subject to the restriction

$$Im(\rho) > 0, \qquad Im(\sigma) > 0, \qquad (Im\,\rho)\,(Im\,\sigma) > (Im\,v)^2\,. \tag{2.8}$$

The modular group $Sp(2, \mathbb{Z})$ transformation acts on this as

$$\Omega \to \Omega' = (A\Omega + B)(C\Omega + D)^{-1}\,, \tag{2.9}$$

where A, B, C, D are 2×2 matrices satisfying

$$AB^T = BA^T, \qquad CD^T = DC^T, \qquad AD^T - BC^T = I\,. \tag{2.10}$$

Let us denote by G the subgroup of $Sp(2,\ \mathbb{Z})$ generated by the 4×4 matrices

$$\begin{aligned}
g_1(a,b,c,d) &\equiv \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad ad-bc=1, \quad c=0 \bmod N, \quad a,d=1 \bmod N \\
g_2 &\equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
g_3(\lambda,\mu) &\equiv \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \lambda,\mu \in \mathbb{Z}.
\end{aligned} \tag{2.11}$$

From the definition it follows that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G \quad \to \quad C = \mathbf{0} \bmod N, \quad \det A = 1 \bmod N, \quad \det D = 1 \bmod N\ . \tag{2.12}$$

We suspect that the reverse is also true, i.e. if $C=\mathbf{0} \bmod N$ and $\det A = 1 \bmod N$ (which implies that $\det D = 1 \bmod N$) then the $Sp(2,\ \mathbb{Z})$ matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ belongs to G. However at present we do not have a proof of this, and we shall not use this result in our analysis.

The group G has the following properties:

1. It contains all elements of the form:

$$g_1(a,b,c,d)g_2g_1(a,-b,-c,d)(g_2)^{-1} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & -b \\ c & 0 & d & 0 \\ 0 & -c & 0 & d \end{pmatrix},$$
$$ad-bc=1, \qquad c=0 \bmod N, \qquad a,d=1 \bmod N\,. \tag{2.13}$$

   We shall denote by H the subgroup of G containing elements of the form (2.13). It is isomorphic to $\Gamma_1(N)$ introduced in (2.5).

2. As shown in appendix B, there is a modular form[4] $\Phi_k(\Omega)$ of G of weight k obeying

[4]Throughout this paper we shall refer to as modular forms the functions of ρ, σ, v which transform as (2.14) under a subgroup of $Sp(2,\ \mathbb{Z})$ but which may have poles in the Siegel upper half plane or at the cusps. Thus these modular forms are not necessarily entire.

the usual relations

$$\Phi_k\Big((A\Omega+B)(C\Omega+D)^{-1}\Big) = \{\det(C\Omega+D)\}^k\,\Phi_k(\Omega)\,, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G\,. \quad (2.14)$$

In fact as has been shown in eqs.(B.14), (B.15), the transformation law (2.14) holds also for the $Sp(2,\ \mathbb{Z})$ element

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & -b \\ c & 0 & d & 0 \\ 0 & -c & 0 & d \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\,, \quad (2.15)$$

where $\Gamma_0(N)$ is defined as the collection of 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a,b,c,d \in \mathbb{Z}, \quad ad-bc=1, \quad c=0 \bmod N\ . \quad (2.16)$$

It has also been shown in appendix B that as $v\to 0$,

$$\Phi_k(\Omega) \simeq 4\pi^2\, v^2\, f^{(k)}(\rho)\, f^{(k)}(\sigma) + \mathcal{O}(v^4)\,, \quad (2.17)$$

where

$$f^{(k)}(\tau) = (\eta(\tau))^{k+2}\,(\eta(N\tau))^{k+2} \quad (2.18)$$

and $\eta(\tau)$ is the Dedekind η function. $f^{(k)}(\tau)$ is the unique cusp form of $\Gamma_1(N)$ of weight $k+2 = 24/(N+1)$[22]:

$$\begin{aligned} & f^{(k)}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+2} f^{(k)}(\tau)\,, \\ & \lim_{\tau\to i\infty} f^{(k)}(\tau) = 0\,, \quad \lim_{\tau\to p/q} f^{(k)}(\tau) = 0 \quad \text{for} \quad p,q\in \mathbb{Z}\,. \end{aligned} \quad (2.19)$$

An algorithm for constructing the modular form Φ_k in terms of the cusp forms $f^{(k)}(\tau)$ of $\Gamma_1(N)$ can be found along the lines of the analysis performed in [23, 24, 25] and has been described in appendices A and B. The results have already been summarized in eqs.(1.3)-(1.7).

Neither the group G nor the modular form Φ_k will be used directly in writing down our proposal for $d(Q)$. Instead we shall use them to define a modular form $\widetilde{\Phi}_k$ of a different subgroup $\widetilde{G}$ of $Sp(2,\ \mathbb{Z})$ which is related to G by a conjugation. It will be this modular form $\widetilde{\Phi}_k$ that will be used in writing down our proposal.

2.3 The modular form $\widetilde{\Phi}_k$

We now introduce the $Sp(2, \mathbb{Z})$ matrices:

$$U_0 \equiv \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \qquad U_0^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{2.20}$$

and denote by $\widetilde{G}$ the subgroup of $Sp(2, \mathbb{Z})$ containing elements of the form:

$$U_0 U U_0^{-1}, \qquad U \in G\,. \tag{2.21}$$

Clearly $\widetilde{G}$ is isomorphic to G. We also define

$$\widetilde{\Omega} = (A_0\Omega + B_0)(C_0\Omega + D_0)^{-1} \equiv \begin{pmatrix} \widetilde{\rho} & \widetilde{v} \\ \widetilde{v} & \widetilde{\sigma} \end{pmatrix}, \tag{2.22}$$

and

$$\widetilde{\Phi}_k(\widetilde{\Omega}) = \{\det(C_0\Omega + D_0)\}^k \Phi_k(\Omega) = (2v - \rho - \sigma)^k\, \Phi_k(\Omega)\,. \tag{2.23}$$

Using (2.14) and (2.21)-(2.23) one can show that for

$$\widetilde{\Omega}' = (\widetilde{A}\widetilde{\Omega} + \widetilde{B})(\widetilde{C}\widetilde{\Omega} + \widetilde{D})^{-1}, \qquad \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} \in \widetilde{G}\,, \tag{2.24}$$

we have

$$\widetilde{\Phi}_k(\widetilde{\Omega}') = \{\det(\widetilde{C}\widetilde{\Omega} + \widetilde{D})\}^k\, \widetilde{\Phi}_k(\widetilde{\Omega})\,. \tag{2.25}$$

The group $\widetilde{G}$ includes the translation symmetries:

$$(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) \to (\widetilde{\rho}+1, \widetilde{\sigma}, \widetilde{v}), \qquad (\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) \to (\widetilde{\rho}, \widetilde{\sigma}+N, \widetilde{v}), \qquad (\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) \to (\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}+1)\,, \tag{2.26}$$

under which $\widetilde{\Phi}_k$ is invariant. The corresponding elements of G are

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} &= g_1(1,1,0,1) g_2 g_1(1,1,0,1)(g_2)^{-1} g_3(0,1) \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -N & N & 1 & 0 \\ N & -N & 0 & 1 \end{pmatrix} &= g_2 g_3(-1,0) g_2 g_1(1,0,-N,1) g_2 g_3(1,0) g_2 \\ \text{and} \quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} &= (g_2)^{-1} g_3(-2,0)\,, \end{aligned} \tag{2.27}$$

respectively, with the g_i's as defined in (2.11).

$\widetilde{G}$ also contains a subgroup $\widetilde{H}$ consisting of elements of the form:

$$U_0 U U_0^{-1}, \qquad U \in H\,. \tag{2.28}$$

Since the elements of H are of the form (2.13), we see using (2.20) that the elements of $\widetilde{H}$ are of the form:

$$\begin{pmatrix} a & -b & b & 0 \\ -c & d & 0 & c \\ 0 & 0 & d & c \\ 0 & 0 & b & a \end{pmatrix}, \qquad ad - bc = 1, \qquad a, d = \ 1 \bmod N, \qquad c = 0 \bmod N\,. \tag{2.29}$$

In fact, due to the result described in (2.14), (2.15) the modular transformation law (2.25) holds for a more general class of $Sp(2,\ \mathbb{Z})$ elements of the form:

$$\begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} = \begin{pmatrix} a & -b & b & 0 \\ -c & d & 0 & c \\ 0 & 0 & d & c \\ 0 & 0 & b & a \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\,. \tag{2.30}$$

Using (2.7), (2.20) and (2.22) we see that the variables $(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ are related to the variables (ρ, σ, v) via the relations:

$$\rho = \frac{\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2}{\widetilde{\sigma}}, \qquad \sigma = \frac{\widetilde{\rho}\widetilde{\sigma} - (\widetilde{v} - 1)^2}{\widetilde{\sigma}}, \qquad v = \frac{\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v}}{\widetilde{\sigma}}, \tag{2.31}$$

or equivalently,

$$\widetilde{\rho} = \frac{v^2 - \rho\sigma}{2v - \rho - \sigma}, \qquad \widetilde{\sigma} = \frac{1}{2v - \rho - \sigma}, \qquad \widetilde{v} = \frac{v - \rho}{2v - \rho - \sigma}\,. \tag{2.32}$$

(2.23) now gives

$$\widetilde{\Phi}_k(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) = \widetilde{\sigma}^{-k} \Phi_k \left(\widetilde{\rho} - \frac{\widetilde{v}^2}{\widetilde{\sigma}}, \widetilde{\rho} - \frac{(\widetilde{v} - 1)^2}{\widetilde{\sigma}}, \widetilde{\rho} - \frac{\widetilde{v}^2}{\widetilde{\sigma}} + \frac{\widetilde{v}}{\widetilde{\sigma}} \right). \tag{2.33}$$

Using the invariance of Φ_k under the transformation $g_3(-1, 0)$:

$$\Phi_k(\rho, \sigma, v) = \Phi_k(\rho, \sigma + \rho - 2v, v - \rho)\,, \tag{2.34}$$

we can rewrite (2.33) as

$$\widetilde{\Phi}_k(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) = \widetilde{\sigma}^{-k} \Phi_k \left(\widetilde{\rho} - \frac{\widetilde{v}^2}{\widetilde{\sigma}}, -\frac{1}{\widetilde{\sigma}}, \frac{\widetilde{v}}{\widetilde{\sigma}} \right). \tag{2.35}$$

Eq.(2.31) shows that the region $v \simeq 0$ corresponds to $\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v} \simeq 0$. (2.17) and (2.33) now give

$$\widetilde{\Phi}_k(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) \simeq 4\pi^2\, \widetilde{\sigma}^{-k-2}(\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v})^2 f^{(k)}(\rho) f^{(k)}(\sigma) + \mathcal{O}\left((\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v})^4\right) . \tag{2.36}$$

On the other hand (2.35) shows that near $\widetilde{v} = 0$,

$$\widetilde{\Phi}_k(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) \simeq 4\pi^2\, \widetilde{\sigma}^{-k-2}\widetilde{v}^2 f^{(k)}(\widetilde{\rho}) f^{(k)}(-\widetilde{\sigma}^{-1}) + \mathcal{O}(\widetilde{v}^4) . \tag{2.37}$$

Using the definition of $f^{(k)}(\tau)$ given in (2.18), and the modular transformation law of $\eta(\tau)$:

$$\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau) , \tag{2.38}$$

we can rewrite (2.37) as

$$\widetilde{\Phi}_k(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) \simeq (i\sqrt{N})^{-k-2}\, 4\pi^2\, \widetilde{v}^2 f^{(k)}(\widetilde{\rho}) f^{(k)}(\widetilde{\sigma}/N) + \mathcal{O}(\widetilde{v}^4) . \tag{2.39}$$

2.4 The dyon spectrum

Our proposal for $d(Q)$ involves the modular form $\widetilde{\Phi}_k$ and can be stated as follows:

$$d(Q) = g\left(\frac{1}{2}Q_m^2, \frac{1}{2}Q_e^2, Q_e \cdot Q_m\right) , \tag{2.40}$$

where $g(m, n, p)$ is defined through the Fourier expansion:

$$\frac{1}{\widetilde{\Phi}_k(\widetilde{\Omega})} = \frac{1}{N\, K} \sum_{\substack{m, Nn, p \in \mathbb{Z} \\ m \geq -1, n \geq -1/N}} e^{2i\pi(m\widetilde{\rho} + n\widetilde{\sigma} + p\widetilde{v})} g(m, n, p) , \tag{2.41}$$

K being an appropriate normalization factor. The multiplicative factor of $1/N$ has been included for later convenience; it could have been absorbed into the definition of K. From eq.(2.26) we see that the sum over m and p run over integer values, whereas the sum over n run over integer multiples of $1/N$. Eq.(2.40) then indicates that $Q_m^2/2$ and $Q_e \cdot Q_m$ are quantized in integer units and $Q_e^2/2$ is quantized in units of $1/N$. This is consistent with the analysis given below (2.4).

Eqs.(2.40), (2.41) may be rewritten as

$$d(Q) = K \int_{\mathcal{C}} d^3\widetilde{\Omega}\, e^{-i\pi Q^T \cdot \widetilde{\Omega} Q}\, \frac{1}{\widetilde{\Phi}_k(\widetilde{\Omega})} , \tag{2.42}$$

where the integration runs over the three cycle $\mathcal{C}$ defined as

$$Im(\widetilde{\rho}),\, Im(\widetilde{\sigma}),\, Im(\widetilde{v}) = \text{fixed}, \qquad 0 \le Re(\widetilde{\rho}) \le 1, \quad 0 \le Re(\widetilde{\sigma}) \le N, \quad 0 \le Re(\widetilde{v}) \le 1\,, \tag{2.43}$$

$$d^3\widetilde{\Omega} = d\widetilde{\rho}\, d\widetilde{\sigma}\, d\widetilde{v}\,, \tag{2.44}$$

and

$$Q^T \widetilde{\Omega} Q = \widetilde{\rho} Q_m^2 + \widetilde{\sigma} Q_e^2 + 2\widetilde{v} Q_e \cdot Q_m\,. \tag{2.45}$$

3 Consistency checks

In this section we shall subject our proposal (2.42) to various consistency checks. In section 3.1 we check the S-duality invariance of (2.42). In section 3.2 we check that $d(Q)$ defined in section 2.4 are integers. In section 3.3 we verify that for large charges (2.42) reproduces the black hole entropy to first non-leading order.

3.1 S-duality invariance of $d(Q)$

First we shall prove the S-duality invariance of the formula (2.42). For the CHL models considered here, the S-duality group is $\Gamma_1(N)$ under which the electric and magnetic charges transform to;

$$Q_e \to Q_e' = aQ_e + bQ_m, \qquad Q_m \to Q_m' = cQ_e + dQ_m\,, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)\,. \tag{3.1}$$

Let us define

$$\widetilde{\Omega}' \equiv \begin{pmatrix} \widetilde{\rho}' & \widetilde{v}' \\ \widetilde{v}' & \widetilde{\sigma}' \end{pmatrix} = (\widetilde{A}\widetilde{\Omega} + \widetilde{B})(\widetilde{C}\widetilde{\Omega} + \widetilde{D})^{-1}, \qquad \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} = \begin{pmatrix} a & -b & b & 0 \\ -c & d & 0 & c \\ 0 & 0 & d & c \\ 0 & 0 & b & a \end{pmatrix} \in \widetilde{H}\,. \tag{3.2}$$

This gives

$$\begin{aligned} \widetilde{\rho}' &= a^2\widetilde{\rho} + b^2\widetilde{\sigma} - 2ab\widetilde{v} + ab\,, \\ \widetilde{\sigma}' &= c^2\widetilde{\rho} + d^2\widetilde{\sigma} - 2cd\widetilde{v} + cd\,, \\ \widetilde{v}' &= -ac\widetilde{\rho} - bd\widetilde{\sigma} + (ad+bc)\widetilde{v} - bc\,. \end{aligned} \tag{3.3}$$

Using (3.1), (3.3) and the quantization laws of Q_e^2, Q_m^2 and $Q_e \cdot Q_m$ one can easily verify that

$$e^{i\pi Q^T \cdot \widetilde{\Omega} Q} = e^{i\pi Q'^T \cdot \widetilde{\Omega}' Q'}, \tag{3.4}$$

and

$$d^3\widetilde{\Omega} = d^3\widetilde{\Omega}' . \tag{3.5}$$

On the other hand, eqs.(2.25), (3.2) give

$$\widetilde{\Phi}_k(\widetilde{\Omega}') = \widetilde{\Phi}_k(\widetilde{\Omega}) . \tag{3.6}$$

Finally we note that under the map (3.3) the three cycle $\mathcal{C}$, which is a three torus lying along the real $\widetilde{\rho}$, $\widetilde{\sigma}$ and $\widetilde{v}$ axes, – with length 1 along $\widetilde{\rho}$ and $\widetilde{v}$ axis and length N along $\widetilde{\sigma}$ axis, – gets mapped to itself up to a shift that can be removed with the help of the shift symmetries (2.26). Thus eqs.(3.4)-(3.6) allow us to express (2.42) as

$$d(Q) = K \int_{\mathcal{C}} d^3\widetilde{\Omega}' \, e^{-i\pi Q'^T \cdot \widetilde{\Omega}' Q'} \, \frac{1}{\widetilde{\Phi}_k(\widetilde{\Omega}')} = d(Q') . \tag{3.7}$$

This proves invariance of $d(Q)$ under the S-duality group $\Gamma_1(N)$.

Due to eq.(2.30) the formula for $d(Q)$ is actually invariant under a bigger group $\Gamma_0(N)$. This indicates that the full U-duality group of the theory may contain $\Gamma_0(N)$ as a subgroup. The physical origin of this $\Gamma_0(N)$ is best understood in the description of the theory as a Z_N orbifold of type II string theory on $K3 \times S^1 \times \widetilde{S}^1$. In this case $\Gamma_0(N)$ acts as a T-duality transformation on $\widetilde{S}^1 \times S^1$. Hence, acting on $1/N$ unit of shift along $\widetilde{S}^1$, represented by the vector $\begin{pmatrix} 1/N \\ 0 \end{pmatrix}$, the $\Gamma_0(N)$ element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gives a vector $\begin{pmatrix} a/N \\ 0 \end{pmatrix}$ modulo integer lattice vectors, representing a/N unit of shift along $\widetilde{S}^1$. This would seem to take this to a different theory where the $\mathbb{Z}_N$ generator acts as a/N units of shift along $\widetilde{S}^1$ but has the same action on $K3$. Of course this group also has an element that contains $1/N$ unit of shift (mod 1) along $\widetilde{S}^1$, but its action on $K3$ is a power of the action of the original $\mathbb{Z}_N$ generator. Thus in order that $\Gamma_0(N)$ is a symmetry, its action must be accompanied by an internal symmetry transformation in $K3$ that permutes the action of different $\mathbb{Z}_N$ elements on $K3$.

3.2 Integrality of $d(Q)$

In this section we shall show that $d(Q)$ defined through (2.40), (2.41) are integers. This is a necessary condition for them to be interpreted as the degeneracy of dyonic states in this theory. For this we begin with the expansion of $\widetilde\Phi_k(\widetilde\rho,\widetilde\sigma,\widetilde v)$ described in appendix C:[5]

$$\begin{aligned}\widetilde\Phi_k(\widetilde\rho,\widetilde\sigma,\widetilde v) &= C\, e^{2\pi i\widetilde\rho+2\pi i\widetilde\sigma/N+2\pi i\widetilde v}\Big(1-2e^{-2\pi i\widetilde v}+e^{-4\pi i\widetilde v} \\ &\quad + \sum_{\substack{q,r,s\in \mathbb{Z}\\ q,r\ge 0, q+r\ge 1}} b(q,r,s)\, e^{2\pi i q\widetilde\rho+2\pi i r\widetilde\sigma/N+2\pi i s\widetilde v}\Big)\,, \end{aligned} \tag{3.8}$$

with the coefficients $b(q,r,s)$ being integers and

$$C = -(i\sqrt{N})^{-k-2}\,. \tag{3.9}$$

This gives

$$\begin{aligned}\frac{N\,K}{\widetilde\Phi_k(\widetilde\rho,\widetilde\sigma,\widetilde v)} &= \frac{N\,K}{C}\, e^{-2\pi i\widetilde\rho-2\pi i\widetilde\sigma/N-2\pi i\widetilde v} \\ &\quad \Big[1+\sum_{l=1}^\infty(-1)^l\Big(-2e^{-2\pi i\widetilde v}+e^{-4\pi i\widetilde v} \\ &\quad + \sum_{\substack{q,r,s\in \mathbb{Z}\\ q,r\ge 0, q+r\ge 1}} b(q,r,s)\, e^{2\pi i q\widetilde\rho+2\pi i r\widetilde\sigma/N+2\pi i s\widetilde v}\Big)^l\Big]\,. \end{aligned} \tag{3.10}$$

We can now expand the terms inside the square bracket to get an expansion in positive powers of $e^{2\pi i\widetilde\rho}$ and $e^{2\pi i\widetilde\sigma/N}$ and both positive and negative powers of $e^{2\pi i\widetilde v}$. (For $\widetilde\rho$ and $\widetilde\sigma$ independent terms inside the square bracket the expansion has only negative powers of $e^{2\pi i\widetilde v}$.) Since $b(q,r,s)$ are integers, each term in this expansion will also be integers. Comparing (3.10) with (2.40), (2.41) we see that $d(Q)$ are integers as long as we choose K such that $\frac{N\,K}{C}$ is an integer. In particular if we choose

$$K = \frac{C}{N}\,, \tag{3.11}$$

then this would correspond to counting the degeneracy in multiples of degeneracy of states with a Q for which $Q_e^2/2=-1/N$, $Q_m^2/2=-1$ and $Q_e\cdot Q_m=-1$.

Comparison of (3.10) and (2.41) also shows that in order to get non-zero $d(Q)$ we must have

$$\frac{1}{2}Q_e^2 \ge -\frac{1}{N}, \qquad \frac{1}{2}Q_m^2 \ge -1\,. \tag{3.12}$$

[5]For definiteness we have chosen the expansion in a form that will be suitable for the $Im(\widetilde v) < 0$ region. For $Im(\widetilde v) > 0$ we need to change the sign of $\widetilde v$ in this expansion.

The condition $\frac{1}{2}Q_e^2 \geq -\frac{1}{N}$ may sound surprising at first sight, since for ordinary electrically charged states this condition, arising from the level matching condition, takes the form $\frac{1}{2}Q_e^2 \geq -1$. The -1 in the right hand side of this inequality is the L_0 eigenvalue of the left-moving ground state of the world-sheet theory. We should note however that here we are considering states whose electric charge arises in the twisted sector and in applying the level matching condition we must use the ground state energy of the twisted sector. In our example, the total number of twisted scalar fields is given by

$$(28 - r) = 24\,\frac{N-1}{N+1}\,. \tag{3.13}$$

This corresponds to $12(N-1)/(N+1)$ complex scalars. These can be divided into $24/(N+1)$ sets, each set containing $(N-1)/2$ complex scalars on which the orbifold group acts as a rotation by angles $\phi_1 = 2\pi/N$, $\phi_2 = 4\pi/N$, $\ldots$ $\phi_{(N-1)/2} = \pi(N-1)/N$.[6] The net L_0 eigenvalue of the twisted sector ground state then takes the form:

$$-1 + \frac{24}{N+1}\,\frac{1}{2}\sum_{j=1}^{(N-1)/2}\frac{j}{N}\left(1-\frac{j}{N}\right) = -\frac{1}{N}\,. \tag{3.14}$$

This gives the restriction

$$\frac{1}{2}\,Q_e^2 \geq -\frac{1}{N}\,. \tag{3.15}$$

This is consistent with (3.12).

3.3 Black hole entropy

We shall now show that in the limit of large charges the expression for $d(Q)$ given in (2.42) reproduces the extremal black hole entropy carrying the same charges. The analysis proceeds as in [1, 2]. First of all, following the procedure of [1] one can deform the integration contour $\mathcal{C}$ in (2.42), picking up contribution from the residues at various poles of the integrand. These poles occur at the zeroes of $\widetilde{\Phi}_k$, which, according to (2.36) (or (2.37)) occur on the divisor $\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v} = 0$ and its images under $Sp(2, \mathbb{Z})$.[7] For large charges the dominant contribution comes from the pole at which the exponent in (2.42) takes maximum value at its saddle point; the contribution from the other poles are

[6] For $N = 2$ the counting is slightly different. In this case each set contains a $\mathbb{Z}_2$ even real scalar and a $\mathbb{Z}_2$ odd real scalar. The final result is the same as (3.14).

[7] We are assuming that $\widetilde{\Phi}_k$ does not have additional zeroes other than at the images of $\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v} = 0$, or that even if such zeroes are present, their contribution to the entropy is subdominant for large charges.

exponentially suppressed. The analysis of [1] showed that the divisor that gives dominant contribution corresponds to

$$\mathcal{D}: \quad \widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v} = 0\,. \tag{3.16}$$

We now carry out the $\widetilde{v}$ integral in (2.42) using Cauchy's integration formula. Let us denote by $\widetilde{v}_\pm$ the zeroes of $(\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v})$:

$$\widetilde{v}_\pm = \frac{1}{2} \pm \Lambda(\widetilde{\rho}, \widetilde{\sigma}), \qquad \Lambda(\widetilde{\rho}, \widetilde{\sigma}) = \sqrt{\frac{1}{4} + \widetilde{\rho}\widetilde{\sigma}}\,. \tag{3.17}$$

The contour integral in $\widetilde{v}$ plane is evaluated by keeping $\widetilde{\rho}$ and $\widetilde{\sigma}$ fixed. The zeros $\widetilde{v}_\pm$ are distinct for generic but fixed $\widetilde{\rho}$ and $\widetilde{\sigma}$. However, in the $\widetilde{\rho}$, $\widetilde{\sigma}$ and $\widetilde{v}$ space the divisor $\mathcal{D}$ is a continuous locus connecting these zeros. Therefore deformation of the integration contour through the divisor picks up the contribution from the pole only once; we can choose this pole to be at either $\widetilde{v}_+$ or $\widetilde{v}_-$. We will consider the contribution coming from $\widetilde{v} = \widetilde{v}_-$. We see from (2.42), (2.36) that near $\widetilde{v} = \widetilde{v}_-$ the integrand behaves as:

$$K\, \exp\left(-i\pi(\widetilde{\rho}Q_m^2 + \widetilde{\sigma}Q_e^2 + 2\widetilde{v}Q_m\cdot Q_e)\right) (4\pi^2)^{-1}\widetilde{\sigma}^{k+2}(\widetilde{v}-\widetilde{v}_+)^{-2}(\widetilde{v}-\widetilde{v}_-)^{-2} f^{(k)}(\rho)^{-1} f^{(k)}(\sigma)^{-1} + \mathcal{O}\left((\widetilde{v}-\widetilde{v}_-)^0\right)\,. \tag{3.18}$$

Using the results

$$\frac{\partial\rho}{\partial\widetilde{v}} = -\frac{2\widetilde{v}}{\widetilde{\sigma}}, \qquad \frac{\partial\sigma}{\partial\widetilde{v}} = -\frac{2(\widetilde{v}-1)}{\widetilde{\sigma}}, \tag{3.19}$$

which follow from (2.31), we get the result of the contour integration around $\widetilde{v}_-$ to be

$$d(Q) = (-1)^{Q_e\cdot Q_m}\,(2\pi)^{-1} i\, K \int d\widetilde{\rho} d\widetilde{\sigma}\, e^{i\pi X(\widetilde{\rho},\widetilde{\sigma}) + \ln\Delta(\widetilde{\rho},\widetilde{\sigma})}\,, \tag{3.20}$$

where

$$X(\widetilde{\rho},\widetilde{\sigma}) = -\widetilde{\rho}Q_m^2 - \widetilde{\sigma}Q_e^2 + 2\Lambda(\widetilde{\rho},\widetilde{\sigma})\, Q_e\cdot Q_m + \frac{k+2}{i\pi}\ln\widetilde{\sigma} - \frac{1}{i\pi}\ln f^{(k)}(\rho) - \frac{1}{i\pi}\ln f^{(k)}(\sigma)\,, \tag{3.21}$$

$$\Delta(\widetilde{\rho},\widetilde{\sigma}) = \frac{1}{4\Lambda(\widetilde{\rho},\widetilde{\sigma})^2}\left[-2\pi i Q_e\cdot Q_m + \frac{1}{\Lambda(\widetilde{\rho},\widetilde{\sigma})} + \frac{2}{\widetilde{\sigma}}\left\{\widetilde{v}_-\frac{f^{(k)\prime}(\rho)}{f^{(k)}(\rho)} - \widetilde{v}_+\frac{f^{(k)\prime}(\sigma)}{f^{(k)}(\sigma)}\right\}\right]\,. \tag{3.22}$$

In eqs.(3.21), (3.22) ρ and σ are to be regarded as functions of $\widetilde{\rho}$ and $\widetilde{\sigma}$ via eq.(2.32) at $v = 0$:

$$\widetilde{\rho} = \frac{\rho\sigma}{\rho+\sigma}, \qquad \widetilde{\sigma} = -\frac{1}{\rho+\sigma}\,. \tag{3.23}$$

We shall now use eqs.(3.20)-(3.22) to calculate the statistical entropy

$$S_{stat} = \ln|d(Q)|\,, \tag{3.24}$$

for large values of the charges. For this we shall carry out the integration over $\tilde{\rho}$ and $\tilde{\sigma}$ using a saddle point approximation, keeping terms in the entropy to leading order as well as first non-leading order in the charges. As we shall see, at the saddle point $\tilde{\rho}$ and $\tilde{\sigma}$ take finite values. Thus the leading order contribution to the entropy comes from the first three terms in X quadratic in the charges. In order to evaluate the first non-leading correction, we need to evaluate the order Q^0 contribution from the other terms at the saddle point; but determination of the saddle point itself can be done with the leading terms in X since an error of order ϵ in the location of the saddle point induces an error of order ϵ^2 in the entropy. At this level, we must also include the contribution to the entropy coming from the $\tilde{\rho}$, $\tilde{\sigma}$ integration around the saddle point:

$$-\frac{1}{2}\ln\left|\det\begin{pmatrix}\partial^2 X/\partial\tilde{\rho}^2 & \partial^2 X/\partial\tilde{\rho}\partial\tilde{\sigma}\\ \partial^2 X/\partial\tilde{\rho}\partial\tilde{\sigma} & \partial^2 X/\partial\tilde{\sigma}^2\end{pmatrix}\right| \simeq -\ln|Q_e\cdot Q_m| + \ln(\frac{1}{4}+\tilde{\rho}\tilde{\sigma}) + \text{constant}\,. \quad (3.25)$$

On the other hand, to order Q^0, we have

$$\ln\Delta(\tilde{\rho},\tilde{\sigma}) \simeq \ln|Q_e\cdot Q_m| - \ln(\frac{1}{4}+\tilde{\rho}\tilde{\sigma}) + \text{constant}\,. \quad (3.26)$$

Thus we see that to this order the contributions (3.25) and (3.26) cancel exactly, leaving behind the contribution to the entropy

$$\begin{aligned} S_{stat} &\simeq i\pi X(\tilde{\rho},\tilde{\sigma}) + \text{constant} \\ &= -i\pi\tilde{\rho}Q_m^2 - i\pi\tilde{\sigma}Q_e^2 + 2i\pi\Lambda(\tilde{\rho},\tilde{\sigma})\,Q_e\cdot Q_m \\ &\quad +(k+2)\ln\tilde{\sigma} - \ln f^{(k)}(\rho) - \ln f^{(k)}(\sigma) + \text{constant}\,, \end{aligned} \quad (3.27)$$

evaluated at the saddle point.

Although the saddle point is to be evaluated by extremizing the leading terms in the expression for $X(\tilde{\rho},\tilde{\sigma})$, clearly to this order we could also use the full expression for X to determine the location of the extremum. Thus the entropy S_{stat} can be regarded as the value of S_{stat} given in (3.27) at the extremum of this expression. The extremization may be done either by regarding $\tilde{\rho}$, $\tilde{\sigma}$ as independent variables, or by regarding ρ, σ defined via (3.23) as independent variables. We shall choose to regard ρ, σ as independent variables. If we now define complex variables a and S through the equations:

$$\rho = a + iS, \qquad \sigma = -a + iS\,, \quad (3.28)$$

then from (3.17)

$$\Lambda(\tilde{\rho},\tilde{\sigma}) = -\frac{1}{2}\,\frac{\rho-\sigma}{(\rho+\sigma)} = \frac{1}{2}\,i\,\frac{a}{S}\,. \quad (3.29)$$

Notice that (3.17) has a square root in the expression for Λ and there is an ambiguity in choosing the sign of the square root. This ambiguity is resolved with the help of (2.32) which tells us that near $v = 0$ we have

$$\widetilde{v} \simeq \frac{\rho}{\rho + \sigma} \,. \tag{3.30}$$

This should be identified with $\widetilde{v}_- = \frac{1}{2} - \Lambda$ since we have assumed that our contour encloses the pole at $\widetilde{v} = \widetilde{v}_-$. This fixes the sign of Λ to be the one given in (3.29). Eqs.(3.23), (3.27) and (3.29) now give

$$\begin{aligned} S_{stat} &= \frac{\pi}{2}\Big[\frac{a^2 + S^2}{S} Q_m^2 + \frac{1}{S} Q_e^2 - 2\,\frac{a}{S}\, Q_e \cdot Q_m + 128\,\pi\,\phi(a, S)\Big] \\ &\quad + \text{constant} + \mathcal{O}(Q^{-2})\,, \end{aligned} \tag{3.31}$$

where

$$\phi(a, S) = -\frac{1}{64\pi^2}\Big\{(k+2)\,\ln S + \ln f^{(k)}(a + iS) + \ln f^{(k)}(-a + iS)\Big\} \tag{3.32}$$

The values of a and S are to be determined by extremizing (3.31) with respect to a and S. This gives rise to four equations for four real parameters coming from a and S. For generic values of a and S, $\phi(a, S)$ is complex valued. However, for real values of a and S, $\phi(a, S)$ is real due to the identity $f^{(k)}(-a + iS)^* = f^{(k)}(a + iS)$ for real a, S, and hence the expression for S_{stat} given in (3.31) is real. Thus by restricting a and S to be real we end up with two equations for two real parameters whose solution gives the saddle point values of a and S on the real axis. Substituting these values of a and S in (3.31) we get the value of the statistical entropy S_{stat}. This matches exactly the entropy of an extremal dyonic black hole given in eq.(4.11) of [19]. The specific form of $\phi(a, S)$ given in (3.32) agrees with the explicit construction of the effective action of CHL models as given in appendix B of [18] (see also [26]). Thus we see that in the large charge limit the degeneracy formula for the dyon, given in (2.42), correctly produces the entropy of the black hole carrying the same charges, not only to the leading order but also to the first non-leading order in the inverse power of the charges.[8]

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[8]Note that up to this order the black hole entropy agrees with the entropy calculated using microcanonical ensemble, i.e. with the logarithm of the degeneracy of states.

A (Weak) Jacobi forms of $\Gamma_1(N)$

In this appendix we shall review the construction and some properties of (weak) Jacobi forms of $\Gamma_1(N)$ following [23, 22, 24]. These will be used in appendix B in the construction of the modular form Φ_k of the subgroup G of $Sp(2, \mathbb{Z})$.

We begin by defining, following [24],

$$\phi_{k,1}(\tau, z) = \eta(\tau)^{-6}\, f^{(k)}(\tau)\, \vartheta_1(\tau, z)^2 = (\eta(\tau))^{k-4}\, (\eta(N\tau))^{k+2}\, \vartheta_1(\tau, z)^2\,, \tag{A.1}$$

where

$$\vartheta_1(\tau, z) = \sum_{n\in \mathbb{Z}} (-1)^{n-\frac{1}{2}} e^{i\pi\tau(n+\frac{1}{2})^2} e^{2\pi i z(n+\frac{1}{2})}\,, \tag{A.2}$$

is a Jacobi theta function. From (A.1), (A.2) we get

$$\phi_{k,1}(\tau, z) = \eta(\tau)^{-6}\, f^{(k)}(\tau) \sum_{\substack{r,s\in \mathbb{Z}\\ r-s=odd}} (-1)^r e^{i\pi\tau(s^2+r^2)/2} e^{2\pi i r z}\,. \tag{A.3}$$

From (A.3) it follows that if

$$f^{(k)}(\tau)\eta(\tau)^{-6} = \sum_{n\geq 1} f_n^{(k)} e^{2\pi i\tau(n-\frac{1}{4})}\,, \tag{A.4}$$

then[9]

$$\phi_{k,1}(\tau, z) = \sum_{\substack{l,r\in \mathbb{Z}\\ r^2<4l, l\geq 1}} C(4l - r^2) e^{2\pi i l\tau} e^{2\pi i r z}\,, \tag{A.5}$$

where

$$C(m) = (-1)^m \sum_{\substack{s,n\in \mathbb{Z}\\ n\geq 1}} f_n^{(k)} \delta_{4n+s^2-1,m}\,. \tag{A.6}$$

Note that $C(m)$ vanishes for $m \leq 0$.

From the definition (A.3), the modular transformation law of ϑ_1, and the fact that $f^{(k)}(\tau)$ transforms as a cusp form of weight $(k+2)$ under $\Gamma_1(N)$, it follows that

$$\begin{aligned} \phi_{k,1}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) &= (c\tau+d)^k \exp(2\pi i c z^2/(c\tau+d))\phi_{k,1}(\tau, z) \\ &\quad \text{for} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)\,, \\ \phi_{k,1}(\tau, z+\lambda\tau+\mu) &= \exp\left(-2\pi i(\lambda^2\tau + 2\lambda z)\right) \phi_{k,1}(\tau, z) \quad \text{for} \quad \lambda, \mu \in \mathbb{Z}\,. \end{aligned} \tag{A.7}$$

[9]Notice that we have dropped the requirement $r-s$=odd that appeared in (A.3) since this follows from the relation $4l - r^2 = 4n + s^2 - 1$ and $l, n \in \mathbb{Z}$.

This shows that $\phi_{k,1}(z,\tau)$ transforms as a weak Jacobi form of $\Gamma_1(N)$ of weight k and index 1[23].

It also follows from the definition of a cusp form and that $f^{(k)}(\tau)$ is a cusp form of weight $(k+2)$ of $\Gamma_1(N)$ that $(c\tau+d)^{-k-2}f^{(k)}((a\tau+b)/(c\tau+d))$ for any element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SL(2,Z)$ has a series expansion involving strictly positive powers of $e^{2\pi i\tau}$. These powers are integral multiples of $1/N$. We now introduce the coefficients $\widehat{f}_n^{(k)}$ through the expansion:

$$\eta(\tau)^{-6}\,(c\tau+d)^{-k-2}\,f^{(k)}((a\tau+b)/(c\tau+d)) = \sum_{\substack{n>0\\ nN\in\mathbb{Z}}} \widehat{f}_n^{(k)}(a,b,c,d)e^{2\pi i\tau(n-\frac{1}{4})}\,. \tag{A.8}$$

From this and the modular transformation properties and series expansion of $\eta(\tau)$ and $\vartheta_1(\tau,z)$ it then follows, in a manner similar to the one that led to eq.(A.5), that

$$\begin{aligned} \widetilde{\phi}_{k,1}(\tau,z) &\equiv (c\tau+d)^{-k}\exp(-2\pi i c z^2/(c\tau+d))\phi_{k,1}((a\tau+b)/(c\tau+d), z/(c\tau+d)) \\ &= \sum_{\substack{Nl,r\in\mathbb{Z}\\ r^2\le 4l+1-\frac{4}{N}, l>0}} \widehat{C}(4l-r^2;a,b,c,d)e^{2\pi i l\tau+2\pi i r z}\,, \qquad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{Z})\,, \end{aligned} \tag{A.9}$$

where

$$\widehat{C}(m;a,b,c,d) = \sum_{\substack{s,nN\in\mathbb{Z}\\ n>0}} (-1)^{s+1}\,\widehat{f}_n^{(k)}(a,b,c,d)\,\delta_{4n+s^2-1,m}\,. \tag{A.10}$$

From (A.10) it follows that $\widehat{C}(m;a,b,c,d)$ vanishes for $m<\frac{4}{N}-1$. For $N=1,2,3$ this restricts m to strictly positive values. Eqs.(A.7) and (A.9) then shows that $\phi_{k,1}(\tau,z)$ is actually a Jacobi cusp form[23]. For $N=5,7$ the argument m of $\widehat{C}$ can be negative, and thus $\phi_{k,1}$ is only a weak Jacobi form.

For $N=1,2,3$ and 5, the values of k are even and $f^{(k)}(\tau)$ is actually a cusp form of the bigger group $\Gamma_0(N)$[22] defined in (2.16). For $N=7$ the value of k is odd and hence under a $\Gamma_0(N)$ transformation $f^{(k)}(\tau)$ transforms as a cusp form only up to a sign. This can be seen by considering the $\Gamma_0(7)$ matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ for which the two sides of the first equation in (2.19) differ by -1. In general $f^{(k)}(\tau)$ transforms under $\Gamma_0(N)$ as:

$$f^{(k)}\left(\frac{a\tau+b}{c\tau+d}\right) = \{\chi(a)\}^{-1}\,(c\tau+d)^{k+2}f^{(k)}(\tau) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(N)\,, \tag{A.11}$$

where $\chi(a) = 1$ or -1 determined by the value of a mod N[22]. It follows easily from (A.11) and that $\chi(a)$ depends only on a mod N, that

$$\chi(a)\,\chi(a') = \chi(aa')\,. \tag{A.12}$$

$\chi(a)$ describes a homomorphism map from $\Gamma_0(N)$ to $\mathbb{Z}_2$ and is known as a Dirichlet character mod 2 of $\Gamma_0(N)$. For $N = 1, 2, 3, 5$ we have $\chi(a) = 1$ for all a since $f^{(k)}(\tau)$ is actually a cusp form of $\Gamma_0(N)$. For $N = 7$ we know from the action of $\pm I$ on $f^{(k)}(\tau)$ that $\chi(-1) = -1$. The only possible $\chi(a)$ which is consistent with this and (A.12) is:

$$\begin{aligned} \chi(a) &= 1 \quad \text{for} \quad a = 1, 2, 4 \bmod 7\,, \\ &= -1 \quad \text{for} \quad a = 3, 5, 6 \bmod 7\,. \end{aligned} \tag{A.13}$$

From (A.11) and the modular transformation properties of $\eta(\tau)$, $\vartheta_1(\tau, z)$ it follows that

$$\begin{aligned} \phi_{k,1}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) &= (\chi(a))^{-1}\,(c\tau+d)^k\, \exp(2\pi i c z^2/(c\tau+d))\phi_{k,1}(\tau, z) \\ &\qquad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\,. \end{aligned} \tag{A.14}$$

We shall now construct a family of other (weak) Jacobi forms of weight k and index m by applying appropriate operators (known as Hecke operators) on $\phi_{k,1}(\tau, z)$[23, 22]. Let S_m be the set of 2×2 matrices satisfying the following relations:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in S_m \quad \text{if} \quad \alpha, \beta, \gamma, \delta \in \mathbb{Z}, \quad \gamma = 0 \bmod N, \quad \text{g.c.d.}(\alpha, N) = 1\,, \quad \alpha\delta - \beta\gamma = m\,. \tag{A.15}$$

The set S_m has the property that it is invariant under left multiplication by any element of $\Gamma_0(N)$. For proving this we need to use the fact that for a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, a satisfies the condition g.c.d.$(a, N) = 1$. This follows automatically from the conditions $ad - bc = 1$ and $c = 0$ mod N.

We denote by $\Gamma_0(N)\backslash S_m$ the left coset of S_m by $\Gamma_0(N)$ and define:

$$\phi_{k,m}(\tau, z) = m^{k-1} \sum_{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N)\backslash S_m} \chi(\alpha)(\gamma\tau+\delta)^{-k}\, e^{-2\pi i m\gamma z^2/(\gamma\tau+\delta)} \phi_{k,1}\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}, \frac{mz}{\gamma\tau+\delta}\right)\,. \tag{A.16}$$

For this definition of $\phi_{k,m}$ to be sensible, it must be independent of the representative matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ that we choose to describe an element of $\Gamma_0(N)\backslash S_m$. A different choice

$\begin{pmatrix} \widehat{\alpha} & \widehat{\beta} \\ \widehat{\gamma} & \widehat{\delta} \end{pmatrix}$ of the representative element, corresponding to multiplying $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ from the left by an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, amounts to replacing the summand in (A.16) by

$$\begin{aligned} & \chi(\widehat{\alpha})(\widehat{\gamma}\tau+\widehat{\delta})^{-k}\, e^{-2\pi i m\widehat{\gamma}z^2/(\widehat{\gamma}\tau+\widehat{\delta})}\phi_{k,1}\left(\frac{\widehat{\alpha}\tau+\widehat{\beta}}{\widehat{\gamma}\tau+\widehat{\delta}}, \frac{mz}{\widehat{\gamma}\tau+\widehat{\delta}}\right) \\ = \;& \chi(\widehat{\alpha})(c\tau'+d)^{-k}(\gamma\tau+\delta)^{-k}\, e^{-2\pi i c z'^2/(c\tau'+d)} e^{-2\pi i m\gamma z^2/(\gamma\tau+\delta)}\phi_{k,1}\left(\frac{a\tau'+b}{c\tau'+d}, \frac{z'}{c\tau'+d}\right) \\ & \qquad \tau' \equiv \frac{\alpha\tau+\beta}{\gamma\tau+\delta}, \quad z' \equiv \frac{mz}{\gamma\tau+\delta}\,. \end{aligned} \tag{A.17}$$

Using eq.(A.14) and the relation $\chi(\widehat{\alpha}) = \chi(\alpha)\chi(a)$ that follows from (A.12), we can easily show that (A.17) is equal to the summand in (A.16). Hence the right hand side of (A.16) is indeed independent of the representative element of $\Gamma_0(N)\backslash S_m$ that we use for the computation.

It is also straightforward to study the transformation laws of $\phi_{k,m}$ under a modular transformation by an element of $\Gamma_1(N)$, and under shift of z by $(\lambda\tau+\mu)$ for integer λ, μ. First consider the effect of modular transformation of the form

$$\tau \to \frac{a\tau+b}{c\tau+d}, \qquad z \to \frac{z}{c\tau+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)\,, \tag{A.18}$$

for which $\chi(a) = 1$. Defining

$$\begin{pmatrix} \widetilde{\alpha} & \widetilde{\beta} \\ \widetilde{\gamma} & \widetilde{\delta} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, \tag{A.19}$$

for which $\chi(\widetilde{\alpha}) = \chi(\alpha)\chi(a) = \chi(\alpha)$, one can show using (A.16), (A.7) that

$$\begin{aligned} \phi_{k,m}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) &= (c\tau+d)^k\, \exp(2\pi i m c z^2/(c\tau+d)) \\ \times m^{k-1} & \sum_{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in\Gamma_0(N)\backslash S_m} \chi(\widetilde{\alpha})\,(\widetilde{\gamma}\tau+\widetilde{\delta})^{-k}\, e^{-2\pi i m\widetilde{\gamma}z^2/(\widetilde{\gamma}\tau+\widetilde{\delta})}\phi_{k,1}\left(\frac{\widetilde{\alpha}\tau+\widetilde{\beta}}{\widetilde{\gamma}\tau+\widetilde{\delta}}, \frac{mz}{\widetilde{\gamma}\tau+\widetilde{\delta}}\right)\,. \end{aligned} \tag{A.20}$$

Since the sum over $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ runs over all the representative elements of $\Gamma_0(N)\backslash S_m$, we can reinterpret the sum over $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ as a sum over inequivalent choices of $\begin{pmatrix} \widetilde{\alpha} & \widetilde{\beta} \\ \widetilde{\gamma} & \widetilde{\delta} \end{pmatrix}$.

Comparing (A.20) with (A.16) we then get

$$\phi_{k,m}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k \exp(2\pi i m c z^2/(c\tau+d))\phi_{k,m}(\tau,z)\,. \tag{A.21}$$

Also, using (A.7) and (A.16) one can easily show that

$$\phi_{k,m}(\tau, z+\lambda\tau+\mu) = \exp\left(-2\pi i m(\lambda^2\tau+2\lambda z)\right)\phi_{k,m}(\tau,z)\,, \quad \lambda,\mu\in \mathbb{Z}\,. \tag{A.22}$$

This shows that $\phi_{k,m}(\tau,z)$ transforms as a weak Jacobi form of weight k and index m under $\Gamma_1(N)$.

As an aside, we note that under a $\Gamma_0(N)$ transformation $\phi_{k,m}$ transforms as

$$\phi_{k,m}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (\chi(a))^{-1}(c\tau+d)^k \exp(2\pi i m c z^2/(c\tau+d))\phi_{k,m}(\tau,z)\,. \tag{A.23}$$

This can be proven easily by keeping track of the $\chi(a)$ factors in (A.20), (A.21) instead of setting it to 1.

We shall now show that we can label the representative elements of $\Gamma_0(N)\backslash S_m$ by matrices of the form:

$$\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}, \quad \alpha,\beta,\delta\in \mathbb{Z}, \quad \alpha\delta=m, \quad \alpha>0, \quad \text{g.c.d.}(\alpha,N)=1, \quad 0\le\beta\le\delta-1\,. \tag{A.24}$$

For this let us consider the product:

$$\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix} \equiv \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \qquad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_0(N), \quad \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in S_m\,. \tag{A.25}$$

This gives:

$$\gamma' = c\alpha + d\gamma\,. \tag{A.26}$$

If we now choose

$$c = -\frac{\gamma}{\text{g.c.d.}(\alpha,\gamma)}, \qquad d = \frac{\alpha}{\text{g.c.d.}(\alpha,\gamma)}\,, \tag{A.27}$$

we get

$$\gamma' = 0\,. \tag{A.28}$$

Since γ is a multiple of N, and since α and N do not have any common factor, it follows that c given in (A.27) is a multiple of N. Furthermore from (A.27) it follows that c and d do not have any common factor. Hence it is always possible to find integers a and b

satisfying $ad - bc = 1$. This shows that by multiplying $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ by an appropriate $\Gamma_0(N)$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ from the left we can bring it to the form

$$\begin{pmatrix} \alpha' & \beta' \\ 0 & \delta' \end{pmatrix} . \tag{A.29}$$

If α' is negative, we can multiply (A.29) from the left by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ to make α' positive. We can also multiply (A.29) from the left by the $\Gamma_0(N)$ matrix

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \qquad k \in \mathbb{Z}, \tag{A.30}$$

to transform

$$\beta' \to \beta' + k\,\delta' , \tag{A.31}$$

preserving the $\gamma' = 0$ and $\alpha' > 0$ conditions. Thus by choosing k appropriately we can bring β' in the range $0 \le \beta' \le \delta' - 1$. Since the final matrix $\begin{pmatrix} \alpha' & \beta' \\ 0 & \delta' \end{pmatrix}$ must still be an element of S_m, it must have determinant m and g.c.d.$(\alpha', N) = 1$. This establishes that the representative elements of $\Gamma_0(N)\backslash S_m$ can be chosen as in (A.24).

Using (A.24) and (A.5) we can rewrite (A.16) as

$$\begin{aligned} \phi_{k,m}(\tau, z) &= m^{k-1} \sum_{\substack{\alpha,\delta \in \mathbb{Z}; \alpha>0 \\ \alpha\delta=m,\, \text{g.c.d.}(\alpha,N)=1}} \chi(\alpha)\, \delta^{-k} \sum_{\beta=0}^{\delta-1} \phi_{k,1}((\alpha\tau+\beta)\delta^{-1}, mz\delta^{-1}) \\ &= m^{k-1} \sum_{\substack{\alpha,\delta \in \mathbb{Z}; \alpha>0 \\ \alpha\delta=m,\, \text{g.c.d.}(\alpha,N)=1}} \chi(\alpha)\, \delta^{-k} \sum_{\beta=0}^{\delta-1} \sum_{\substack{n,r \in \mathbb{Z} \\ n\ge 1, r^2<4n}} C(4n-r^2) e^{2\pi i (n\delta^{-1}(\alpha\tau+\beta) + r\delta^{-1} mz)} . \end{aligned} \tag{A.32}$$

For fixed α, n, r, the sum over β is equal to δ if $n = 0 \bmod \delta$, and vanishes otherwise. Thus we get

$$\phi_{k,m}(\tau, z) = m^{k-1} \sum_{\substack{\alpha,\delta \in \mathbb{Z}; \alpha>0 \\ \alpha\delta=m,\, \text{g.c.d.}(\alpha,N)=1}} \chi(\alpha)\, \delta^{1-k} \sum_{\substack{n,r \in \mathbb{Z} \\ n\ge 1, r^2<4n, n\delta^{-1} \in \mathbb{Z}}} C(4n-r^2) e^{2\pi i (n\delta^{-1}\alpha\tau + r\delta^{-1} mz)} . \tag{A.33}$$

We now define:

$$n' = n\alpha/\delta, \qquad r' = mr/\delta . \tag{A.34}$$

Since $n\delta^{-1} \in \mathbb{Z}$, $m = \alpha\delta$ and $n, m \geq 1$ we see that

$$n', r' \in \mathbb{Z}, \qquad \alpha | (m, n', r'), \qquad n' \geq 1 \,. \tag{A.35}$$

Furthermore (A.35) is sufficient to find integers $n \geq 1$, r satisfying (A.34). Thus we can replace the sum over n and r in (A.33) by sum over n' and r' subject to the restriction given in (A.35):

$$\phi_{k,m}(\tau, z) = \sum_{\substack{n', r' \in \mathbb{Z} \\ n' \geq 1, r'^2 < 4mn'}} \exp(2\pi i(n'\tau + r'z)) \sum_{\substack{\alpha \in \mathbb{Z}; \alpha > 0 \\ \alpha | (m, n', r'),\, \text{g.c.d.}(\alpha, N) = 1}} \chi(\alpha)\, \alpha^{k-1}\, C\left(\frac{4mn' - r'^2}{\alpha^2}\right) . \tag{A.36}$$

This can be rewritten as

$$\phi_{k,m}(\tau, z) = \sum_{\substack{n, r \in \mathbb{Z} \\ n \geq 1,\, r^2 < 4mn}} a(n, m, r)\, e^{2\pi i(n\tau + rz)} \,, \tag{A.37}$$

where

$$a(n, m, r) = \sum_{\substack{\alpha \in \mathbb{Z}; \alpha > 0 \\ \alpha | (n, m, r),\, \text{g.c.d.}(\alpha, N) = 1}} \chi(\alpha)\, \alpha^{k-1}\, C\left(\frac{4mn - r^2}{\alpha^2}\right) . \tag{A.38}$$

Since $C(s)$ vanishes for $s \leq 0$, we have

$$a(n, m, r) = 0 \,, \qquad \text{for} \quad r^2 \geq 4mn \,. \tag{A.39}$$

B Proof of modular property of Φ_k

In this appendix we shall prove, following [23, 24], that Φ_k defined in (1.6):

$$\Phi_k(\rho, \sigma, v) = \sum_{\substack{n, m, r \in \mathbb{Z} \\ n, m \geq 1,\, r^2 < 4mn}} a(n, m, r)\, e^{2\pi i(n\rho + m\sigma + rv)} \,, \tag{B.1}$$

transforms as a modular form of weight k under the group G defined in (2.11), and also that for small v it has the factorization property described in (2.17).[10] We use (A.37) to rewrite (B.1) as

$$\Phi_k(\rho, \sigma, v) = \sum_{m \geq 1} \phi_{k,m}(\rho, v) e^{2\pi i m\sigma} \,, \tag{B.2}$$

[10]Note that due to eq.(A.39), the exponent appearing in (B.1) has strictly negative real part for (ρ, σ, v) satisfying (2.8), and hence the Fourier expansion (B.1) is sensible in the region (2.8) for large imaginary values of ρ and σ.

and study the transformation law of $\Phi_k(\rho,\sigma,v)$ under various special $Sp(2,\ \mathbb{Z})$ transformations:

$$\Omega \to (A\Omega+B)(C\Omega+D)^{-1}, \qquad \Omega \equiv \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} . \tag{B.3}$$

1. First consider the transformation generated by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad ad-bc=1, \qquad c = 0 \text{ mod } N\,, \quad a,d=1 \text{ mod } N\,. \tag{B.4}$$

   This corresponds to

$$\rho \to \rho' = \frac{a\rho+b}{c\rho+d}, \qquad v \to v' = \frac{v}{c\rho+d}\,, \qquad \sigma \to \sigma' = \sigma - \frac{cv^2}{c\rho+d}\,,$$
$$\det(C\Omega+D) = (c\rho+d)\,. \tag{B.5}$$

   Hence it follows from (B.2), (A.21) that

$$\Phi_k(\rho',\sigma',v') = \det(C\Omega+D)^k \Phi_k(\rho,\sigma,v)\,. \tag{B.6}$$

   As an aside we note that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, then due to (B.2), (A.23),

$$\Phi_k(\rho',\sigma',v') = (\chi(a))^{-1} \det(C\Omega+D)^k \Phi_k(\rho,\sigma,v)\,. \tag{B.7}$$

2. Next consider the transformation:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \lambda,\mu \in \ \mathbb{Z}\,. \tag{B.8}$$

   which induces the transformation:

$$\rho \to \rho' = \rho, \qquad v \to v' = v+\lambda\rho+\mu\,, \qquad \sigma \to \sigma' = \sigma + 2\lambda v + \lambda^2 \rho + \lambda\mu\,. \tag{B.9}$$

   It follows from (B.2), (A.22) that under this transformation:

$$\Phi_k(\rho',\sigma',v') = \Phi_k(\rho,\sigma,v) = \det(C\Omega+D)^k \Phi_k(\rho,\sigma,v)\,, \tag{B.10}$$

   since $\det(C\Omega+D)=1$.

3. The matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{B.11}$$

induces the transformation

$$\rho \to \rho' = \sigma, \qquad \sigma \to \sigma' = \rho, \qquad v \to v' = -v\,. \tag{B.12}$$

From (A.38), (B.1) it follows that $\Phi_k(\rho, \sigma, v)$ is symmetric under the exchange of ρ and σ and also under $v \to -v$. As a result we get, under (B.12)

$$\Phi_k(\rho', \sigma', v') = \Phi_k(\rho, \sigma, v) = \det(C\Omega + D)^k \Phi_k(\rho, \sigma, v)\,, \tag{B.13}$$

since $\det(C\Omega + D) = 1$.

This shows that $\Phi_k(\rho, \sigma, v)$ transforms as a modular form of weight k under the transformations (B.4), (B.8) and (B.11). Since these transformations generate the group G, it follows that Φ_k transforms as a modular form of weight k under the entire group G.

From (B.7) and the $\rho \leftrightarrow \sigma$ exchange symmetry (B.12) it also follows that under an $Sp(2,\ \mathbb{Z})$ transformation:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & -b \\ c & 0 & d & 0 \\ 0 & -c & 0 & d \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\,, \tag{B.14}$$

Φ_k transforms as

$$\Phi_k\left((A\Omega + B)(C\Omega + D)^{-1}\right) = \det(C\Omega + D)^k \Phi_k(\Omega)\,, \tag{B.15}$$

since the $\chi(a)$ factors arising from the $\Gamma_0(N)$ transformations acting on ρ and σ cancel.

We shall now turn to the study of Φ_k for small v and indicate the proof of eq.(2.17). For this we note from (A.1) and the relation

$$\vartheta_1(\tau, z) \simeq 2\pi\eta(\tau)^3 z + \mathcal{O}(z^3)\,, \tag{B.16}$$

that

$$\phi_{k,1}(\tau, z) = 4\pi^2\, f^{(k)}(\tau)\, z^2 + \mathcal{O}(z^4)\,, \tag{B.17}$$

for small z. Using eq.(A.16) we now see that each $\phi_{k,m}(\tau, z)$ also vanishes as z^2 for small z. (B.2) then gives, for small v,

$$\Phi_k(\rho, \sigma, v) = v^2 F_k(\rho, \sigma) + \mathcal{O}(v^4) \, , \tag{B.18}$$

where

$$F_k(\rho, \sigma) = 4\pi^2 \sum_{m \geq 1} e^{2\pi i m\sigma} \, m^{k+1} \sum_{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N) \backslash S_m} \chi(\alpha) (\gamma\rho + \delta)^{-k-2} \, f^{(k)} \left(\frac{\alpha\rho + \beta}{\gamma\rho + \delta} \right) . \tag{B.19}$$

It follows from the modular transformation property of $\Phi_k(\rho, \sigma, v)$ that

$$\begin{aligned} & F_k((a\rho + b)(c\rho + d)^{-1}, \sigma) = (c\rho + d)^{k+2} F_k(\rho, \sigma) \, , \\ & F_k(\rho, (a\sigma + b)(c\sigma + d)^{-1}) = (c\sigma + d)^{k+2} F_k(\rho, \sigma), \\ & \qquad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \, . \end{aligned} \tag{B.20}$$

Finally, since $f^{(k)}(\tau)$ vanishes at the cusps in the τ-plane, it follows from (B.19) that for fixed σ, $F_k(\rho, \sigma)$ vanishes at the cusps in the ρ plane. Due to the $\rho \leftrightarrow \sigma$ symmetry the same result is true in the σ-plane for fixed ρ. Hence $F_k(\rho, \sigma)$ vanishes at the cusps in the ρ as well as in the σ-plane.

The above results show that for fixed σ we can regard $F_k(\rho, \sigma)$ as a cusp form of $\Gamma_1(N)$ of weight $k + 2$ in the ρ-plane. Similarly for fixed ρ we can regard $F_k(\rho, \sigma)$ as a cusp form of $\Gamma_1(N)$ of weight $k + 2$ in the σ plane. Since these cusp forms are known to be unique and proportional to $f^{(k)}(\rho)$ and $f^{(k)}(\sigma)$ respectively[22], we get

$$F_k(\rho, \sigma) = C_0 \, f^{(k)}(\rho) f^{(k)}(\sigma) \tag{B.21}$$

where C_0 is a constant of proportionality. By examining the behaviour of both sides for large imaginary values of ρ and σ one can verify that $C_0 = 4\pi^2$. This proves (2.17):

$$\Phi_k(\rho, \sigma, v) \simeq 4\pi^2 \, v^2 \, f^{(k)}(\rho) \, f^{(k)}(\sigma) + \mathcal{O}(v^4) \, . \tag{B.22}$$

C Fourier expansion of $\widetilde{\Phi}_k$

In this appendix we shall show that $\widetilde{\Phi}_k(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ given in (2.35) has an expansion of the form described in (3.8) with integer coefficients $b(q, r, s)$. For this we use the Fourier expansion of Φ_k given in (B.2) with ρ, σ exchanged to express (2.35) as

$$\widetilde{\Phi}_k(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) = \sum_{m\geq 1} e^{2\pi i m\widetilde{\rho}}\,\widetilde{\sigma}^{-k}\,e^{-2\pi i m\widetilde{v}^2/\widetilde{\sigma}}\,\phi_{k,m}(-1/\widetilde{\sigma},\widetilde{v}/\widetilde{\sigma})\,. \tag{C.1}$$

Thus in order to find the Fourier expansion of $\widetilde{\Phi}_k$ in $\widetilde{\rho}$, $\widetilde{\sigma}$ and $\widetilde{v}$, we need to find the Fourier expansion of $\widetilde{\sigma}^{-k}\,e^{-2\pi i m\widetilde{v}^2/\widetilde{\sigma}}\,\phi_{k,m}(-1/\widetilde{\sigma},\widetilde{v}/\widetilde{\sigma})$ in $\widetilde{\sigma}$ and $\widetilde{v}$.

We first analyze $\phi_{k,1}$. From eqs.(A.1), (2.18), and the known modular transformation properties of $\eta(\tau)$ and $\vartheta_1(\tau,z)$:

$$\eta(-\frac{1}{\tau}) = (-i\tau)^{1/2}\eta(\tau)\,, \qquad \vartheta_1(-\frac{1}{\tau},\frac{z}{\tau}) = -i\,(-i\tau)^{1/2}e^{i\pi z^2/\tau}\,\vartheta_1(\tau,z)\,, \tag{C.2}$$

we see that

$$\phi_{k,1}(-1/\tau,z/\tau) = (i\sqrt{N})^{-k-2}\,\tau^k\,e^{2\pi i z^2/\tau}\,\eta(\tau)^{-6}\,f^{(k)}(\tau/N)\,\vartheta_1(\tau,z)^2\,. \tag{C.3}$$

Thus

$$\widetilde{\sigma}^{-k}\,e^{-2\pi i\widetilde{v}^2/\widetilde{\sigma}}\,\phi_{k,1}(-1/\widetilde{\sigma},\widetilde{v}/\widetilde{\sigma}) = (i\sqrt{N})^{-k-2}\,\eta(\widetilde{\sigma})^{-6}\,f^{(k)}(\widetilde{\sigma}/N)\,\vartheta_1(\widetilde{\sigma},\widetilde{v})^2\,. \tag{C.4}$$

If we define the coefficients $\widetilde{f}_n^{(k)}$ through

$$\eta(\tau)^{-6}f^{(k)}(\tau/N) = \sum_{\substack{n>0\\ nN\in\mathbb{Z}}} \widetilde{f}_n^{(k)} e^{2\pi i\tau(n-\frac{1}{4})}\,, \tag{C.5}$$

then due to eq.(A.2) $\eta(\tau)^{-6}f^{(k)}(\tau/N)\vartheta_1(\tau,z)^2$ has a Fourier expansion of the form:

$$\begin{aligned}\eta(\tau)^{-6}\,f^{(k)}(\tau/N)\,\vartheta_1(\tau,z)^2 &= \sum_{\substack{r,s\in\mathbb{Z}\\ r\geq 1}} d(r,s)e^{2\pi i r\tau/N+2\pi i s z}\\ \equiv\quad -e^{2\pi i\tau/N}e^{2\pi i z}\Big(1-2e^{-2\pi i z}+e^{-4\pi i z} &+ \sum_{\substack{r,s\in\mathbb{Z}\\ r\geq 1}} c(r,s)e^{2\pi i r\tau/N+2\pi i s z}\Big)\end{aligned} \tag{C.6}$$

where

$$d(r,s) = \widetilde{C}(4r/N-s^2)\,, \tag{C.7}$$

$$\widetilde{C}(m) = \sum_{\substack{l,nN\in\mathbb{Z}\\ n>0}} (-1)^{l+1}\,\widetilde{f}_n^{(k)}\delta_{4n+l^2-1,m}\,. \tag{C.8}$$

Since $\widetilde{f}_n^{(k)}$ are integers, $c(r,s)$ and $d(r,s)$ are also integers. Using (C.4) and (C.6) we get

$$\widetilde{\sigma}^{-k}\, e^{-2\pi i\widetilde{v}^2/\widetilde{\sigma}}\, \phi_{k,1}(-1/\widetilde{\sigma}, \widetilde{v}/\widetilde{\sigma}) = (i\sqrt{N})^{-k-2} \sum_{\substack{r,s\in \mathbb{Z}\\ r\geq 1}} d(r,s) e^{2\pi i r\widetilde{\sigma}/N + 2\pi i s\widetilde{v}} \,. \tag{C.9}$$

We now turn to the analysis of $\phi_{k,m}$. We begin with (A.16):

$$\phi_{k,m}(\tau,z) = m^{k-1} \sum_{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma_0(N)\backslash S_m} \chi(\alpha)\,(\gamma\tau+\delta)^{-k}\, e^{-2\pi i m\gamma z^2/(\gamma\tau+\delta)} \phi_{k,1}\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}, \frac{mz}{\gamma\tau+\delta}\right) . \tag{C.10}$$

Here $\alpha,\beta,\gamma,\delta$ satisfy

$$\alpha,\beta,\gamma,\delta\in \mathbb{Z}, \qquad \alpha\delta-\beta\gamma = m, \qquad \gamma = 0 \text{ mod } N, \qquad g.c.d.(\alpha, N) = 1\,. \tag{C.11}$$

We shall analyze (C.10) by choosing appropriate representative matrices $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$. However the representatives chosen in appendix A, which were useful for studying the Fourier expansion of Φ_k, will not be useful for studying the Fourier expansion of $\widetilde{\Phi}_k$, and we need to proceed differently. First consider the case $m\neq 0$ mod N. In this case we can find a matrix $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_0(N)$ such that

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha' & 0\\ \gamma' & \delta'\end{pmatrix}, \quad \alpha' > 0\,. \tag{C.12}$$

This is done by choosing

$$a = \mp\frac{\delta}{g.c.d.(\beta,\delta)}, \qquad b = \pm\frac{\beta}{g.c.d.(\beta,\delta)}\,, \tag{C.13}$$

where the overall sign of a and b is chosen so that $\alpha' = a\alpha + b\gamma > 0$. As long as $m\neq 0$ mod N, the conditions (C.11) ensures that $\delta\neq 0$ mod N. As a result a determined from (C.13) satisfies $a\neq 0$ mod N. In this case it is always possible to choose integers c and d satisfying

$$ad - bc = 1, \qquad c = 0 \text{ mod } N\,. \tag{C.14}$$

We shall use this freedom to choose the representative matrices in the sum in (C.10) to be of the form

$$\begin{pmatrix}\alpha & 0\\ \gamma & \delta\end{pmatrix}, \qquad \alpha\delta = m, \qquad \alpha > 0\,, \qquad \gamma = 0 \text{ mod } N, \qquad g.c.d.(\alpha, N) = 1\,. \tag{C.15}$$

Furthermore using the freedom of multiplying this matrix from the left by the $\Gamma_0(N)$ matrix

$$\begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}, \tag{C.16}$$

which shifts γ by αN, we can restrict γ to be of the form:

$$\gamma = \gamma_0 N, \qquad 0 \le \gamma_0 \le \alpha - 1 \, . \tag{C.17}$$

This gives

$$\begin{aligned} \phi_{k,m}(\tau, z) &= m^{k-1} \sum_{\substack{\alpha,\delta;\alpha>0 \\ \alpha\delta=m, g.c.d.(\alpha,N)=1}} \chi(\alpha) \sum_{\gamma_0=0}^{\alpha-1} (\gamma_0 N\tau + \delta)^{-k} e^{-2\pi i m \gamma_0 N z^2/(\gamma_0 N\tau+\delta)} \\ & \qquad \times \phi_{k,1}\left(\frac{\alpha\tau}{\gamma_0 N\tau + \delta}, \frac{mz}{\gamma_0 N\tau + \delta} \right) . \end{aligned} \tag{C.18}$$

The sum over α, δ, γ_0 run over integer values only. From now on this is the convention we shall be using for all summation indices unless mentioned otherwise. We now use (C.3) to reexpress (C.18) as

$$\begin{aligned} \phi_{k,m}(\tau, z) &= (i\sqrt{N})^{-k-2}\, m^{k-1} \sum_{\substack{\alpha,\delta;\alpha>0 \\ \alpha\delta=m, g.c.d.(\alpha,N)=1}} \chi(\alpha) \sum_{\gamma_0=0}^{\alpha-1} (-\alpha\tau)^{-k} e^{-2\pi i m z^2/\tau} \\ & \qquad \eta^{-6}\left(-\frac{\gamma_0 N\tau + \delta}{\alpha\tau} \right) f^{(k)}\left(-\frac{\gamma_0 N\tau + \delta}{N\alpha\tau} \right) \vartheta_1^2\left(-\frac{\gamma_0 N\tau + \delta}{\alpha\tau}, -\frac{mz}{\alpha\tau} \right) \end{aligned} \tag{C.19}$$

Setting $\tau = -1/\widetilde{\sigma}$, $z = \widetilde{v}/\widetilde{\sigma}$ in (C.19) we get

$$\begin{aligned} & e^{-2\pi i m \widetilde{v}^2/\widetilde{\sigma}}\, \widetilde{\sigma}^{-k}\, \phi_{k,m}\left(-\frac{1}{\widetilde{\sigma}}, \frac{\widetilde{v}}{\widetilde{\sigma}} \right) \\ = \; & (i\sqrt{N})^{-k-2} \sum_{\substack{\alpha,\delta;\alpha>0 \\ \alpha\delta=m, g.c.d.(\alpha,N)=1}} \chi(\alpha) \sum_{\gamma_0=0}^{\alpha-1} \alpha^{-1}\delta^{k-1}\eta^{-6}\left(-\frac{\gamma_0 N}{\alpha} + \frac{\delta}{\alpha}\,\widetilde{\sigma} \right) f^{(k)}\left(-\frac{\gamma_0}{\alpha} + \frac{\delta}{N\alpha}\,\widetilde{\sigma} \right) \\ & \times \vartheta_1^2\left(-\frac{\gamma_0 N}{\alpha} + \frac{\delta}{\alpha}\,\widetilde{\sigma}, \delta\,\widetilde{v} \right) . \end{aligned} \tag{C.20}$$

Using (C.6) we can express (C.20) as

$$\begin{aligned} & e^{-2\pi i m \widetilde{v}^2/\widetilde{\sigma}}\, \widetilde{\sigma}^{-k}\, \phi_{k,m}\left(-\frac{1}{\widetilde{\sigma}}, \frac{\widetilde{v}}{\widetilde{\sigma}} \right) \\ = \; & (i\sqrt{N})^{-k-2} \sum_{\substack{\alpha,\delta;\alpha>0 \\ \alpha\delta=m, g.c.d.(\alpha,N)=1}} \chi(\alpha) \sum_{\gamma_0=0}^{\alpha-1} \alpha^{-1}\delta^{k-1} \sum_{\substack{r,s\in \mathbb{Z} \\ r\ge 1}} d(r,s)\, e^{2\pi i r(-\alpha^{-1}\gamma_0 + \delta\alpha^{-1}\widetilde{\sigma}/N) + 2\pi i \delta s \widetilde{v}} . \end{aligned} \tag{C.21}$$

Performing the sum over γ_0 for fixed α, δ, r and s we see that the sum vanishes unless $r/\alpha \in \mathbb{Z}$, and is equal to α if $r/\alpha \in \mathbb{Z}$. Writing $r = r_0\alpha$, we get from (C.21):[11]

$$\begin{aligned} & e^{-2\pi i m \widetilde{v}^2/\widetilde{\sigma}}\, \widetilde{\sigma}^{-k}\, \phi_{k,m}\left(-\frac{1}{\widetilde{\sigma}}, \frac{\widetilde{v}}{\widetilde{\sigma}}\right) \\ = & \ (i\sqrt{N})^{-k-2} \sum_{\substack{\alpha,\delta;\alpha>0 \\ \alpha\delta=m, g.c.d.(\alpha,N)=1}} \chi(\alpha)\, \delta^{k-1} \sum_{\substack{r_0, s\in \mathbb{Z} \\ r_0\delta>0}} d(r_0\alpha, s)\, e^{2\pi i \delta r_0 \widetilde{\sigma}/N + 2\pi i \delta\, s\widetilde{v}}\,. \end{aligned} \tag{C.22}$$

Finally we turn to the case where $m = 0 \bmod N$. In this case eq.(C.11) gives $\delta = 0 \bmod N$. We split the sum in (C.10) into two parts:

$$\phi_{k,m}(\tau, z) = I_1(\tau, z) + I_2(\tau, z)\,, \tag{C.23}$$

where I_1 and I_2 represent the sum over $(\alpha, \beta, \gamma, \delta)$ restricted as follows:[12]

$$\begin{aligned} I_1 &: \frac{\delta}{g.c.d.(\beta,\delta)} \neq 0 \bmod N\,, \\ I_2 &: \frac{\delta}{g.c.d.(\beta,\delta)} = 0 \bmod N\,. \end{aligned} \tag{C.24}$$

For I_1 we can bring the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ into the form given in (C.15), (C.17) using transformations given in (C.12)-(C.14). The analysis now proceeds exactly as in the $m \neq 0 \bmod N$ case, and we get the analog of (C.22)

$$\begin{aligned} & e^{-2\pi i m \widetilde{v}^2/\widetilde{\sigma}}\, \widetilde{\sigma}^{-k}\, I_1(-1/\widetilde{\sigma}, \widetilde{v}/\widetilde{\sigma}) \\ = & \ (i\sqrt{N})^{-k-2} \sum_{\substack{\alpha,\delta;\alpha>0 \\ \alpha\delta=m, g.c.d.(\alpha,N)=1}} \chi(\alpha)\, \delta^{k-1} \sum_{\substack{r_0, s\in \mathbb{Z} \\ r_0\delta>0}} d(r_0\alpha, s)\, e^{2\pi i \delta r_0 \widetilde{\sigma}/N + 2\pi i \delta s\widetilde{v}}\,. \end{aligned} \tag{C.25}$$

For I_2 we cannot choose a as in (C.13) since this will give $a = 0 \bmod N$ and it will be impossible to find integers b, c satisfying (C.14). Instead we choose

$$c = -\frac{\delta}{g.c.d.(\beta,\delta)}, \qquad d = \frac{\beta}{g.c.d.(\beta,\delta)}\,, \tag{C.26}$$

[11]The condition $r_0\delta > 0$ in (C.22) follows from $r_0 m/\delta = r_0\alpha = r \geq 1$, and $m \geq 1$.

[12]It is easy to verify that the conditions on (β, δ) appearing in (C.24) are preserved under left multiplication of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ by an element of $\Gamma_0(N)$ and hence the splitting of the sum into I_1 and I_2 is independent of the choice of representative elements of $\Gamma_0(N)\backslash S_m$.

to bring the representative matrices in the sum in (C.10) to the form

$$\begin{pmatrix} \alpha & -\beta \\ \gamma & 0 \end{pmatrix}, \qquad \beta\gamma = m, \qquad \beta > 0, \qquad \gamma = 0 \text{ mod N} \qquad g.c.d.(\alpha, N) = 1\,. \tag{C.27}$$

Furthermore using the freedom of multiplying this matrix from the left by the $\Gamma_0(N)$ matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{C.28}$$

which shifts α by γ, we restrict α to the range $0 \le \alpha \le \gamma - 1$. Defining integers m_0 and γ_0 through

$$m_0 = m/N, \qquad \gamma_0 = \gamma/N\,, \tag{C.29}$$

we get, using eqs.(C.10), (C.24), (C.27),

$$I_2(\tau, z) = (m_0\, N)^{k-1} \sum_{\substack{\beta,\gamma_0;\beta>0 \\ \beta\gamma_0=m_0}} \sum_{\substack{\alpha=1 \\ g.c.d.(\alpha,N)=1}}^{\gamma_0 N-1} \chi(\alpha)\,(\gamma_0 N\tau)^{-k} e^{-2\pi i m z^2/\tau} \phi_{k,1}\left(\frac{\alpha\tau-\beta}{\gamma_0 N\tau}, \frac{m_0 z}{\gamma_0\tau}\right). \tag{C.30}$$

We now set $\tau = -1/\widetilde{\sigma}$, $z = \widetilde{v}/\widetilde{\sigma}$, and write $\alpha = s + pN$ with $1 \le s \le N-1$, $0 \le p \le \gamma_0 - 1$. This gives:

$$I_2(-1/\widetilde{\sigma}, \widetilde{v}/\widetilde{\sigma}) = \sum_{\substack{\beta,\gamma_0;\beta>0 \\ \beta\gamma_0=m_0}} \sum_{s=1}^{N-1} \sum_{p=0}^{\gamma_0-1} \chi(s)(-\widetilde{\sigma})^k \beta^{k-1} (\gamma_0 N)^{-1} e^{2\pi i m\widetilde{v}^2/\widetilde{\sigma}}\, \phi_{k,1}\left(\frac{s+pN}{\gamma_0 N} + \frac{\beta}{\gamma_0 N}\widetilde{\sigma}, -\beta\widetilde{v}\right). \tag{C.31}$$

Using (A.5) this may be expressed as:

$$\begin{aligned} & e^{-2\pi i m\widetilde{v}^2/\widetilde{\sigma}}\, \widetilde{\sigma}^{-k}\, I_2(-1/\widetilde{\sigma}, \widetilde{v}/\widetilde{\sigma}) \\ = \; & (-1)^k N^{-1} \sum_{\substack{\beta,\gamma_0;\beta>0 \\ \beta\gamma_0=m_0}} \beta^{k-1}\, (\gamma_0)^{-1} \sum_{s=1}^{N-1} \sum_{p=0}^{\gamma_0-1} \chi(s) \sum_{\substack{l,r \\ r^2<4l, l\ge 1}} C(4l-r^2) e^{2\pi i l \gamma_0^{-1} N^{-1}(s+pN+\beta\widetilde{\sigma}) - 2\pi i r\beta\widetilde{v}}\,. \end{aligned} \tag{C.32}$$

The sum over p produces a factor of γ_0 if $l = l_0\gamma_0$, $l_0 \in \mathbb{Z}$ and vanishes otherwise. This gives

$$\begin{aligned} & e^{-2\pi i m\widetilde{v}^2/\widetilde{\sigma}}\, \widetilde{\sigma}^{-k}\, I_2(-1/\widetilde{\sigma}, \widetilde{v}/\widetilde{\sigma}) \\ = \; & (-1)^k N^{-1} \sum_{\substack{\beta,\gamma_0;\beta>0 \\ \beta\gamma_0=m_0}} \beta^{k-1} \sum_{\substack{l_0,r \\ r^2<4l_0\gamma_0, l_0\gamma_0\ge 1}} C(4l_0\gamma_0 - r^2) e^{2\pi i l_0\beta\widetilde{\sigma}/N - 2\pi i r\beta\widetilde{v}} \sum_{s=1}^{N-1} e^{2\pi i l_0 s/N} \chi(s)\,. \end{aligned} \tag{C.33}$$

In this sum $l_0\beta = l_0 m/\gamma = l_0 m/\gamma_0 N > 0$ since $l_0\gamma_0 > 0$, $m > 0$, $N > 0$. Thus $\widetilde\sigma$ in the exponent always appears with positive coefficient. We now consider the cases $N \neq 7$ and $N = 7$ separately. For $N \neq 7$ we have $\chi(s) = 1$ and the sum over s gives $(N-1)$ if $l_0 = 0$ mod N and -1 if $l_0 \neq 0$ mod N. Since in this case $k \geq 2$ and even, and $C(s)$ are integers, we can express $I_2(-1/\widetilde\sigma, \widetilde v/\widetilde\sigma)$ as:

$$e^{-2\pi i m\widetilde v^2/\widetilde\sigma}\, \widetilde\sigma^{-k}\, I_2(-1/\widetilde\sigma, \widetilde v/\widetilde\sigma) = -(i\sqrt{N})^{-k-2} \sum_{\substack{r,p\in \mathbb{Z}\\ p\geq 1}} e_N(m_0,p,r)\, e^{2\pi i p\widetilde\sigma/N + 2\pi i r\widetilde v}\,, \quad \text{(C.34)}$$

with integer coefficients $e_N(m_0,p,r)$. For $N = 7$, we can use the identity:

$$\begin{aligned}\sum_{s=1}^{6} e^{2\pi i l_0 s/7}\chi(s) &= 0 \quad \text{for} \quad l_0 = 0 \mod N \\ &= i\sqrt{7}\,\chi(l_0) \quad \text{for} \quad l_0 \neq 0 \mod N\,, \end{aligned} \quad \text{(C.35)}$$

which follows from the values of $\chi(a)$ given in (A.13). Substituting this into (C.33), and noting that $k = 1$ for $N = 7$, we see that even in this case we can express $I_2(-1/\widetilde\sigma, \widetilde v/\widetilde\sigma)$ as:

$$e^{-2\pi i m\widetilde v^2/\widetilde\sigma}\, \widetilde\sigma^{-k}\, I_2(-1/\widetilde\sigma, \widetilde v/\widetilde\sigma) = -(i\sqrt{N})^{-k-2} \sum_{\substack{r,p\in \mathbb{Z}\\ p\geq 1}} e_7(m_0,p,r)\, e^{2\pi i p\widetilde\sigma/N + 2\pi i r\widetilde v}\,, \quad \text{(C.36)}$$

with integer coefficients $e_7(m_0,p,r)$.

It is now time to collect all the information together. Using eqs.(C.1), (C.9), (C.22)-(C.25), (C.34), (C.36) we get

$$\begin{aligned} &\widetilde\Phi_k(\widetilde\rho,\widetilde\sigma,\widetilde v) \\ = \;& (i\sqrt{N})^{-k-2} \sum_{m\geq 1} e^{2\pi i m\widetilde\rho} \sum_{\substack{\alpha,\delta;\alpha>0\\ \alpha\delta=m, g.c.d.(\alpha,N)=1}} \chi(\alpha)\,\delta^{k-1} \sum_{\substack{r_0,s\\ r_0\delta>0}} d(r_0\alpha,s)\, e^{2\pi i\delta r_0\widetilde\sigma/N + 2\pi i\delta\, s\widetilde v} \\ & -(i\sqrt{N})^{-k-2} \sum_{m_0\geq 1} e^{2\pi i m_0 N\widetilde\rho} \sum_{\substack{r,p\\ p\geq 1}} e_N(m_0,p,r) e^{2\pi i p\widetilde\sigma/N + 2\pi i r\widetilde v}\,. \end{aligned} \quad \text{(C.37)}$$

In the above expression the sum over the indices m, α, δ, r_0, s, r, p run over integer values. Using the integrality of the coefficients $d(r,s)$, $e_N(m_0,p,r)$ and the expression for $d(1,s)$ implicit in (C.6) we can rewrite this as

$$\begin{aligned}\widetilde\Phi_k(\widetilde\rho,\widetilde\sigma,\widetilde v) &= C\, e^{2\pi i\widetilde\rho + 2\pi i\widetilde\sigma/N + 2\pi i\widetilde v} \Big(1 - 2e^{-2\pi i\widetilde v} + e^{-4\pi i\widetilde v} \\ &\quad + \sum_{\substack{q,r,s\in \mathbb{Z}\\ q,r\geq 0, q+r\geq 1}} b(q,r,s)\, e^{2\pi i q\widetilde\rho + 2\pi i r\widetilde\sigma/N + 2\pi i s\widetilde v}\Big)\,, \end{aligned} \quad \text{(C.38)}$$

where $b(q,r,s)$ are integers and

$$C = -(i\sqrt{N})^{-k-2} \,. \qquad \text{(C.39)}$$

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