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A C_0 CONTINUOUS FOUR-NODED CYLINDRICAL SHELL ELEMENT

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Abstract—A C_0 continuous four-noded cylindrical shell element with independent interpolations for in-plane displacements u and v; transverse displacement w and face rotations θ_x and θ_y are made efficient by using substitute smoothed shape functions for w in membrane strain evaluation. This removes "membrane locking", making it the simplest efficient quadrilateral cylindrical shell element available.

1. INTRODUCTION

A series of very interesting studies have appeared recently [1-4], which show that it is possible to improve dramatically, the behaviour of doubly curved thin shell elements by using a "reduced interaction" method in which the membrane strains are evaluated using a reduced interpolation for the transverse displacement w. In this paper we demonstrate that for a cylindrical shell element, this can be achieved by using a smoothed strain interpolation for the membrane strains. The dramatic improvement In accuracy is due to the removal of membrane locking, a phenomenon which has been recently studied and established for thin curved team and arch elements [5–7].

The difficulty of deriving simple doubly curved or cylindrical deep shell elements, based on independently interpolated low order displacement elements, has been excellently reviewed in Ref. [8]. Successful elements required either high-order interpolations [9–11] or had to be based on independent-strain models and hence coupled displacement fields [12].

The dramatic success achieved with the use of a substitute reduced interpolation for the transverse displacement w in the membrane strain energy evaluation of a doubly curved deep shell element [1] seems to indicate that this is due to the removal, thereby, of membrane locking; a phenomenon that was shown to exist for thin curved beam elements [5–7]. This suggests that a very simple four-noded rectangular cylindrical shell element of 20 dof, using independent bi-linear interpolations for the five external nodal displacements u, v, w, θ_x, and θ_y, can be made competitive with the well-known coupled-displacement field independent strain 20 dof model of Ashwell and Sabir [12] and even with the higher order 48 dof model of Bogner, Fox, and Schmidt [13]. This is made possible by the use of substitute smoothed interpolation functions in the membrane and shear strain evaluations so that both membrane and shear locking are removed. The pinched cylinder problem is used as a test example and the nature of membrane locking is established.

2. ELEMENT DESCRIPTION

The cylindrical shell element presented here is a conforming (i.e. C_0 continuous) shear-flexible rectangular version of a nonconforming (in relation to the C_1 continuity required) cylindrical thin-shell element presented in Ref. [12]. The five external nodal displacements chosen at each node are w, v, w, θ_x, and θ_y (see Fig. 1). The membrane strains, curvatures, and shear strains are then

$$\begin{aligned}
\epsilon_x &= u_{,x} + w/R_x, \\
\epsilon_y &= v_{,y}, \\
\epsilon_{xy} &= u_{,y} + v_{,x}, \\
\chi_x &= \theta_{x,x}, \\
\chi_y &= \theta_{y,y}, \\
\chi_{xy} &= \theta_{x,y} + \theta_{y,x} - u_{,y}/R_x, \\
\gamma_{xz} &= \theta_x - w_{,x} + u/R_x, \\
\gamma_{yz} &= \theta_y - w_{,y},
\end{aligned} \quad (1)$$

so that in the thin shell limit where $\gamma_{xz} = \gamma_{yz} = 0$, we recover the strain-displacement equations given in Timoshenko and Woinowsky-Krieger [14].

The strain energy density is now

$$W = \frac{Et}{2(1-\nu^2)}\left\{(\epsilon_x+\epsilon_y)^2 - 2(1-\nu)\left(\epsilon_x\epsilon_y - \frac{\epsilon_{xy}^2}{4}\right) + \frac{t^2}{12}[(\chi_x+\chi_y)^2 - 2(1-\nu)(\chi_x\chi_y - \frac{\chi_{xy}^2}{4})]\right\} + \frac{kGT}{2}(\gamma_{xz}^2+\gamma_{yz}^2). \quad (2)$$

Figure 2 shows the isoparametric system used. In order to derive the stiffness matrix, it is necessary to interpolate these strains in terms of the displacement variables u, v, w, θ_x, and θ_y. In this paper

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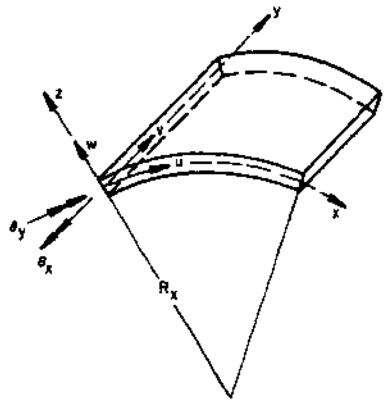

Fig, 1. Geometry of cylindrical shell element.

we restrict attention to a case where the global axes system—(x, y) and the local system—(r, s) are orthogonal and exactly aligned. For this system the shape functions are

$$\begin{aligned} h_1 &= (1-r)(1-s)/4, \\ h_2 &= (1+r)(1-s)/4, \\ h_3 &= (1+r)(1+s)/4, \\ h_4 &= (1-r)(1+s)/4. \end{aligned} \tag{3}$$

If (x_i, y_i), $i = 1,4$ are the coordinates of the four nodes shown in Fig. 2, the coordinate interpolations are quite simply

$$\begin{aligned} x(r, s) &= \sum_{i=1}^{4} x_i h_i, \\ y(r, s) &= \sum_{i=1}^{4} y_i h_i. \end{aligned} \tag{4}$$

These same low-order functions are used to interpolate the five field functions that define the deformation of the shear flexible shallow shell—namely, the in-plane displacements u and v, the normal displacement w, and the face rotations θ_x and θ_y. We shall now examine how the use of such low-order polynomials leads to the locking phenomenon.

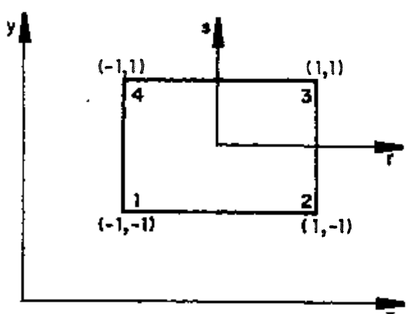

Fig. 2. Global and local isoparametric coordinate system.

3. FIELD CONSISTENCY AND SPURIOUS CONSTRAINING

As the phenomenon of shear locking is quite well established, we shall consider first, the effect of the introduction of these simple shape functions into one of the shear strains. We notice that for a rectangular element aligned with the global (x, y) axes, we can write

$$\begin{aligned} \theta_x &= a_0 + a_1 x + a_2 y + a_3 xy, \\ w &= b_0 + b_1 x + b_2 y + b_3 xy, \end{aligned} \tag{5}$$

so that the shear strain γ_{xz} becomes

$$\gamma_{xz} = (a_0 - b_1) + (a_2 - b_3)y + a_1 x + a_3 xy. \tag{6}$$

We notice now that the coefficients associated with the constant and linear in y terms comprise quantities from both interpolation functions. However, the coefficients associated with the linear in x and with the product xy terms originate only from the interpolation for the face rotation θ_x. If an exact integration of the shear strain energy had been performed, then one can expect that in the penalty function limits that are reached as the thickness of the plate or shell is made very small, the following four constraints are in some manner enforced:

$$\begin{aligned} a_0 - b_1 &= 0, \\ a_2 - b_3 &= 0, \\ a_1 &= 0, \\ a_3 &= 0. \end{aligned} \tag{7}$$

Clearly, the first two are true Kirchhoff constraints relating the face rotations to the slope of the middle surface and the remaining two are spurious constraints that impose unnecessary restrictions on the behaviour of the θ_x field. An examination along these lines showed that the shear locking emerged from these spurious constraints, and that shear locking can be removed, either by using optimal integration rules so that the "field-inconsistent" terms such as $a_1 x$ and $a_3 xy$ in eqn (6) are removed or by using substitute smoothed shape functions for θ_x etc. of the form

$$\bar{\theta}_x = \bar{a}_0 + \bar{a}_1 y, \tag{8}$$

such that the shear strains are consistently interpolated with terms from both interpolation functions [2, 3].

It is now easy to extend this interpretation to the anticipated phenomenon of membrane locking. We see that, starting with

$$\begin{aligned} u &= c_0 + c_1 x + c_2 y + c_3 xy, \\ w &= b_0 + b_1 x + b_2 y + b_3 xy, \end{aligned} \tag{9}$$

we have, e.g.

$$\epsilon_x = \left(c_0 + \frac{b_0}{R_x}\right) + \left(c_3 + \frac{b_2}{R_x}\right)y + \frac{b_1}{R_x}x + \frac{b_3}{R_x}xy\,. \qquad (10)$$

In a very thin shell, we can expect an inextensional bending that will emphasise the emergence of the following constraints:

$$c_1 + \frac{b_0}{R_x} = 0, \quad c_3 + \frac{b_2}{R_x} = 0, \quad \frac{b_1}{R_x} = 0, \quad \frac{b_3}{R_x} = 0. \qquad (11)$$

Clearly, the last two constraints will now introduce a spurious in-plane stiffening state of stress, that was described as "in-plane" or membrane locking and this was clearly demonstrated for the curved beam and arch element [5, 6]. In the next section, we shall establish for a simple shallow shell element that by using smoothed interpolation functions for w in the membrane strains, and for θ_x and θ_y in the shear strains, we can arrive at an element that is free of both membrane and shear locking.

4. SMOOTHED INTERPOLATION FUNCTIONS

In Refs. [5-7], we have seen that the spurious constraints can be eliminated by an optimal integration strategy that will use separate numerical integration rules for the different strain energy components. Thus, as the spurious constraints originate from the membrane energy, these terms are integrated by an order low enough to remove the inconsistent terms but accurate enough to retain all consistent terms. The bending energy is, of course, evaluated exactly.

Alternatively, if a properly chosen substitute smoothed function procedure is used, then a single and exact integration strategy can be used uniformly for all energy terms, and this will ensure the presence of all true constraints. We shall adopt this strategy here. From eqns (1), we notice that the strains that must be consistently interpolated are ϵ_x, γ_{xz}, and γ_{yz} so that we need a substitute function for w consistent with $u_{,x}$ in ϵ_x, substitute functions for u and θ_x consistent with $w_{,x}$ for γ_{xz}, and a substitute function for θ_y consistent with $w_{,y}$ for γ_{yz}. We require, therefore, two substitute functions of the form

$$h_x = b_0 + b_1 y, \quad h_y = c_0 + c_1 x, \qquad (12)$$

which will, in the appropriate strain definitions, replace the original interpolation function

$$h = a_0 + a_1 x + a_2 y + a_3 xy. \qquad (13)$$

This can be obtained by choosing h_x and h_y to be least-squares approximation of h over the element domain. In this case, as the element is rectangular, the operation is simple and results in the substitute interpolation formulas

$$h_{x1} = (1 - s)/4, \quad h_{x2} = (1 - s)/4, \quad h_{x3} = (1 + s)/4, \quad h_{x4} = (1 + s)/4 \qquad (14)$$

and

$$h_{y1} = (1 - r)/4, \quad h_{y2} = (1 + r)/4, \quad h_{y3} = (1 + r)/4, \quad h_{y4} = (1 - r)/4. \qquad (15)$$

These interpolations form the basis of the reduced interpolation technique adopted here so that both membrane and shear locking are obviated.

5. NUMERICAL EXPERIMENTS

The element stiffness matrix is now computed in a straightforward manner, using uniformly, a 2 x 2 Gaussian integration scheme. Options are introduced, which allow various reduced interpolation schemes to be investigated. Of particular interest in the present study are the following three schemes:

Scheme A—the element proposed in this paper, without any locking by using reduced interpolations for membrane and shear strains.

Scheme B—smoothed interpolation of shear strains alone is introduced, thus producing an element in which membrane locking is present.

Scheme C—original shape functions are used throughout; and so the element has both membrane and shear locking.

The test problem chosen is the pinched cylinder shown in Fig. 3. This has been used most often as

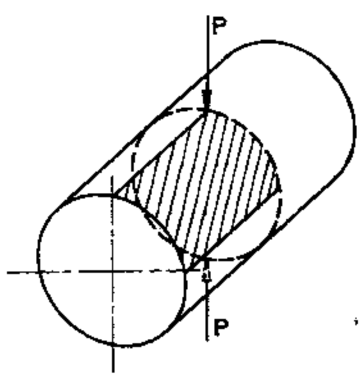

Fig. 3. The pinched cylinder.

Table 1. Deflections under load for a thick pinched cylinder (t = 0.094 in.) for 1 x N mesh

	Reference [13]		Present analysis		
N	dof	w	dof	w_A	w_C
1	18	0.0025	8	0.0018	0.0001
2	36	0.0802	16	0.0626	0.0002
3	54	0.1026	...	...	...
4	72	0.1087	32	0.0971	0.0005
8	...	...	64	0.1078	0.0014
Exact (Ref. [14]) w = 0.1084					

the standard for evaluating cylindrical shell elements. One octant is considered and the geometrical and elastic properties considered are: radius = 4.953 in., thickness t_1 = 0.094 or t_2 = 0.01548 in., length = 10.35 in., E = 10.5 x 10^6 lbf. in^{-2}, ν = 0.3125, and a load of P = 100 lbf. is applied in both the thick and thin cylinders. The exact solution [14] is based on an inextensional theory and gives the deflections under this load as w_1 = 0.1084 in. and w_2 = 0.2428 x 10^2 in., respectively.

Table 1 shows the results obtained as the octant is idealised by 1 x N elements, where N is the number of divisions along the curved edge of the octant and only one division is used along the length. This offers a ready comparison with identical idealisations used for the 48 dof element of Bogner *et al.* [13]. We notice that the reduced interpolated element (scheme A) is as efficient as the high precision element on a degree-of-freedom basis. Also noteworthy, is the very poor behaviour shown by the original element (scheme C)—the locking makes the results virtually meaningless. Table 2 compares the locking effect in the thick (t = 0.094 in.) and thin (t = 0.01548 in.) cases, respectively. The additional stiffening parameter due to locking is defined as

$$e = \frac{w_{theory}}{w_C} - 1$$

and the very large values of e reflect errors approaching 100%. The remarkable agreement between the ratio of additional stiffening parameters for the thick and thin cases—(e_2/e_1) and $(t_1/t_2)^2$

Table 2. Nature of shear + membrane locking in a thick and a thin pinched cylinder

	t_1 = 0.094 in.		t_2 = 0.01548		
N	w_C	e_1	w_C	e_2	e_2/e_1
1	0.9664 x 10^{-4}	1121	0.5870 x 10^{-3}	41362	36.9
2	0.2068 x 10^{-3}	523	0.1257 x 10^{-2}	19315	36.9
4	0.4747 x 10^{-3}	227	0.2891 x 10^{-2}	8397	37.0
8	0.1405 x 10^{-2}	76	0.8629 x 10^{-2}	2813	37.0
w*	0.1084	—	0.2428 x 10^2	—	—
			$(t_1/t_2)^2$		36.9

*Reference [14].

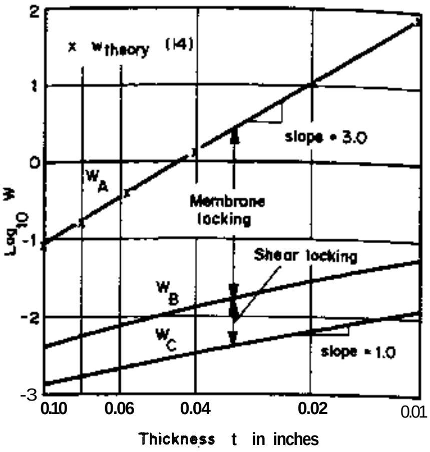

Fig. 4. Logarithmic plot of deflection under load vs thickness.

shows that the errors introduced when scheme C is used are essentially those due to locking, a typical feature of which is that they are exaggerated in a $1/t^2$ fashion, as seen here.

We next examine the element for the membrane locking behaviour. This can be demonstrated by comparing the results obtained from schemes A, B, and C, as the thickness of the shell is varied from a reasonably thick to a reasonably thin value. Figure 4 plots on logarithmic scales, the theoretically predicted deflection under the load w(theory), and the finite element results w_A, w_B, and w_C from a 1 x 8 mesh. It is seen that the w_A line is very close to the exact values w(theory), and that it follows very accurately a t^{-3} relationship that is to be expected when the bending is of an inextensional type. We also find that w_B and w_C follow a t^{-1} rule, showing that an additional locking stiffness of t^2 nature is introduced in the membrane and shear locking mechanisms. Clearly, the original element without reduced interpolation or even with only shear locking removed, will be too poor in performance to be used in any practical case.

6. CONCLUSIONS

We have demonstrated here, how a simple Co continuous rectangular cylindrical shell element, based on the lowest order bi-linear polynomial interpolation, without the trignometric rigid body terms, can be made useful by a reduced interpolation technique for the membrane and shear strains. This also explains the remarkable improvement noticed in the "reduced interaction" technique of Mohr [1–4] where the use of a similar substitute lower order function for w in the membrane strains results in a better element. It is quite simple to extend these concepts to derive a doubly curved shell element using simple polynomial interpolations and with optimal reduced interpolation so that there is no membrane or shear locking.

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REFERENCES

1. G. A. Mohr, Numerically integrated triangular element for doubly curved thin shells. *Comput. Struct.* **11**, 565–571 (1980).
2. G. A. Mohr, A doubly curved isoparametric triangular shell element. *Comput. Struct.* 14, 9-13 (1981).
3. G. A. Mohr, Application of penalty factors to a doubly curved quadratic shell element. *Comput. Struct.* 14, 15–19 (1980).
4. G. A. Mohr and N. B, Paterson, A natural numerical differential geometry scheme for a doubly curved shell element. *Comput. Struct.* 18, 433-439 (1984).
5. G. Prathap and G. R. Bhashyam, Reduced integration and the shear flexible beam element. *Int. J. Numer. Meth. Engng* 18, 195–210 (1982).
6. H. Stolarski and T. Belytschko. Membrane locking and reduced integration for curved elements. *J. Appl. Mech.* **49**, 172–176 (1982).
7. G. Prathap, The curved beam/deep arch/finite ring element revisited. NAL-TM-ST-501/257/83 (1983).
8. D. G. Ashwell and R. H. Gallagher, Finite Elements for Thin Shells and Curved Members. Wiley. London (1976).
9. J. H. Argyris and D. W. Scharpf. The SHEBA family of shell elements for the matrix displacement method. *J. R. Aeronaut. Soc.* 72, 873-883 (1968).
10. G. R. Cowper, G. M. Lindberg and M. D. Olson, A shallow shell finite element of triangular shape. *Int. J. Solids Struct.* 6, 1133–1156 (1970).
11. D. J. Dawe, High order finite element for shell analysis. *Int. J. Solids Struct.* 11, 1097–1112 (1975).
12. D. G. Ashwell and A. B. Sabir. A new cylindrical shell finite element based on simple independent strain functions. *Int. J. Mech, Sci.* 14, 171–185 (1972).
S3. F. K. Bogner, R. L. Fox and L. A. Schmit. A cylindrical shell discrete element. *AIAA J.* 5, 745-750 (1967).
14. S. Timoshenko and S. Woinowsky-Krieger. *Theory of Plates and Shells.* 2nd Edn. pp. 501–506. McGraw-Hill, New York (1959).