hep-th/0209122

Time and Tachyon

Ashoke Sen

Harish-Chandra Research Institute
Chhatnag Road, Jhusi, Allahabad 211019, INDIA
and
Abdus Salam International Centre for Theoretical Physics
Strada Costiera 11, 34014, Trieste, Italy
E-mail: asen@thwgs.cern.ch, sen@mri.ernet.in

Abstract

Recent analysis suggests that the classical dynamics of a tachyon on an unstable D-brane is described by a scalar Born-Infeld type action with a runaway potential. The classical configurations in this theory at late time are in one to one correspondence with the configuration of a system of non-interacting (incoherent), non-rotating dust. We discuss some aspects of canonical quantization of this field theory coupled to gravity, and explore, following earlier work on this subject, the possibility of using the scalar field (tachyon) as the definition of time in quantum cosmology. At late 'time' we can identify a subsector in which the scalar field decouples from gravity and we recover the usual Wheeler - de Witt equation of quantum gravity.

Contents

1 Introduction and Summary

Recent studies in time dependent solutions involving the tachyon[1, 2, 3] (see also [4, 5, 6, 7, 8, 9, 10, 11]) indicates that open string field theory on a non-BPS D-brane or a brane-antibrane pair has time dependent, spatially homogeneous classical solutions of arbitrary low energy density measured from the tachyon vacuum. Whereas the energy density associated with these solutions remains constant in time due to the usual conservation law, the pressure goes to zero asymptotically. This, in turn, suggests a specific form of the low energy effective action describing the dynamics of the tachyon near the minimum of the tachyon potential. If in particular we consider the dynamics of the tachyon on a space-filling D-brane system, then this low energy effective action will be an action in (9+1) or (25+1) dimensions in superstring and bosonic string theories respectively, and has the form

$$S = -\int d^{p+1}x V(T)\sqrt{1+\eta^{\mu\nu}\partial_\mu T\partial_\nu T}\,, \tag{1.1}$$

where $p = 9$ for type IIA or IIB superstring theories and 25 for the bosonic string theory. (Independent of this analysis, this form of the action has been suggested from other considerations[12, 13].) We are using the convention $\alpha' = \hbar = c = 1$ and $\eta^{\mu\nu} = \text{diag}(-1, 1, \ldots 1)$. The tachyon potential $V(T)$ has a maximum at $T = 0$, and behaves for large T as $e^{-\alpha T/2}$ where $\alpha = 1$ for bosonic string theory and $\sqrt{2}$ for the superstring theory[14].[1] Thus the minimum of the potential is at $T = \infty$. Unlike in the case of ordinary scalar field, there is no physical scalar particle associated with the perturbative quantization of the tachyon field T around the minimum of its potential[14], consistent with the conjecture put forward in [15].

The space-time independent classical solutions in this effective field theory include solutions with fixed energy density and asymptotically vanishing pressure, exactly as we see in explicit string calculation. On the other hand this effective field theory also contains other inhomogeneous classical solutions, and at late time these solutions are in one to one correspondence with configurations of non-interacting, non-rotating dust[14, 18].[2] These inhomogeneous solutions of the effective field theory can also be identified with specific classical solutions in string field theory on space-filling D-brane system as follows. First of all, beginning with sufficient number of space filling D-branes or brane anti-brane pairs, we can construct, as classical solutions, an arbitrary distribution of D0-$\bar{\text{D}}$0-brane pairs (for IIA string theory) or non-BPS D0-branes (for IIB or bosonic string theory) [15, 21]. We can now repeat the construction of time dependent solutions with arbitrary energy on each of these 0-branes. This gives a distribution of point objects, each carrying arbitrary energy, and by choosing an appropriate distribution of these objects with appropriate energies, we can construct a configuration of arbitrary energy density and zero pressure, exactly as we obtain from the classical solutions of the field theory described by the action (1.1). Thus it would seem that the effective action (1.1) captures at least to some extent the classical dynamics of open string field theory around the tachyon vacuum.

In this paper we consider coupling of the action (1.1) to gravity, and discuss some aspects of the quantum theory describing this coupled system.[3] Before we summarize the results, let us make a few cautionary remarks:

1. The tachyon effective action (1.1) has not been derived from first principles; it is only consistent with the time evolution of the energy-momentum tensor of various classical solutions in string theory at late time.

[1]This apparently differs from the potential computed in boundary string field theory[16], but as has been argued in [17], this could simply be an effect of field redefinition which includes derivative terms.

[2]Various other aspects of the action (1.1) have been discussed recently in [19, 20].

[3]Much work has been done recently on (semi-)classical analysis of this coupled system, keeping in mind possible application to cosmology[22].

2. Although the energy momentum tensor associated with the above-mentioned classical solutions in string theory approach finite limit asymptotically, the sources for some massive closed string states created by time-like oscillators acting on the vacuum grow exponentially with time[23, 17].[4] It is possible to add appropriate coupling of these massive closed string fields to the tachyon T appearing in (1.1) to reproduce these results; however the physical meaning of this exponential growth is not completely clear at present. In our analysis we shall only include the coupling of (1.1) to gravity. Some comments on the possible role of these exponentially growing terms in the boundary state will be made in section 4.

3. Even if the effective action (1.1) describes correctly the classical tachyon dynamics, quantizing this classical field theory coupled to gravity may not describe correctly the physics in quantum string theory.

Nevertheless, we take the point of view that the classical field theory described by the action (1.1) is an interesting field theory, and studying quantum properties in this theory is an interesting problem in its own right. It is in this spirit that the results of this paper should be interpreted.[5] We shall continue to refer to the scalar field T as tachyon for convenience.

The main results of our analysis are summarized below:

1. As already mentioned, at the classical level solutions of the equations of motion of the field theory described in (1.1) at late time are in one to one correspondence with configuration of non-rotating, non-interacting dust. Furthermore at late time (large x^0) the 'classical vacuum' solution of the equations of motion approach $T \simeq x^0$. This makes T a candidate for describing time at this classical level. We construct a more general class of field theories, of which the action (1.1) is a special case, all of which have the same property. This class of field theories have been analyzed independently in [19, 20] in the context of tachyon dynamics. (See also [25] for related studies.)

2. As is well-known, there is no natural notion of intrinsic time in classical general relativity, and this leads to a number of conceptual difficulties in formulating a quantm theory of gravity. (See [26, 27, 28, 29, 30], references therein, and citations

[4] A different kind of exponential growth, – that of the mass of the open string states on the original D-brane, – has been discussed recently in [24]. The relationship of this to the results of [23, 17] is also not clear.

[5] Of course (1.1) coupled to gravity is a non-renormalizable field theory, and so much of the analysis will have to be at a formal level. If this analysis turns out to be relevant for string theory then the ultraviolet divergence will be automatically regularized by string theory.

to these papers.) It has been argued earlier[27, 28] that some of these problems can be resolved in a theory with dust coupled to gravity, where one uses dust world-lines to foliate space-time, and takes the proper time along the world-line passing through any given space-time point as the definition of time at that point. Since the classical low energy configurations of the tachyon field theory can be represented as non-interacting, non-rotating dust, we could use this to give a definition of time. We show that this procedure leads to the identification of the value of the tachyon field at a given space-time point as the time coordinate associated to this point. Given this natural definition of time, one can rewrite the Hamiltonian constraints on the wave-functional (the Wheeler - de Witt equations) so that they take the form of many fingured time Schrodinger equation.

3. At late 'time', *i.e.* for large values of the field T, our results are identical to that of [27, 28] for non-rotating, incoherent dust coupled to gravity. In particular at late 'time' the quantum theory of the tachyon coupled to gravity has a subsector where the solution to the Wheeler - de Witt equation is independent of T. Hence the tachyon field decouples from gravity. and we recover the usual vacuum gravity. However this does not happen at early 'time', *i.e.* for small or finite values of T, and the solution to the Wheeler - de Witt equation necessarily has a non-trivial dependence on T in this region. This shows that in this theory we are inevitably led to 'time' dependent wave-functionals of the universe.

The rest of the paper is organised as follows. In section 2 we review the classical low energy effective action describing the tachyon dynamics, discuss the region of validity of this effective action and (approximate) conservation laws following from this effective action. We also argue, based on classical dynamics of the tachyon effective action, that it is natural to identify the tachyon as time coordinate. In section 3 we couple the tachyon effective action to gravity and analyze the constraints of general relativity for this coupled system. In particular, following [28] we show how the Hamiltonian constraints of general relativity can be simplified for this system, and the Wheeler - de Witt equations can be rewritten as many fingured time Schrodinger equation. We conclude the paper in section 4 with a few comments on various issues related to this paper.

2 Tachyon Without Gravity

We begin this section by reviewing the salient features of the classical field theory describing the tachyon field without coupling to gravity.

2.1 The action

The field theory proposed in [2, 14] describing the dynamics of the tachyon near the minimum of its potential is given by:

$$\begin{aligned} S &= \int d^{p+1}x\, \mathcal{L}\,, \\ \mathcal{L} &= -V(T)\sqrt{1+\eta^{\mu\nu}\partial_\mu T\partial_\nu T} \;=\; -V(T)\sqrt{-\det A}\,, \end{aligned} \tag{2.1}$$

where

$$A_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu T\partial_\nu T\,, \tag{2.2}$$

and

$$V(T) \simeq e^{-\alpha T/2} \quad \text{for large } T\,, \tag{2.3}$$

with

$$\begin{aligned} \alpha &= 1 \qquad \text{for bosonic string theory} \\ &= \sqrt{2} \qquad \text{for superstring theory}\,. \end{aligned} \tag{2.4}$$

The form of the potential $V(T)$ was derived by requiring that if we let the tachyon roll down from its maximum, then at late time the pressure associated with the configuration falls off as $K_0\exp(-\alpha x^0)$ for some constant K_0. This result, in turn, is derived from a stringy analysis of the boundary state associated with the rolling tachyon solution. Unless stated otherwise, in the rest of the analysis we shall take $(p+1)$ to be equal to the dimension of space-time (10 for superstring and 26 for bosonic string theory), so that (2.1) describes the dynamics of the tachyon on a space-filling brane system.

The energy momentum tensor computed from the action (2.1) is given by:

$$T_{\mu\nu} = \frac{V(T)\,\partial_\mu T\partial_\nu T}{\sqrt{1+\eta^{\rho\sigma}\partial_\sigma T\partial_\rho T}} - V(T)\,\eta_{\mu\nu}\sqrt{1+\eta^{\rho\sigma}\partial_\rho T\partial_\sigma T}\,. \tag{2.5}$$

2.2 Hamiltonian Formulation and Classical solutions

The solutions to the equations of motion described by the action (2.1) can be found easily by working in the hamiltonian formalism[31, 32, 33, 34, 14, 20]. Defining the momentum conjugate to T as:

$$\Pi(x) = \frac{\delta S}{\delta(\partial_0 T(x))} = \frac{V(T)\partial_0 T}{\sqrt{1-(\partial_0 T)^2+(\vec\nabla T)^2}}\,, \tag{2.6}$$

we can construct the Hamiltonian H:

$$H = \int d^p x \, (\Pi \partial_0 T - \mathcal{L}) \equiv \int d^p x \mathcal{H}, \qquad \mathcal{H} = T_{00} = \sqrt{\Pi^2 + (V(T))^2} \, \sqrt{1 + (\vec{\nabla} T)^2} \, . \quad (2.7)$$

The equations of motion derived from this hamiltonian take the form:

$$\partial_0 \Pi(x) = -\frac{\delta H}{\delta T(x)} = \partial_j \Big(\sqrt{\Pi^2 + V^2} \, \frac{\partial_j T}{\sqrt{1 + (\vec{\nabla} T)^2}} \Big) - \frac{V(T) V'(T)}{\sqrt{\Pi^2 + V^2}} \sqrt{1 + (\vec{\nabla} T)^2} \, , \quad (2.8)$$

$$\partial_0 T(x) = \frac{\delta H}{\delta \Pi(x)} = \frac{\Pi}{\sqrt{\Pi^2 + V^2}} \sqrt{1 + (\vec{\nabla} T)^2} \, . \quad (2.9)$$

In the limit of large T (*i.e.* near the tachyon vacuum) at fixed Π, we can ignore the $V^2 \simeq e^{-\alpha T}$ term, and the Hamiltonian and the equations of motion take the form:

$$H = \int d^p x \, |\Pi| \sqrt{1 + (\vec{\nabla} T)^2} \, , \quad (2.10)$$

$$\partial_0 \Pi(x) = \partial_j \Big(|\Pi| \, \frac{\partial_j T}{\sqrt{1 + (\vec{\nabla} T)^2}} \Big) \, , \quad (2.11)$$

$$\partial_0 T(x) = \frac{\Pi}{|\Pi|} \sqrt{1 + (\vec{\nabla} T)^2} \, . \quad (2.12)$$

From (2.12), we see that in this limit we have $(\partial_0 T)^2 - (\vec{\nabla} T)^2 = 1$.

These equations can be rewritten in a suggestive form by defining[14]

$$u_\mu \equiv -\partial_\mu T, \qquad \epsilon(x) \equiv |\Pi(x)| / \sqrt{1 + (\vec{\nabla} T)^2} \, . \quad (2.13)$$

Eqs.(2.11), (2.12) then take the form:

$$\eta^{\mu\nu} u_\mu u_\nu = -1, \qquad \partial_\mu (\epsilon(x) u^\mu) = 0 \, . \quad (2.14)$$

Expressed in terms of these new variables, $T_{\mu\nu}$ given in (2.5) take the form:

$$T_{\mu\nu} = \epsilon(x) u_\mu u_\nu \, , \quad (2.15)$$

where we have used the small $V(T)$ approximation and used the equations of motion (2.11), (2.12). These are precisely the equations governing the motion of non-rotating, non-interacting dust, with u_μ interpreted as the local $(p+1)$-velocity vector[14], and $\epsilon(x)$ interpreted as the local rest mass density. Conversely, any configuration describing flow of non-rotating, non-interacting dust can be interpreted as a solution of the equations of motion (2.11), (2.12).

2.3 Tachyon as time

Particular solutions of the equations (2.11), (2.12) can be obtained by taking

$$\Pi(x) = f(\vec{x}) \,, \qquad T(x) = x^0 \,, \tag{2.16}$$

where $f(\vec{x})$ is any arbitrary positive definite function of the spatial coordinates. The energy density associated with such a solution is proportinal to $f(\vec{x})$. If we further take $f(\vec{x})$ to be independent of $\vec{x}$, then we get a spatially uniform energy density.

Solution (2.16) naturally leads to a definition of the *classical vacuum* solution. Since the hamiltonian is positive semi-definite, the classical vacuum must have zero energy density. However we shall define this by a limiting procedure. If the tachyon begins rolling at some large value of T with a small potential energy ε, then no matter how small ε is, as the tachyon rolls down the potential hill towards large T, the system will eventually settle down to the configuration:

$$\Pi \simeq \varepsilon, \qquad T \simeq x^0 - \frac{1}{\alpha} \ln(2\alpha C \varepsilon^2) + C e^{-\alpha x^0}, \qquad V(T) \simeq \varepsilon \sqrt{2\alpha C}\, e^{-\alpha x^0/2} \tag{2.17}$$

for some constant C, so that the energy density $V(T)/\sqrt{1-(\partial_0 T)^2}$ remains constant at ε, and approaches $\Pi/\partial_0 T$ as $T \to \infty$[2]. We now define the classical vacuum solution as the $x^0 \to \infty$, $\varepsilon \to 0^+$ limit of this configuration. In this limit:

$$\Pi \to 0^+, \qquad \partial_0 T \to 1, \qquad V(T)/\Pi \to 0 \,. \tag{2.18}$$

For the vacuum solution $T(x) = x^0$ at late time. This indicates that it is natural to take $T(x)$ as the definition of the time coordinate in general. This definition of cosmic time agrees with a proposal of [27, 28] for defining the time coordinate in general relativity by coupling it to a system of incoherent, non-rotating dust. According to this proposal, we foliate the space-time by world-lines of dust, and the proper time along a dust world-line is to be interpreted as the time coordinate of various space-time points lying along that world-line. In our case, according to (2.13), the local $(p+1)$-velocity of the dust is given by $u_\mu = -\partial_\mu T$. Thus the proper time τ measured along a dust world-line satisfies:

$$d\tau = -u_\mu dx^\mu = dT \,. \tag{2.19}$$

Hence we see that we can identify τ with T. Thus T measures the proper time along any of the dust world-lines and hence can be identified with the cosmic time coordinate associated with any space-time point.

This analysis also shows that for any $V(T)/\Pi > 0$, however small, if we go sufficiently far back in time then the time dependence of the solution becomes non-trivial and $T = x^0$ is no longer a valid solution. Thus the identification of $T(x)$ as time breaks down at early time. We shall encounter a quantum version of this phenomenon in the next section.

2.4 Lorentz transformation properties

It is easy to derive the Lorentz transformation property of Π. Since $\partial_\mu T$ transforms as a covariant 4-vector, and $\eta^{\mu\nu}\partial_\mu T \partial_\nu T$ transforms as a scalar, we see, from eq.(2.9) that $\sqrt{(\Pi^2 + V^2)/(1 + (\vec{\nabla}T)^2)}$ transforms as a scalar, and $(\Pi\sqrt{1 + (\vec{\nabla}T)^2}/\sqrt{\Pi^2 + V^2}, \partial_i T)$ transforms as a covariant 4-vector. Thus

$$\left(\Pi, \frac{\sqrt{\Pi^2 + V^2}\,\partial_i T}{\sqrt{1 + (\vec{\nabla}T)^2}}\right) , \tag{2.20}$$

transforms as a covariant 4-vector. In the limit of large T where V vanishes, we see that

$$\left(\Pi, \frac{|\Pi|\,\partial_i T}{\sqrt{1 + (\vec{\nabla}T)^2}}\right) \tag{2.21}$$

transforms as a covariant 4-vector, and $|\Pi|/\sqrt{1 + (\vec{\nabla}T)^2}$ transforms as a scalar. This is consistent with the interpretation of $|\Pi|/\sqrt{1 + (\vec{\nabla}T)^2}$ as the local rest mass density, and of (2.21) as the local $(p+1)$-momentum density of the dust.

2.5 Approximate conservation laws

Let us work in the region $|\Pi| >> V(T)$, so that the equations (2.10)-(2.12) give valid approximation to the classical dynamics.[6] In this approximation the dynamics of the system has a peculiar property. Eq.(2.12) shows that

$$|\partial_0 T| \geq 1 . \tag{2.22}$$

Continuity of T in space-time will then imply that if some region in space-time has positive (negative) $\partial_0 T$, then all space-time points must have positive (negative) $\partial_0 T$. Eq.(2.12) also shows that the sign of $\partial_0 T$ is the same as the sign of Π. Hence $\Pi(x)$ must either be positive everywhere in space-time or negative everywhere in space-time.[7] Since the effect

[6] Since $T \simeq x^0$, this requires the energy density contained in the tachyon field to be large compared to $e^{-\alpha x^0/2}$. If we take x^0 to be the present age of the universe, then $e^{-\alpha x^0/2}$ is an incredibly small number, and an energy density of this order will be undetectable for all practical purpose.

[7] This argument of course is somewhat circular, since we have assumed that $|\Pi| > V(T)$ to begin with. What we want to emphasize here is that once we have arrived at (2.10)-(2.12) by taking this limit, then we can forget about the restriction $|\Pi| > V(T)$, and still configurations which are continuous in space-time must have $\Pi(x)$ positive or negative everywhere. See section 4 for more discussion on this. We also note that for a generic initial condition, time evolution of T typically leads to cautics[18] where the approximation leading to the original action (2.1) itself breaks down. Hence all our classical analysis will be applicable only in those regions of space-time which are free from these singularities.

of the potential $V(T) \propto \exp(-\alpha T/2)$ is to drive Π towards positive value, for studying late time dynamics of the system we can restrict $\Pi(x)$ to be positive everywhere in space-time as long as the field configuration remains non-singular. Eqs.(2.10)-(2.12) can then be rewritten as

$$H = \int d^p x \, \Pi \sqrt{1 + (\vec{\nabla} T)^2} \,, \tag{2.23}$$

$$\partial_0 \Pi(x) = \partial_j \Big(\Pi(x) \, \frac{\partial_j T}{\sqrt{1 + (\vec{\nabla} T)^2}} \Big) \,, \tag{2.24}$$

$$\partial_0 T(x) = \sqrt{1 + (\vec{\nabla} T)^2} \,. \tag{2.25}$$

The local rest mass density $\epsilon(x)$ is now given by $\Pi(x)/\sqrt{1 + (\vec{\nabla} T)^2}$, and preservation of the condition $\Pi > 0$ by the equations of motion (2.24), (2.25) is simply the statement that the local rest mass density of the dust remains positive during the time evolution. Consistency of this restriction on Π is also confirmed by noting from (2.21) that $\Pi(x)$ is the time component of a time-like vector field, and hence the condition $\Pi(x) > 0$ is Lorentz invariant.

In this limit, the quantity,

$$\Pi_0(x^0) \equiv \int d^p x \, \Pi(x) \,, \tag{2.26}$$

is conserved. Indeed, We see from (2.23) that the Poisson bracket of H and Π_0 vanishes:

$$\{H, \Pi_0\} = 0 \,. \tag{2.27}$$

2.6 Region of validity of (2.1)

Of course, (2.1) cannot be the complete answer for the tachyon effective action in string theory; it can only be a good approximation in some limit. This effective action ignores all quantum corrections, and even at the classical level the form of this action has not been derived from first principles. We conjecture that this is a good approximation to the full classical effective action if the following criteria are satisfied:

1. The tachyon field T must be large. Only in this limit we expect the lagrangian density to have a factorized form as given in (2.1).

2. The second and higher derivatives of the tachyon must be small. In particular even when T is large, we do not expect (2.1) (or more generally (2.30)) to be a valid description of the dynamics of the tachyon unless the second and higher derivatives

of the tachyon are small. This can be seen as follows. For uniform tachyon field, (2.5) gives:

$$T_{00} = V(T)/\sqrt{1-(\partial_0 T)^2}\,, \qquad T_{ij} = -V(T)\,\sqrt{1-(\partial_0 T)^2}\,\delta_{ij}\,, \qquad T_{i0} = 0\,. \quad (2.28)$$

Thus the energy-momentum tensor of the system at an instant when $\partial_0 T = 0$ must be of the form $-V(T)\,\eta_{\mu\nu}$. This in particular implies that the pressure p is equal to the negative of the energy density ρ at this instant. This result also holds for the more general action (2.30) discussed later. This however does not agree with the exact stringy formula for the evolution of p and ρ calculated from string theory[2]:[8]

$$\rho(x^0) = \frac{1}{2}\,\mathcal{T}_p\,(1+\cos(2\widetilde{\lambda}\pi)), \qquad p(x^0) = -\mathcal{T}_p\left[\frac{1}{1+\sin(\widetilde{\lambda}\pi)e^{x^0}} + \frac{1}{1+\sin(\widetilde{\lambda}\pi)e^{-x^0}} - 1\right], \quad (2.29)$$

where $\mathcal{T}_p = V(0)$ is the energy density (tension) of the static unstable configuration at the top of the tachyon potential, and $\widetilde{\lambda}$ is a parameter labelling the total energy density of the system. From this we see that at $x^0 = 0$, where the tachyon begins rolling with $\partial_0 T = 0$, the ratio p/ρ does not approach -1 even for $\widetilde{\lambda} \simeq 1/2$ which corresponds to low energy density, and hence large initial value of the tachyon field T. This in turn implies that the higher derivative terms must be contributing to the effective action. Only at late time, when $\partial_0 T$ approaches 1, and hence the second and higher derivatives of T go to zero, the effective action (2.1) is valid, and gives $p \propto e^{-\alpha x^0}$. For $\alpha = 1$ this corrctly reproduces (2.29) for large x^0.

Note that the constraint that the second and higher derivatives of the tachyon must be small implies that this description is expected to be valid as long as $\partial_\mu u_\nu$ is small in magnitude (in string units), where $u_\mu = -\partial_\mu T$ is the local velocity vector of the dust. As pointed out in [18], however, actual classical evolution of the equations of motion beginning with a generic inhomogeneous tachyon field configuration runs into the problem that $\partial_\mu u_\nu$ becomes large during the course of evolution, and hence we will need to invoke stringy modification of the dynamics to follow the evolution of the system near these points.

3. We also expect that the description of the system by a low energy effective action will be meaningful only when the energy density of the system is small. This will require $V(T)/\sqrt{1-(\partial_0 T)^2}$ to be small.

[8]Here we are quoting the results for bosonic string theory, but the results for superstring theory lead to a similar conclusion.

2.7 Other Lagrangians leading to the same limiting Hamiltonian

Before we conclude this section, we would like to point out that the limiting form (2.10) of the Hamiltonian can in fact be obtained from a more general class of Lagrangians than considered here. (This analysis has been done independently in [20], see also [19].) In fact, based on the analysis in boundary string field theory, a different form of the action for describing the classical tachyon dynamics was proposed in refs.[5, 6]. The action (2.1), as well as the action discussed in [5, 6], can be thought of as special cases of a general class of actions of the form:

$$S = -\int d^{p+1}x\, V(T)\, F(\eta^{\mu\nu}\partial_\mu T\partial_\nu T)\,, \tag{2.30}$$

where V and F are two functions with the property that $V(T)$ goes to zero as $T\to\infty$ from positive side, and $F(u)$ and its derivatives are well defined in the range $-1<u\le 0$, and has a singularity at $u=-1$ where $F'(u)\to+\infty$, and $F(u)/F'(u)$ vanishes. (The location of the singularity of $F(u)$ can be changed by a rescaling of the field T; we have chosen a convenient normalization so that the singularity is at -1.) The energy-momentum tensor computed from this action is given by:

$$T_{\rho\sigma} = 2V(T)F'(\eta^{\mu\nu}\partial_\mu T\partial_\nu T)\partial_\rho T\partial_\sigma T - V(T)F(\eta^{\mu\nu}\partial_\mu T\partial_\nu T)\eta_{\rho\sigma}\,. \tag{2.31}$$

For spatially uniform tachyon field T_{00} must be conserved. Since as T rolls towards infinity, $V(T)\to 0$, in order to keep T_{00} fixed $\partial_0 T$ must approach its critical value 1 where $F'(\eta^{\mu\nu}\partial_\mu T\partial_\nu T)$ blows up. Eq.(2.31) then shows that the pressure, being proportional to $-V(T)F(\eta^{\mu\nu}\partial_\mu T\partial_\nu T)\propto -T_{00}F(\eta^{\mu\nu}\partial_\mu T\partial_\nu T)/F'(\eta^{\mu\nu}\partial_\mu T\partial_\nu T)$ in this limit, vanishes. By adjusting the form of $V(T)$ for large T and the behaviour of the function $F(u)$ near $u\simeq -1$ we can ensure that the pressure falls off as $\exp(-\alpha x^0)$ for large x^0.

The momentum conjugate to T, computed from the action (2.30), is given by:

$$\Pi = 2V(T)\,\partial_0 T\, F'(\eta^{\mu\nu}\partial_\mu T\partial_\nu T)\,. \tag{2.32}$$

The Hamiltonian is given by:

$$H \equiv \int d^p x\, \mathcal{H}(x)\,, \tag{2.33}$$

where

$$\mathcal{H}(x) = T_{00}(x) = 2V(T)F'(\eta^{\mu\nu}\partial_\mu T\partial_\nu T)(\partial_0 T)^2 + V(T)F(\eta^{\mu\nu}\partial_\mu T\partial_\nu T)\,. \tag{2.34}$$

For $\eta^{\mu\nu}\partial_\mu T\partial_\nu T\simeq -1$, we have

$$\partial_0 T = \pm\sqrt{1+(\vec\nabla T)^2}\,, \tag{2.35}$$

and the first term in (2.34) dominates the second one. Using (2.32) and (2.35) we now get:

$$\mathcal{H} \simeq |\Pi| \sqrt{1 + (\vec{\nabla} T)^2}\,, \tag{2.36}$$

which is the same as the one given in (2.10). Thus the equations of motion near this critical point are identical to those given in (2.11), (2.12). The energy momentum tensor, expressed in terms of $\Pi(x)$ also has the same form as given in eqs.(2.13)-(2.15) in this limit, since the second term on the right hand side of (2.31) can be neglected. Thus the system near its critical point describe non-rotating, non-interacting dust exactly as the system analyzed earlier. For explicit analysis we shall continue to work with the action (2.1)-(2.3), but many of our results will be valid also for this more general class of field theories.

3 Coupling Tachyon Effective Action to Gravity

The coupling of the tachyon effective action (2.1) to supergravity fields (including the world-volume gauge and fermionic fields on the unstable D-brane) has been described in [35, 13, 17, 7]. In this section we restrict ourselves to an analysis of the coupling of (2.1) to gravitational field, discuss how it modifies the constraints of general relativity, and rewrite the Wheeler - de Witt equation for this coupled system as many fingured time Schodinger equation. We follow closely the analysis of [28].

3.1 The action

For simplicity we shall work with the action (2.1), but identical results can be derived for the general action (2.30) near the critical point where T is large and $g^{\mu\nu}\partial_\mu T \partial_\nu T$ is close to -1. If $g_{\mu\nu}$ denotes the space-time metric, then the action (2.1) coupled to gravity takes the form:

$$S = -\int d^{p+1}x\, V(T)\, \sqrt{-\det g}\, \sqrt{1 + g^{\mu\nu}\partial_\mu T \partial_\nu T}\,. \tag{3.1}$$

We now consider foliation of space-time by space-like hypersurfaces Σ parametrized by coordinates ξ^a $(1 \le a \le p)$ and label the different leaves of foliation by the parameter t. Then the space-time coordinates x^μ can be thought of as functions of (t, ξ^a). $\partial_a x^\mu$ for $a = 1, \ldots p$) are the p vectors tangential to Σ. We define by n^μ the unit time-like vector normal to Σ:

$$n^\mu g_{\mu\nu} \partial_a x^\nu = 0, \qquad g_{\mu\nu} n^\mu n^\nu = -1\,. \tag{3.2}$$

We also define the lapse function $N^\perp$ and the shift vector N^a through the decomposition:

$$\partial_t x^\mu = N^\perp n^\mu + N^a \partial_a x^\mu\,, \tag{3.3}$$

and denote by

$$h_{ab} = g_{\mu\nu}\partial_a x^\mu \partial_b x^\nu \tag{3.4}$$

the induced metric on Σ, and by h^{ab} the matrix inverse of h_{ab}. All information about the metric $g_{\mu\nu}$ is contained in h_{ab}, N^a and $N^\perp$. We have

$$g^{\mu\nu} = -n^\mu n^\nu + h^{ab}\partial_a x^\mu \partial_b x^\nu\,, \qquad \sqrt{-\det g}\, d^{p+1}x = N^\perp \sqrt{\det h}\, dt\, d^p\xi\,. \tag{3.5}$$

Using these relations we can rewrite (3.1) as

$$\begin{aligned} S &= \int dt \int_\Sigma d^p\xi\, \mathcal{L}\,, \\ \mathcal{L} &= -N^\perp \sqrt{\det h}\, V(T)\sqrt{1-(N^\perp)^{-2}(\partial_t T - N^a\partial_a T)(\partial_t T - N^b\partial_b T) + h^{ab}\partial_a T\partial_b T}\,, \end{aligned} \tag{3.6}$$

where $\partial_t T = \partial_t x^\mu \partial_\mu T$ and $\partial_a T = \partial_a x^\mu \partial_\mu T$.

3.2 The Hamiltonian and the constraints

From (3.6) we can compute the momentum $\Pi(t,\xi)$ conjugate to T as

$$\Pi(t,\vec{\xi}) = \frac{\delta S}{\delta(\partial_t T)} = \frac{\sqrt{\det h}\, V(T)(N^\perp)^{-1}(\partial_t T - N^c\partial_c T)}{\sqrt{1-(N^\perp)^{-2}(\partial_t T - N^a\partial_a T)(\partial_t T - N^b\partial_b T) + h^{ab}\partial_a T\partial_b T}}\,. \tag{3.7}$$

The Hamiltonian associated with the tachyon effective action in the background gravitational field is given by:

$$H^T = \int d^p\xi\, (\Pi(t,\vec{\xi})\partial_t T - \mathcal{L}) = \int d^p\xi\, (N^\perp \mathcal{H}^T_\perp + N^a \mathcal{H}^T_a)\,, \tag{3.8}$$

where

$$\mathcal{H}^T_\perp = \sqrt{\Pi^2 + V(T)^2 \det h}\,\sqrt{1 + h^{ab}\partial_a T\partial_b T}\,, \qquad \mathcal{H}^T_a = \Pi\,\partial_a T\,. \tag{3.9}$$

The full hamiltonian, when we treat the metric also as a dynamical variable, and include the standard Einstein-Hilbert term in the action, is given by

$$H^{total} = \int d^p\xi\,[N^\perp(\mathcal{H}^T_\perp + \mathcal{H}^G_\perp) + N^a(\mathcal{H}^T_a + \mathcal{H}^G_a)] \,+\, \text{boundary terms}\,, \tag{3.10}$$

where $\mathcal{H}^G_\perp$ and $\mathcal{H}^G_a$ are the contributions to $\mathcal{H}_\perp$ and $\mathcal{H}_a$ from the Einstein's action:

$$\mathcal{H}^G_\perp = G_{abcd}p^{ab}p^{cd} - \sqrt{\det h}\, R(h)\,, \qquad \mathcal{H}^G_a = -2D_b p^b_a\,. \tag{3.11}$$

Here p^{ab} is the momentum conjugate to h_{ab}, $R(h)$ is the Ricci scalar on Σ with metric h_{ab}, D_a denotes covariant derivative along Σ, and

$$G_{abcd} = \frac{1}{2}(\det h)^{-1/2}\,(h_{ac}h_{bd} + h_{ad}h_{bc} - \frac{2}{p-1}\,h_{ab}h_{cd})\,. \tag{3.12}$$

The boundary terms appearing in (3.10) have been discussed in [36]. In our analysis we shall not be careful in keeping track of these terms. For a discussion of boundary terms for non-interacting, non-rotating dust coupled to gravity see, for example, ref.[37]. We can include in $H^G_\perp$ and H^G_a the contribution from any other standard matter field coupled to gravity without affecting the rest of the analysis.

The constraints of general relativity can now be written as

$$\mathcal{H}^T_\perp + \mathcal{H}^G_\perp = 0\,, \tag{3.13}$$

and

$$\mathcal{H}^T_a + \mathcal{H}^G_a = 0\,. \tag{3.14}$$

Since from (3.9) we see that $H^T_\perp$ is positive, (3.13) forces $H^G_\perp$ to be negative.

For large T, we can ignore the $V(T)$ term in eq.(3.9). The hamiltonian equations of motion for T then gives

$$\partial_0 T = \frac{\Pi}{|\Pi|}\,N^\perp\,\sqrt{1 + h^{ab}\partial_a T\partial_b T}\, + N^a\partial_a T\,. \tag{3.15}$$

This, together with (3.3), (3.5), gives $g^{\mu\nu}\partial_\mu T\partial_\nu T = -1$. As in section 2, this allows us to restrict $\partial_\mu T$ to be a future pointing time-like vector field.[9] On the other hand, using (3.3), (3.7), we see that Π is given by a positive quantity multiplying $n^\mu\partial_\mu T$. This, being an inner product of a future pointing contravriant vector n^μ and a future pointing covariant vector $\partial_\mu T$ (see footnote 9 for convention) is positive definite. This allows us to restrict $\Pi(\vec{\xi}, t)$ to be positive. Using (3.9) we can now write the hamiltonian constraint (3.13) in this approximation as

$$\Pi + (1 + h^{ab}\partial_a T\partial_b T)^{-1/2}\mathcal{H}^G_\perp = 0\,. \tag{3.16}$$

We can also rewrite this using (3.14) as[28]

$$\mathcal{H}_\uparrow \equiv \Pi + H^G_\uparrow = 0, \qquad H^G_\uparrow \equiv -\sqrt{(\mathcal{H}^G_\perp)^2 - h^{ab}\mathcal{H}^G_a\mathcal{H}^G_b} = 0\,, \tag{3.17}$$

where we take the positive square root of $((\mathcal{H}^G_\perp)^2 - h^{ab}\mathcal{H}^G_a\mathcal{H}^G_b)$ in (3.17) so that $H^G_\uparrow$ is negative. In this form, Π appears linearly in this constraint, and there is no T-dependence.

[9] We are using the convention that either for covariant or a contravariant vector, a future pointing time-like vector will have its time component positive in a locally inertial frame of reference. In this convention if n^μ is a future pointing time-like vector, then $-n_\mu$ will be a future pointing time-like vector.

3.3 Constraint algebra

The algebra of the costraints $\mathcal{H}_\uparrow$ and $\mathcal{H}_a$ can be easily computed. In particular one can show, following [28], that

$$
\begin{aligned}
\{\mathcal{H}^G_\uparrow(t,\vec{\xi}), \mathcal{H}^G_\uparrow(t,\vec{\xi}')\} &= 0\,, \\
\{\mathcal{H}^G_\uparrow(t,\vec{\xi}), \mathcal{H}^G_a(t,\vec{\xi}')\} &= \mathcal{H}^G_\uparrow(t,\vec{\xi}')\partial_a\delta(\vec{\xi}-\vec{\xi}')\,, \\
\{\mathcal{H}^G_a(t,\vec{\xi}), \mathcal{H}^G_b(t,\vec{\xi}')\} &= H^G_b(t,\xi)\partial_a\delta(\vec{\xi}-\vec{\xi}') - H^G_a(t,\xi')\partial'_b\delta(\vec{\xi}-\vec{\xi}')\,. \qquad (3.18)
\end{aligned}
$$

Using this and the fact that Π and T have vanishing Poisson bracket with H^G_a and $H^G_\uparrow$, we get

$$
\begin{aligned}
\{\mathcal{H}_\uparrow(t,\vec{\xi}), \mathcal{H}_\uparrow(t,\vec{\xi}')\} &= 0\,, \\
\{\mathcal{H}_\uparrow(t,\vec{\xi}), \mathcal{H}_a(t,\vec{\xi}')\} &= \mathcal{H}_\uparrow(t,\xi')\partial_a\delta(\vec{\xi}-\vec{\xi}')\,, \qquad (3.19)
\end{aligned}
$$

and

$$
\{\mathcal{H}_a(t,\vec{\xi}), \mathcal{H}_b(t,\vec{\xi}')\} = H_b(t,\xi)\partial_a\delta(\vec{\xi}-\vec{\xi}') - H_a(t,\xi')\partial'_b\delta(\vec{\xi}-\vec{\xi}')\,. \qquad (3.20)
$$

Eqs.(3.19), (3.20) generate the algebra of constraints.

3.4 Wheeler-de Witt equation in the presence of tachyon field

We shall now proceed with the assumption that the 'late time' quantum dynamics of the system will be described by implementing (3.14) and (3.16) (or (3.17)) in the quantum theory.[10] Since these constraints take identical form to those discussed in [28] for non-rotating, incoherent dust, we can proceed exactly as in [28] to analyze the Wheeler - de Witt equation. In particular, the constraint (3.14), applied on the wave-functional $\Psi[T(\vec{\xi}), h_{ab}(\vec{\xi})]$, takes the form:

$$
i\partial_a T\,\frac{\delta\Psi}{\delta T(\vec{\xi})} = \widehat{\mathcal{H}}^G_a(\vec{\xi})\ \Psi\,, \qquad (3.21)
$$

[10] Of course, since the eigenstates of complete $\mathcal{H}^T_\perp$ are given by superpositions of positive and negative Π eigenstates, it is not true that the solutions of eqs.(3.21) and (3.22) or (3.23) will also be solutions of corresponding constraint equations when we use the full $\mathcal{H}^T_\perp$ given in (3.9) without the $\Pi > 0$ restriction. What we are assuming here is that eqs.(3.21) - (3.23) give a quantum theory which is approximately *equivalent* to the quantum theory associated with full $\mathcal{H}^T_\perp$. This is in the same spirit in which we can get a one to one correspondence between the eigenstates of a quantum mechanical operator $\widehat{\sqrt{\Pi^2+V^2}}$ without any restriction on Π with that of $\widehat{\Pi}$ under the restriction $\Pi > 0$, both operators having a continuous spectrum of positive eigenvalues, and having approximately the same density of states at low energy.

where the hat on top of a variable indicates that it is regarded as an operator. On the other hand, the constraint (3.17) gives rise to the 'many fingured time Schrodinger equation' with $T(\vec{\xi})$ interpreted as the time coordinate:

$$i\frac{\delta\Psi}{\delta T(\vec{\xi})} = \widehat{H}^G_\uparrow(\vec{\xi})\ \Psi\,. \tag{3.22}$$

Note that in order that $\widehat{H}^G_\uparrow(\vec{\xi})$ be well defined, we need to restrict the allowed space of functionals $\Psi[\{T(\vec{\xi})\},\{h_{ab}(\vec{\xi})\}]$ to be in the subspace spanned by the eigenvectors of $(\mathcal{H}^G_\perp(\vec{\xi}))^2 - h^{ab}(\vec{\xi})\mathcal{H}^G_a(\vec{\xi})\mathcal{H}^G_b(\vec{\xi}))$ with positive eigenvalue. Various subtleties arising from this restriction have been discussed in [28] and we shall not address these issues here.

We can also use (3.16) instead of (3.17) to replace (3.22) by

$$i\frac{\delta\Psi}{\delta T(\vec{\xi})} = \left[\widehat{\left(1 + h^{ab}(\vec{\xi})\partial_a T(\vec{\xi})\partial_b T(\vec{\xi})\right)^{-1/2}\mathcal{H}^G_\perp(\vec{\xi})}\right]\ \Psi\,. \tag{3.23}$$

In this case we require $\Psi[\{T(\vec{\xi})\},\{h_{ab}(\vec{\xi})\}]$ to be in the subspace of negative $\mathcal{H}^G_\perp(\vec{\xi})$ eigenvalue. Eq.(3.23) then automatically projects Ψ into states of positive $\Pi(\vec{\xi})$ eigenvalue. As discussed in [28], it is not clear whether (3.22) and (3.23) lead to equivalent quantization.

3.5 Recovering the physics without tachyon field

In the tachyon field had been absent, the constraints of general relativity take the form:

$$\widehat{\mathcal{H}}^G_a(\vec{\xi})\ \Psi = 0\,, \qquad \widehat{H}^G_\uparrow(\vec{\xi})\ \Psi = 0\,, \tag{3.24}$$

or in a more conventional form:

$$\widehat{\mathcal{H}}^G_a(\vec{\xi})\ \Psi = 0\,, \qquad \widehat{H}^G_\perp(\vec{\xi})\ \Psi = 0\,. \tag{3.25}$$

Given any solution of the constraints (3.24), we can regard this as a T-independent solution of (3.22), (3.21). On the other hand, any solution of the constraints (3.25) can be regarded as a T-independent solution of (3.21), (3.23). This allows us to recover the theory without tachyon field as a subsector of the theory with tachyon field. Note that this does not happen when we couple a conventional free scalar field to gravity, since even for a free scalar field, the energy-momentum tensor receives contribution not only from the momentum conjugate to the scalar, but also from the spatial gradient terms and the mass terms for the scalar field. Thus acting on an initial wave-functional that is independent of the scalar field, the contribution from the momentum term vanishes, but both the mass^2 term as well as the spatial gradient terms will modify the Wheeler-de

Witt equation. (In perturbation theory this is reflected in the fact that even if we begin with a state containing only gravitons, their scattering can pair produce quanta of this scalar field.)

The decoupling of the tachyon at late 'time' is consistent with the fact that the tachyon does not give rise to a physical particle[14, 15, 35, 38], *i.e.* in the scattering of gravitons we should not be able to produce 'quanta' of the tachyon field.[11] Of course this decoupling of the tachyon occurs only for large T where the potential term can be ignored. Thus the correct statement will be that given a solution of the Wheeler - de Witt equation without the tachyon field, it can be regarded as a T-independent solution of the Wheeler de Witt equation in the presence of tachyon field for large T. For finite T the solution needs to be modified to include the effect of the potential term. Thus all solutions of the Wheeler - de Witt equation will have a T-dependence at early time, showing that 'time' dependent solutions are inevitable in this theory.

3.6 Gauge fixing

We can simplify the hamiltonian constraint by 'choosing a gauge'

$$\partial_a T(\vec{\xi}) = 0 \,. \tag{3.26}$$

In concrete terms this means that we choose to look at the wave-functional $\Psi[\{T(\vec{\xi})\}, \{h_{ab}(\vec{\xi})\}]$ only over the slice of the configuration space for which $T(\vec{\xi}) = T$ is independent of $\vec{\xi}$. On such a slice,

$$\partial'_a \widehat{T}(\vec{\xi}') \, \Psi[T, \{h_{ab}(\vec{\xi})\}] = 0 \,. \tag{3.27}$$

Thus the momentum constraints (3.21) takes the form:

$$\widehat{\mathcal{H}}^G_a(\vec{\xi}') \, \Psi[T, \{h_{ab}(\vec{\xi})\}] = 0 \,, \tag{3.28}$$

whereas the hamiltonian constraint (3.22) or (3.23) take the form, respectively,[12]

$$i\frac{\partial}{\partial T} \, \Psi[T, \{h_{ab}(\vec{\xi})\}] = \int d^p\xi' \, \mathcal{H}^G_\uparrow(\vec{\xi}') \;\, \Psi[T, \{h_{ab}(\vec{\xi}\})] \,, \tag{3.29}$$

or

$$i\frac{\partial}{\partial T} \, \Psi[T, \{h_{ab}(\vec{\xi})\}] = \int d^p\xi' \, \mathcal{H}^G_\perp(\vec{\xi}') \;\, \Psi[T, \{h_{ab}(\vec{\xi}\})] \,. \tag{3.30}$$

[11] This in turn is a consequence of the fact that in this approximation the Hamiltonian involving the tachyon field, including the terms describing graviton-tachyon interaction, has vanishing matrix element between $\Pi(\vec{\xi}) = 0$ states.

[12] Formally, once (3.28) is satisfied, we have $H^G_\perp(\vec{\xi}) \, \Psi = H^G_\uparrow(\vec{\xi}) \, \Psi$, and hence (3.29) and (3.30) give the same equations.

This has the form of usual Schrodinger equation, with T interpreted as time. Note however that this form of the equation is modified at small or finite T where the potential term $V(T)$ is important.

4 Discussion

In this paper we have analyzed the scalar Born-Infeld theory coupled to gravity, described by the action (3.1), and shown that at 'late time' the field T appearing in this action could serve as a satisfactory definition of time in canonical quantum gravity. Since the action (3.1) in a fixed background metric is able to reproduce some features of tachyon dynamics in string theory, our hope is that the field T could arise as some collective mode in open string field theory, and could lead to an identification of an intrinsic time variable in string theory. Clearly much work remains to be done before we have a concrete realization of this idea.

We shall end this paper by commenting on few related issues.

1. In section 2 we have used a continuity argument to show that in the limit of large T, we can restrict $\partial_0 T$ and hence Π to be positive in the whole region of space-time at late time. This argument of course breaks down when we take into account the effect of $V(T)$ and use the complete equation (2.9), since here $\partial_0 T$ can pass from -1 to 1 smoothly as long as $|\Pi|$ passes from a region $\Pi \gtrsim V(T)$ to $\Pi \lesssim -V(T)$. We note however that for large T, when $V(T)$ is exponentially small, this gives a very small window in the Π space. If we want the effective field theory to be valid then the region of space in which this transition from $\partial_0 T \geq 1$ to $\partial_0 T \leq -1$ takes place must have width of order unity or more (measured in string scale) so that $\partial_i \partial_0 T$ is of order one or less. Constraining $|\Pi|$ to be less than $V(T)$ in this spatial region will give rise to large quantum fluctuations in T of order $e^{\alpha T/2}$ (for $V(T) = \exp(-\alpha T/2)$) and will invalidate our classical analysis.

   In string theory one can construct configurations where different regions of space-time have Π with different sign as follows. Consider a pair of non-BPS D0-branes (or two D0-$\bar{\text{D}}$0 pair in the case of type IIA string theory) separated by a distance. On each we set up a rolling tachyon solution given in [2], but we adjust one of the solutions to have a large time lag relative to the other. As a result during a large interval of time, as on one of the D0-branes the tachyon rolls up the hill, giving rise to negative Π, on the other D0-brane the tachyon will roll down the hill and will have positive Π. Thus one could ask if quantum fluctuations around this background in string theory shows any unusual divergence of the kind predicted from the field theory analysis. Since the one loop partition function in the presence of this D-brane system involves computing the inner product of the boundary states associated with these two D0-branes, we see that if the boundary states had been finite in the $x^0 \to \pm\infty$ limit, we would not expect any unusual behaviour of the partition function. However recent analysis shows that coefficients of some terms in the boundary state grows exponentially with time[23, 17]. It is tempting to speculate that this exponential growth is related to the exponentially large quantum fluctuations that we expect from such a configuration in the effective field theory.

2. One question one could ask is whether it is possible to say something concrete about the quantum theory described by the action (1.1) even before we couple this to gravity. Of course this is not a standard renormalizable field theory, but we could in principle use some kind of lattice regularization, and treat this as a theory with a finite cut-off. To begin with we can consider a simpler model, – that for $p = 0$, – for which the Hamiltonian takes the form:

$$\widehat{H} = \sqrt{\widehat{\Pi^2 + (V(T))^2}}\,. \tag{4.1}$$

Since $\widehat{\Pi}^2 + \widehat{V}^2$ is a positive definite hermitian operator, we can define $\widehat{H}$ as the positive square root of $\widehat{\Pi}^2+\widehat{V}^2$. $(\widehat{\Pi}^2+\widehat{V}^2)$ for $\widehat{V} = e^{-\alpha\widehat{T}/2}$ has a continuous spectrum beginning at 0. An eigenstate of $(\widehat{\Pi}^2 + \widehat{V}^2)$ with eigenvalue λ^2 has the asymptotic form proportional to $\sin(\lambda T+\phi(\lambda))$ where $\phi(\lambda)$ is a phase factor. There is, however, no strictly zero energy state, since $\phi(\lambda) \to 0$ as $\lambda \to 0$.

In order to analyze the time evolution of a state under the Hamiltonian (4.1), we can decompose it as a linear combination of the eigenstates of $(\widehat{\Pi}^2 + \widehat{V}^2)$, and multiply the coefficient of the eigenstate with eigenvalue λ^2 by $e^{i\lambda t}$. We shall not do this explicitly here, but from general considerations we expect that if we begin with a gaussian wave-packet, and evolve it according to this scheme, then in the far future the wave-function, *when decomposed in the basis of plane waves on an infinite line*, will consist mostly of right-moving waves, since all the left-moving components will get reflected by the tachyon potential.

If we begin with some wave-function that approaches a constant value at ∞, then the Schrodinger equation

$$i\frac{\partial\Psi(T;x^0)}{\partial x^0} = \sqrt{\widehat{\Pi}^2 + \widehat{V}^2}\,\Psi(T;x^0)\,, \tag{4.2}$$

has the property that for large T, the right hand side of the equation is small and hence the wave-function remains constant. However the wave-function does evolve with time in a non-trivial manner for $T\lesssim 1$, since the true ground state for which the right hand side of (4.2) vanishes is localized at $T=\infty$.

For the case of general p (including the space-filling branes) we can define the Hamiltonian density in a similar way, by taking this to be the positive square root of the operator:

$$(1+\widehat{(\vec{\nabla}T)^2})(\widehat{\Pi}^2+\widehat{V}^2)\,. \tag{4.3}$$

Note that $[\widehat{\Pi}(x), \partial_i\widehat{T}(x)] \propto \delta'(0)=0$. To see this in a properly regularized theory we can use a lattice regularization with lattice sites labelled by s, and identify $\widehat{\Pi}(x)$ with $\widehat{\Pi}_s(x^0)$ and $\partial_i\widehat{T}(x)$ with $(\widehat{T}_{s+1}(x^0)-\widehat{T}_{s-1}(x^0))$ where we are using a shorthand notation where $\widehat{T}_{s\pm 1}$ denotes the lattice site displaced ± 1 unit from the site s along the ith direction. $\widehat{\Pi}_s(t)$ clearly commutes with $(\widehat{T}_{s+1}(x^0)-\widehat{T}_{s-1}(x^0))$. Thus there is no operator ordering problem in defining (4.3) at individual sites and it is a positive definite hermitian operator. This allows us to define its positive definite square root. However this analysis also shows that the hamiltonian densities at different sites do not commute, and hence are not simultaneously diagonalizable. This makes the concrete analysis of the spectrum difficult. Nevertheless this definition of the Hamiltonian allows us to calculate the action of the hamiltonian on a give wave-function explicitly. For example, in the lattice version, in order to calculate $\widehat{\mathcal{H}}_s\Psi$ for a given wave-functional $\Psi(T_1,\ldots T_n)$, we first multiply Ψ by the lattice version of $\sqrt{1+(\vec{\nabla}T)^2}$ associated with the s-th site, then express the result as a linear combination of the eigenfunctions of $(\widehat{\Pi}_s^2+(V(\widehat{T}_s))^2)$, and finally multiply each term in the expansion by the positive square root of the associated eigenvalue. We can repeat this for each site s, and the final action of the full Hamiltonian on Ψ is obtained by adding up the contribution from each lattice site. We can then find the time evolution of the state by solving the time dependent Schrodinger equation:

$$\begin{aligned} i\,\frac{\partial\Psi(T(\vec{x});x^0)}{\partial x^0} &= \int d^px'\,\widehat{\sqrt{(1+(\vec{\nabla}T)^2)\,(\Pi^2+V^2)}}\;\Psi(T(\vec{x});x^0) \\ &= \sum_s \sqrt{(1+(\vec{\nabla}T_s)^2)(\Pi_s^2+V_s^2)}\,\Psi(T_1,\ldots T_n;x^0)\,, \end{aligned} \tag{4.4}$$

where in the last line of the above equation we have the lattice version of the Hamiltonian, with $\vec{\nabla}T_s$ defined by taking appropriate difference between the values of T at neighbouring sites. This equation will have the property that if we take an initial wave-function which approaches a constant for large $\{T(\vec{x})\}$, then it remains constant for large $\{T(\vec{x})\}$ as the wave-function evolves in time. More precisely a configuration where Ψ is a constant independent of $\{T(\vec{x})\}$ and x^0 will satisfy (4.4) approximately in the region $T(\vec{x})\geq M$, $|\vec{\nabla}T| << e^{\alpha M/2}$ for a sufficiently large number M, but fails in the region where $T(\vec{x})$ is finite, or $|\vec{\nabla}T|$ is of order $e^{\alpha T/2}$. If we are only interested in studying the properties of the wave-function for large T and not too large $|\vec{\nabla}T|$, then we could treat this as the 'approximate vacuum state', although strictly it is not an eigenstate of the Hamiltonian.

Acknowledgement: I would like to thank S. Minwalla, E. Rabinovici, A. Strominger and B. Zwiebach for useful discussions. This work was supported in part by a grant from the Eberly College of Science of the Penn State University, and a grant from the NM Rothschild and Sons Ltd at the Isaac Newton institute. I would also like to acknowledge the hospitality of YITP, Stony Brook, and the Center for Theoretical Physics at MIT where part of this work was done.

References

[1] A. Sen, JHEP **0204**, 048 (2002) [arXiv:hep-th/0203211].

[2] A. Sen, JHEP **0207**, 065 (2002) [arXiv:hep-th/0203265].

[3] A. Sen, arXiv:hep-th/0207105.

[4] M. Gutperle and A. Strominger, JHEP **0204**, 018 (2002) [arXiv:hep-th/0202210].

[5] S. Sugimoto, S. Terashima JHEP **0207**, 025 (2002) [arXiv:hep-th/0205085].

[6] J. A. Minahan, JHEP **0207**, 030 (2002) [arXiv:hep-th/0205098].

[7] A. Ishida and S. Uehara, arXiv:hep-th/0206102.

[8] N. Moeller and B. Zwiebach, arXiv:hep-th/0207107.

[9] K. Hashimoto, JHEP **0207**, 035 (2002) [arXiv:hep-th/0204203].

[10] K. Ohta and T. Yokono, arXiv:hep-th/0207004.

[11] J. Kluson, arXiv:hep-th/0208028.

[12] M. R. Garousi, Nucl. Phys. B **584**, 284 (2000) [arXiv:hep-th/0003122]; arXiv:hep-th/0209068; J. Kluson, Phys. Rev. D **62**, 126003 (2000) [arXiv:hep-th/0004106].

[13] E. A. Bergshoeff, M. de Roo, T. C. de Wit, E. Eyras and S. Panda, JHEP **0005**, 009 (2000) [arXiv:hep-th/0003221].

[14] A. Sen, arXiv:hep-th/0204143.

[15] A. Sen, JHEP **9808**, 010 (1998) [arXiv:hep-th/9805019]; JHEP **9809**, 023 (1998) [arXiv:hep-th/9808141]; JHEP **9812**, 021 (1998) [arXiv:hep-th/9812031]; Int. J. Mod. Phys. A **14**, 4061 (1999) [arXiv:hep-th/9902105]; arXiv:hep-th/9904207.

[16] A. A. Gerasimov and S. L. Shatashvili, JHEP **0010**, 034 (2000) [arXiv:hep-th/0009103]; D. Kutasov, M. Marino and G. W. Moore, JHEP **0010**, 045 (2000) [arXiv:hep-th/0009148]; arXiv:hep-th/0010108.

[17] P. Mukhopadhyay and A. Sen, arXiv:hep-th/0208142.

[18] G. N. Felder, L. Kofman and A. Starobinsky, arXiv:hep-th/0208019.

[19] N. D. Lambert and I. Sachs, arXiv:hep-th/0208217.

[20] G. Gibbons, K. Hashimoto and P. Yi, arXiv:hep-th/0209034.

[21] E. Witten, JHEP **9812**, 019 (1998) [arXiv:hep-th/9810188]; P. Horava, Adv. Theor. Math. Phys. **2**, 1373 (1999) [arXiv:hep-th/9812135].

[22] G. W. Gibbons, arXiv:hep-th/0204008; M. Fairbairn and M. H. Tytgat, arXiv:hep-th/0204070; S. Mukohyama, arXiv:hep-th/0204084; A. Feinstein, arXiv:hep-th/0204140; T. Padmanabhan, arXiv:hep-th/0204150; A. Frolov, L. Kofman and A. A. Starobinsky, arXiv:hep-th/0204187; D. Choudhury, D. Ghoshal, D. P. Jatkar and S. Panda, arXiv:hep-th/0204204; X. Li, J. Hao and D. Liu, arXiv:hep-th/0204252; G. Shiu and I. Wasserman, arXiv:hep-th/0205003; T. Padmanabhan and T. R. Choudhury, arXiv:hep-th/0205055; L. Kofman and A. Linde, arXiv:hep-th/0205121; H. B. Benaoum, arXiv:hep-th/0205140; M. Sami, arXiv:hep-th/0205146; M. Sami, P. Chingangbam and T. Qureshi, arXiv:hep-th/0205179; T. Mehen and B. Wecht, arXiv:hep-th/0206212; G. Shiu, S. H. Tye and I. Wasserman, arXiv:hep-th/0207119; Y. S. Piao, R. G. Cai, X. m. Zhang and Y. Z. Zhang, arXiv:hep-ph/0207143; X. z. Li, D. j. Liu and J. g. Hao, arXiv:hep-th/0207146; J. M. Cline, H. Firouzjahi and P. Martineau, arXiv:hep-th/0207156; A. Buchel, P. Langfelder and J. Walcher, arXiv:hep-th/0207235; B. Wang, E. Abdalla and R. K. Su, arXiv:hep-th/0208023; S. Mukohyama, problem in tachyon cosmology," arXiv:hep-th/0208094; M. C. Bento, O. Bertolami and A. A. Sen, arXiv:hep-th/0208124.

[23] T. Okuda and S. Sugimoto, arXiv:hep-th/0208196.

[24] A. Strominger, arXiv:hep-th/0209090.

[25] C. Armendariz-Picon, T. Damour and V. Mukhanov, Phys. Lett. B **458**, 209 (1999) [arXiv:hep-th/9904075].

[26] C. Rovelli, Phys. Rev. D **43**, 442 (1991); Class. Quant. Grav. **8**, 297 (1991); Class. Quant. Grav. **8**, 317 (1991); C. Rovelli and L. Smolin, Phys. Rev. Lett. **72**, 446 (1994) [arXiv:gr-qc/9308002].

[27] K. V. Kuchař and C. G. Torre, Phys. Rev. D **43**, 419 (1991).

[28] J. D. Brown and K. V. Kuchař, Phys. Rev. D **51**, 5600 (1995) [arXiv:gr-qc/9409001].

[29] S. Carlip, Rept. Prog. Phys. **64**, 885 (2001) arXiv:gr-qc/0108040.

[30] C. J. Isham, arXiv:gr-qc/9210011.

[31] G. W. Gibbons, K. Hori and P. Yi, Nucl. Phys. B **596**, 136 (2001) [arXiv:hep-th/0009061].

[32] P. Yi, Nucl. Phys. B **550**, 214 (1999) [arXiv:hep-th/9901159]. O. Bergman, K. Hori and P. Yi, Nucl. Phys. B **580**, 289 (2000) [arXiv:hep-th/0002223]; J. A. Harvey, P. Kraus, F. Larsen and E. J. Martinec, "D-branes and strings as non-commutative solitons," JHEP **0007**, 042 (2000) [arXiv:hep-th/0005031]. F. Larsen, Int. J. Mod. Phys. A **16**, 650 (2001) [arXiv:hep-th/0010181].

[33] U. Lindstrom and R. von Unge, Phys. Lett. B **403**, 233 (1997) [arXiv:hep-th/9704051]; H. Gustafsson and U. Lindstrom, Phys. Lett. B **440**, 43 (1998) [arXiv:hep-th/9807064]; U. Lindstrom, M. Zabzine and A. Zheltukhin, JHEP **9912**, 016 (1999) [arXiv:hep-th/9910159].

[34] A. Sen, J. Math. Phys. **42**, 2844 (2001) [arXiv:hep-th/0010240].

[35] A. Sen, JHEP **9910**, 008 (1999) [arXiv:hep-th/9909062].

[36] T. Regge and C. Teitelboim, Annals Phys. **88**, 286 (1974).

[37] C. Vaz and L. Witten, Phys. Rev. D **60**, 024009 (1999) [arXiv:gr-qc/9811062].

[38] I. Ellwood and W. Taylor, Phys. Lett. B **512**, 181 (2001) [arXiv:hep-th/0103085]; I. Ellwood, B. Feng, Y. H. He and N. Moeller, JHEP **0107**, 016 (2001) [arXiv:hep-th/0105024].