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A QUANTUM ANTI-ZENO PARADOX

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Abstract

We establish an exact differential equation for the operator describing time-dependent measurements continuous in time and obtain a series solution. Suppose the projection operator $E(t) = U(t)EU^\dagger(t)$ is measured continuously from $t = 0$ to T, where E is a projector leaving the initial state unchanged and $U(t)$ a unitary operator obeying $U(0) = 1$ and some smoothness conditions in t. We prove that the probability of always finding $E(t) = 1$ from $t = 0$ to T is unity. If $U(t) \neq 1$, the watched kettle is sure to 'boil'.

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1. Introduction. Ordinary quantum physics specifies probabilities of ideal observations at one instant of time or of a sequence of such observations at different instants[1]. How should one describe the limit of infinitely frequent measurements or continuous observation? One of the earliest approaches to continuous quantum measurements was already suggested by Feynman[2] in his original work on the path integral. The Feynman propagator as modified by measurements is to be calculated by restricting the paths to cross (or not to cross) certain spacetime regions (where space can mean configuration space or phase space). An approximate way of doing this by incorporating Gaussian cut-offs in the phase space path integral was developed by Mensky[3] who also showed its equivalence to the phenomenological master equation approach for open quantum systems using models of system-environment coupling developed by Joos and Zeh and others[4].

On the other hand a completely different approach was initiated by Misra and Sudarshan[5] who asked: what is the rigorous quantum description of ideal continuous measurement of a projector E (time independent in the Schrödinger representation) over a time interval $[0, T]$? Their original motivation[5]: "there does not seem to be any principle, internal to quantum theory, that forbids the duration of a single measurement or the dead time between successive measurements from being arbitrarily small", led them to rigorous confirmation of a seemingly paradoxical conclusion noted earlier[6]. The conclusion "that an unstable particle which is continuously observed to see whether it decays will never be found to decay" or that a "watched kettle never boils" was christened "Zeno's paradox in quantum theory" by Misra and Sudarshan[5]. The paradox has been theoretically scrutinized[7] and experimentally probed[8].

Here we ask a question far more general than that of Misra and Sudarshan: what is the operator (the modified Feynman propagator) corresponding to an ideal continuous measurement of a projection operator $E_s(t)$ which has an arbitrary (but smooth) dependence on time in the Schrödinger representation? We obtain a differential equation for the operator and a series solution which has many applications (to be illustrated in a longer paper[9]). One of them leads us to a new watched kettle paradox which is apparently quite the opposite of the Zeno paradox, but mathematically a far reaching generalization of it. Suppose we continuously measure from $t = 0$ to T the projector $E_s(t) = U(t)EU^\dagger(t)$ where $U(t)$ is a unitary operator obeying $U(0) = 1$ and some smoothness conditions, and E a projector obeying $E\rho(0)E = \rho(0)$, where $\rho(0)$ is the initial density operator. Then the probability of always finding $E_s(t) = 1$ from $t = 0$ to T is unity. For the Misra-Sudarshan case $U(t) = 1$ we recover the usual Zeno paradox that the watched kettle does not boil. Generically $U(t) \neq 1$. Hence, for most ways of watching ($U(t) \neq 1$), the watched kettle is sure to 'boil', an anti-Zeno paradox. If the system is in an eigenstate of E with eigenvalue unity at $t = 0$, it will change its state with time so as to be in an eigenstate of $E_s(t)$ with eigenvalue unity at all future times.

Our computation of modified Feynman propagators corresponding to continuous measurements is in the framework of ordinary quantum mechanics. Exactly the same mathematical expressions for the propagators, albeit with a different physical meaning would arise in the 'consistent histories' or 'sum over histories' quantum mechanics of closed systems[10,11], where there is no notion of measurement. Our computations can therefore be applied also to these history extended quantum mechanics provided that the probability sum rules corresponding to consistency conditions or decoherence conditions are obeyed.

2. Formulation of the Problem: For a quantum system with a self-adjoint Hamiltonian H, an initial state vector $|\psi(0)\rangle$ evolves to a state vector $|\psi(t)\rangle$,

$$|\psi(t)\rangle = \exp(-iHt)|\psi(0)\rangle. \quad (1)$$

More generally, an initial state with density operator $\rho(0)$ has the Schrödinger time evolution

$$\rho(t) = \exp(-iHt)\rho(0)\exp(iHt), \quad (2)$$

which preserves the normalization condition Tr $\rho(t) = 1$. In an ideal instantaneous measurement of a self-adjoint projection operator E, the probability of finding $E = 1$ is $\mathrm{Tr}(E\rho E)$ and on finding the value 1 for E the state collapses according to

$$\rho \to \rho' = E\rho E/\mathrm{Tr}(E\rho E). \quad (3)$$

If projectors $E_1, E_2, \cdots, E_n$ are measured at times $t_1, t_2, \cdots, t_n$ respectively, with Schrödinger evolution in between measurements, the probability $p(h)$ for the sequence of events h,

$$h: \ E_1 = 1 \text{ at } t = t_1; \ E_2 = 1 \text{ at } t = t_2; \cdots; \ E_n = 1 \text{ at } t = t_n \quad (4)$$

is[1]

$$p(h) = ||\psi_h(t')||^2, \ \psi_h(t') = K_h(t')\psi(0), \ t' > t_n. \quad (5)$$

Here $K_h(t')$ is the Feynman propagator modified by the events h

$$K_h(t') = \exp(-iHt')A_h(t_n, t_1) \quad (6)$$

where,

$$A_h(t_n, t_1) = E_H(t_n)E_H(t_{n-1}) \cdots E_H(t_1) = T\prod_{i=1}^{n} E_H(t_i), \quad (7)$$

with T denoting 'time-ordering' and the Heisenberg operators $E_H(t_i)$ are related to the Schrödinger operators by the usual relation

$$E_H(t_i) = \exp(iHt_i)E_s(t_i)\exp(-iHt_i), \ E_s(t_i) \equiv E_i. \quad (8)$$

The state vector of the system at a time t' after the events h is

$$\psi_h(t')/||\psi_h(t')||.$$

Correspondingly, if the initial state is a density operator $\rho(0)$, the probability $p(h)$ for the events h is given by

$$p(h) = \mathrm{Tr}\ K_h(t')\rho(0)K_h^\dagger(t') = \mathrm{Tr}\ A_h(t_n, t_1)\ \rho(0)A_h^\dagger(t_n, t_1), \quad (9)$$

and the state at $t' > t_n$ is

$$K_h(t')\rho(0)K_h^\dagger(t')/\mathrm{Tr}\ (K_h(t')\rho(0)K_h^\dagger(t')).$$

Continuous Measurements. Consider infinitely frequent measurements of the projection operators $E_s(t_i)$ which are values at times t_i of a projection valued function $E_s(t)$. We make the technical assumption that the corresponding Heisenberg operator $E_H(t)$ is weakly analytic. We seek to calculate the modified Feynman propagator

$$K_h(t') = \exp(-iHt')A_h(t, t_1), \tag{10}$$

where

$$A_h(t, t_1) = \lim_{n\to\infty} T \prod_{i=1}^{n} E_H(t_1 + (t - t_1)(i - 1)/(n - 1)) \tag{11}$$

which is the $n \to \infty$ limit of Eq. (7) with a specific choice of the t_i. Let us also introduce the projectors $\bar{E}_i = 1 - E_i$ which are the orthogonal complements of the projectors E_i, and a sequence of events $\bar{h}$ complementary to the sequence h,

$$\bar{h} : \bar{E}_1 = 1 \text{ at } t = t_1;\ \bar{E}_2 = 1 \text{ at } t = t_2, \cdots, \bar{E}_n = 1 \text{ at } t = t_n. \tag{12}$$

Corresponding to Eqs. (6), (7), (10), (11), we have Eqs. with $E \to \bar{E}$, $h \to \bar{h}$. The special interest in $K_{\bar{h}}(t')$ is that it is closely related to the propagator

$$K_{h'}(t') \equiv \exp(-iHt') - K_{\bar{h}}(t') = \exp(-iHt')[1 - A_{\bar{h}}(t, t_1)],\ h' \equiv \underset{i}{U} E_i, \tag{13}$$

which represents the modified Feynman propagator corresponding to the union of the events E_i, i.e. to at least one of the events $E_s(t_i) = 1$ occurring, with t_i lying between t_1 and t. Our object is to obtain exact operator expressions for the propagators K_h, $K_{\bar{h}}$ which have been defined above by formal infinite products. These results will also provide evaluations of the path integral formulae for the propagators in history extended quantum mechanics[10,11].

3. Differential Equation and Series Solution. We see from Eq. (10) that $A_h(t, t_1)$ ($A_{\bar{h}}(t, t_1)$) represents the modification of the Feynman propagator due to the continuous measurement corresponding to the sequence of events $h(\bar{h})$. Consider first the operators $A_h(t_i, t_1)$, $A_{\bar{h}}(t_i, t_1)$ before taking the $n \to \infty$ limit, and note that

$$A_h(t_i, t_1) = E_H(t_i)A_h(t_{i-1}, t_1),\ A_{\bar{h}}(t_i, t_1) = \bar{E_H}(t_i)A_{\bar{h}}(t_{i-1}, t_1). \tag{14}$$

The relation $\bar{E}_H^2 = \bar{E}_H$ implies $A_{\bar{h}}(t_{i-1}, t_1) = \bar{E}_H(t_{i-1})A_{\bar{h}}(t_{i-1}, t_1)$. We thus have

$$A_{\bar{h}}(t_i, t_1) - A_{\bar{h}}(t_{i-1}, t_1) = (\bar{E_H}(t_i) - \bar{E_H}(t_{i-1}))A_{\bar{h}}(t_{i-1}, t_1), \tag{15}$$

and a similar relation for A_h. Dividing by $t_i - t_{i-1} = \delta t$, taking the limit $n \to \infty$ (i.e., $\delta t \to 0$) and assuming that $E_H(t)$ is weakly analytic at $t = 0$ we obtain the differential eqns.,

$$\frac{dA_{\bar{h}}(t, t_1)}{dt} = \frac{d\bar{E}_H(t)}{dt} A_{\bar{h}}(t_-, t_1),\ \frac{dA_h(t, t_1)}{dt} = \frac{dE_H(t)}{dt} A_h(t_-, t_1). \tag{16}$$

where the arguments t_- on the right-hand sides indicate that in case of any ambiguity in defining the operator products the arguments have to be taken as $t - \epsilon$ with $\epsilon \to 0$ from positive values and

$$\frac{dE_H(t)}{dt} = i[H, E_H(t)] + \exp(iHt)\frac{dE_s(t)}{dt}\exp(-iHt). \tag{17}$$

Further $A_{\bar{h}}(t, t_1), A_h(t, t_1)$ must obey the initial conditions

$$A_{\bar{h}}(t_1, t_1) = \bar{E_H}(t_1), \;\; A_h(t_1, t_1) = E_H(t_1). \tag{18}$$

The measurement differential equations (16) are reminiscent of Schrödinger equation for the time evolution operator except for the fact that the operators $d\bar{E}_H/dt$, dE_H/dt are hermitian whereas in Schrödinger theory the antihermitian operator H/i would occur. Using the initial conditions we obtain the explicit solutions,

$$A_h(t, t_1) = T\exp\left(\int_{t_1}^{t} dt' \frac{dE_H(t')}{dt'}\right) E_H(t_1), \tag{19}$$

and a similar equation with $h \to \bar{h}$, $E_h \to \bar{E}_h$, where the time ordered exponentials have the series expansion

$$T\exp\left(\int_{t_1}^{t} dt' \frac{dE_H(t')}{dt'}\right) = 1 + \sum_{n=1}^{\infty}\int_{t_1}^{t} dt'_1 \int_{t_1}^{t'_1} dt'_2 \cdots \int_{t_1}^{t'_{n-1}} dt'_n T\prod_{i=1}^{n}\frac{dE_H(t'_i)}{dt'_i}. \tag{20}$$

In general the time-ordered operator products appearing on the right-hand side are distributions and the series on the right-hand side must be taken as the definition of the exponential on the left-hand side; we may not do the integral of $dE_H(t')/dt'$ on the left-hand side. Multiplying the expressions for $A_{\bar{h}}(t, t_1)$ and $A_h(t, t_1)$ on the left by $\exp(-iHt')$ then completes the evaluation of the modified Feynman propagators $K_{\bar{h}}(t')$ and $K_h(t)$.

4. Quantum Anti-Zeno Paradox. We recall first the usual Zeno paradox. Let the initial state be $|\psi_0>$ and let the projection operator $|\psi_0><\psi_0|$ be measured at times $t_1, t_2, \cdots t_n$ with $t_j - t_{j-1} = (t_n - t_1)/(n-1)$ and $t_n = t$, and let $n \to \infty$. Then, the definition (7) yields,

$$\begin{aligned} A_h(t, t_1) &= \lim_{n\to\infty} e^{iHt}|\psi_0><\psi_0|\exp(-iH(t-t_1)/(n-1))|\psi_0>^{n-1}<\psi_0|e^{-iHt_1} \\ &= \exp(i(H-\bar{H})t)|\psi_0><\psi_0|\exp(-i(H-\bar{H})t_1), \end{aligned} \tag{21}$$

where $\bar{H}$ denotes $<\psi_0|H|\psi_0>$ and we assume that[12] $<\psi_0|\exp(-iH\tau)|\psi_0>$ is analytic at $\tau = 0$. Our differential eqn. also yields exactly this solution for $A_h(t, t_1)$. Taking $t_1 = 0$, we deduce that the probability $p(h)$ of finding the system in the initial state at all times upto t is given by

$$p(h) = ||K_h(t)|\psi_0>||^2 = ||\bar{e}^{i\bar{H}t}|\psi_0>||^2 = 1, \tag{22}$$

which is the Zeno paradox. (The result can also be generalized to the case of an initial state described by a density operator, and the measured projection operator being of arbitrary rank but leaving the initial state unaltered, see below.)

Anti-Zeno Paradox: The above result may suggest that continuous observation inhibits change of state. Now we prove a far more general result which shows that a generic continuous observation actually ensures change of state. Suppose that the initial state is described by a density operator $\rho(0)$, and we measure the projection operator

$$E_s(t') = U(t')EU^\dagger(t') \tag{23}$$

continuously for $t'\epsilon[0,t]$. Here E is an arbitrary projection operator (which need not even be of finite rank) which leaves the initial state unaltered,

$$E\rho(0)E = \rho(0), \tag{24}$$

and $U(t')$ is a unitary operator which coincides with the identity operator at $t'=0$,

$$U^\dagger(t')U(t') = U(t')U^\dagger(t') = 1, U(0) = 1. \tag{25}$$

The Heisenberg operator $E_H(t')$ is then

$$E_H(t') = V(t')EV^\dagger(t'), \quad V(t') = e^{iHt'}U(t'). \tag{26}$$

Clearly $V(t')$ is also a unitary operator. The definition (7) yields, for $t_1 \geq 0$,

$$A_h(t_n,t_1) = V(t_n)(T\prod_{i=1}^{n-1} X(t_i))V^\dagger(t_i),\ n \geq 2 \tag{27}$$

where

$$X(t_i) \equiv EV^\dagger(t_{i+1})V(t_i)E, \tag{28}$$

and $A_h(t_1,t_1) = V(t_1)EV^\dagger(t_1)$. Denoting

$$Y(t_j) = T\prod_{i=1}^{j-1} X(t_i),\ j \geq 2, \tag{29}$$

$Y(t_1) = E$ and noting that $EY(t_{j-1}) = Y(t_{j-1})$, we have

$$Y(t_j) - Y(t_{j-1}) = E(V^\dagger(t_j)V(t_{j-1}) - 1)EY(t_{j-1}). \tag{30}$$

Taking $t_{j-1} = t',\ t_j = t' + \delta t,\ n \to \infty$, we have $\delta t = 0(1/n)$, and

$$E(V^\dagger(t'+\delta t)V(t') - 1)E = \delta t E\frac{dV^\dagger(t')}{dt'}V(t')E + 0(\delta t)^2. \tag{31}$$

To derive that the last term on the right-hand side is $0(\delta t)^2$ in the weak sense (i.e., for matrix elements between any two arbitrary state vectors in the Hilbert space), we make the smoothness assumption that $E(V^\dagger(t'+\tau)V(t') - 1)E$ is analytic in τ at $\tau = 0$ in the weak sense. (It may be seen that this reduces to analyticity of $<\psi_0|\exp(-iH\tau)|\psi_0>$ in the usual Zeno case[12]). Hence the $n \to \infty$ limit yields,

$$A_h(t,t_1) = V(t)Y(t)V^\dagger(t_1), \tag{32}$$

where

$$\frac{dY(t')}{dt'} = E\frac{dV^\dagger(t')}{dt'}V(t')EY(t'). \tag{33}$$

Solving the differential eqn. we obtain,

$$A_h(t,t_1) = V(t)T\exp(\int_{t_1}^{t} dt' E\frac{dV^\dagger(t')}{dt'}V(t')E)EV^\dagger(t_1). \tag{34}$$

It is satisfying to note that this expression indeed solves our basic differential equation (16) as can be verified very easily by direct substitution.

The most crucial point for deriving the anti-Zeno paradox is that the operator

$$T\exp(\int_{t_1}^{t} dt' E\frac{dV^\dagger(t')}{dt'}V(t')E) \equiv W(t,t_1)$$

is unitary, because $(dV^\dagger(t')/dt')V(t')$ is anti-hermitian as a simple consequence of the unitarity of $V(t')$. Taking $t_1 = 0$, Eq. (9) gives the probability of finding $E_s(t') = 1$ for all t' from $t' = 0$ to t as

$$p(h) = \mathrm{Tr}\left(V(t)W(t,0)EV^\dagger(0)\rho(0)V(0)EW^\dagger(t,0)V^\dagger(t)\right) = \mathrm{Tr}\rho(0) = 1, \tag{35}$$

where we have used $V(0) = 1$, $E\rho(0)E = \rho(0)$, the unitarity of $V(t)$ and the unitarity of $W(t,0)$. This completes the demonstration of the anti-Zeno paradox: continuous observation of $E_s(t) = U(t)EU^\dagger(t)$ with $U(t) \neq 1$ ensures that the initial state must change with time such that the probability of finding $E_s(t) = 1$ at all times during the duration of the measurement is unity.

Mathematical remarks. The great generality of the present results with respect to the ordinary Zeno paradox[5] derives from the fact that the unitary operator $V(t)$ need not obey the semigroup law[5] $V(t)V(s) = V(t+s)$. Further, the following remarks about the set of pairs (E,ρ) [with ρ a density operator] fulfilling $E\rho E = \rho$ can be made. The first is that as E and ρ are self-adjoint, this condition is equivalent to either of the requirements $E\rho = \rho$, or $\rho E = \rho$. They mean just that ρ is zero on the range of $(1-E)$. The properties of the pairs (E,ρ) in a finite-dimensional quantum theory are simple. In that case, the density operators, being a convex set, are connected and contractible while the connected components of projectors E consist of all the projectors of the same rank. Thus for fixed rank n of projectors, the allowed pairs (E,ρ) form a connected space with the structure of a fibre bundle, with projectors forming the base and a fibre being a convex set. This bundle is trivial, the fibres being contractible. If the quantum Hilbert space $\mathcal{H}_{n+k}$ is of dimension $n+k$, its unitary group $U(n+k) = \{U\}$ acts on (E,ρ) by conjugation: $E \rightarrow UEU^{-1},\ \rho \rightarrow U\rho U^{-1}$. This action is an automorphism of the bundle. Since any two projectors of the same rank are unitarily related, it is also transitive on the base. The nature of the base follows from this remark. The stability group of E is $U(n)\times U(k)$ where $U(n)$ and $U(k)$ act as identities on the range of $(1-E)$ and E respectively. Thus the base, as is well-known, is the Grassmannian[13] $G_{n,k}(C) = U(n+k)/[U(n)\times U(k)]$. When we pass to quantum physics in infinite dimensions, the space of connected projectors are determined by orbits of infinite-dimensional unitary groups, and, in addition, a projector can itself be of infinite rank. In this manner, general applications of our results will involve infinite-dimensional Grassmannians (on which there are excellent reviews[14]).

Conclusion. It should be stressed that within standard quantum mechanics and its measurement postulates both the usual Zeno paradox and the anti-Zeno paradox derived here are theorems. The two paradoxes appear 'paradoxical' and 'mutually contradictory' only when we forget Bohr's insistence that quantum results depend not only on the quantum state but also on the entire disposition of the experimental apparatus. Indeed the apparatus to measure E and $U(t)EU^\dagger(t)$ are different. It would be challenging to see how these results appear in a quantum theory of closed systems (including the apparatus) in which there is no notion of measurements. It will also be interesting to devise experimental tests of the anti-Zeno effect along lines used to test the ordinary Zeno effect[8].

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