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Distinguished Representations, Base Change, and Reducibility for Unitary Groups

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1 Introduction

A representation (π, V) of a group G is said to be distinguished with respect to a character χ of a subgroup H if there exists a linear form l of V satisfying $l(\pi(h)v) = \chi(h)l(v)$ for all $v \in V$ and $h \in H$. When the character χ is taken to be the trivial character, such representations are also called as distinguished representations of G with respect to H. The concept of distinguished representations can be carried over to a continuous context of representations of real and p-adic Lie groups, as well in a global automorphic context (where the requirement of a nonzero linear form is replaced by the nonvanishing of a period integral). The philosophy, due to Jacquet, is that representations of a group G distinguished with respect to a subgroup H of fixed points of an involution on G are often functorial lifts from another group G'.

In this paper, we consider $G = \mathrm{Res}_{E/F}\,\mathrm{GL}(n)$ and $H = \mathrm{GL}(n)$, where E is a quadratic extension of a non-Archimedean local field F of characteristic zero. In this case, the group G' is conjectured to be the quasisplit unitary group with respect to E/F,

$$G' = U(n) = \left\{ g \in \mathrm{GL}_n(E) \mid gJ^t\bar{g} = J \right\}, \tag{1.1}$$

where $J_{ij} = (-1)^{n-i}\delta_{i,n-j+1}$ and $\bar{g}$ is the Galois conjugate of g. There are two base change

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maps from $U(n)$ to $GL(n)$ over E called the stable and the unstable base change maps (see Section 4.2). We have the following conjecture due to Flicker and Rallis (see[**4**]).

Conjecture 1.1. Let π be an irreducible admissible representation of $GL_n(E)$. If n is odd (resp., even), then π is $GL_n(F)$-distinguished if and only if it is a stable (resp., unstable) base change from $U(n)$. □

When $n = 1$ the above conjecture is just Hilbert's Theorem 90. The case $n = 2$ is established by Flicker [**4**]. The following theorem proves the conjecture for a supercuspidal representation when $n = 3$.

Theorem 1.2. A supercuspidal representation π of $GL_3(E)$ is distinguished with respect to $GL_3(F)$ if and only if it is a stable base change lift from $U(3)$. □

Let G be a reductive p-adic group. Any irreducible tempered representation of G occurs as a component of an induced representation $I(\pi)$, parabolically induced from a square-integrable representation π of the Levi component M of a parabolic subgroup P of G. Thus the tempered spectrum of G is determined from a knowledge of the discrete series representations of the Levi components of different parabolics and knowing the decomposition of induced representations. The decomposition of $I(\pi)$ is governed by the theory of R-groups.

Let $G = U(n, n)$ be the quasisplit unitary group in $2n$ variables over a p-adic field F, defined with respect to a quadratic extension E of F. Let P be a parabolic subgroup of G with a Levi component M isomorphic to $GL_{n_1}(E) \times \cdots \times GL_{n_t}(E)$ for some integers $n_i \geq 1$ satisfying $\sum_{i=1}^{t} n_i = n$. Let π_i, $1 \leq i \leq t$, be discrete series representations of $GL_{n_i}(E)$. Let $\pi = \pi_1 \otimes \cdots \otimes \pi_t$ be the discrete series representation of M. Let $\omega_{E/F}$ denote the quadratic character of F^* associated to the quadratic extension E/F. The following theorem gives a description of the R-group $R(\pi)$ in terms of distinguishedness of the representations π_i.

Theorem 1.3. With the above notation,

$$R(\pi) \simeq (\mathbb{Z}/2\mathbb{Z})^r, \tag{1.2}$$

where r is the number of inequivalent representations π_i which are $\omega_{E/F}$-distinguished with respect to $GL_{n_i}(F)$. □

Corollary 1.4. Let P be a maximal parabolic of $U(n, n)$ with Levi component isomorphic to $GL_n(E)$, and π be a discrete series representation of $GL_n(E)$. Then $I(\pi)$ is reducible if and only if π is $\omega_{E/F}$-distinguished with respect to $GL_n(F)$. □

A particular consequence of the corollary is the following result about the Steinberg representation of $\mathrm{GL}_n(E)$, which is part of a more general conjecture, due to D. Prasad, about the Steinberg representation of $G(E)$, where G is a reductive algebraic group over F [15].

Theorem 1.5. Let π be the Steinberg representation of $\mathrm{GL}_n(E)$. Then π is distinguished with respect to a character $\chi \circ \det$ of $\mathrm{GL}_n(F)$, for a character χ of F^*, if and only if n is odd and χ is the trivial character, or n is even and $\chi = \omega_{E/F}$. $\square$

Our approach to the above theorems is via the theory of Asai L-functions. The Asai L-function, also called the twisted tensor L-function, can be defined in three different ways: one via the local Langlands correspondence and in terms of Langlands parameters denoted by $L(s, \mathrm{As}(\pi))$; via the theory of Rankin-Selberg integrals [3, 5, 12] denoted by $L_1(s, \mathrm{As}(\pi))$; and the Langlands-Shahidi method (applied to a suitable unitary group) [6, 18] denoted by $L_2(s, \mathrm{As}(\pi))$. It is of course expected that all the above three L-functions match.

The main point is that the analytical properties of the different definitions of Asai L-function give different insights about the representation: the Asai L-function defined via the Rankin-Selberg method can be related to distinguishedness with respect to $\mathrm{GL}_n(F)$, whereas the Asai L-function defined via the Langlands-Shahidi method is related to the base change theory from $U(n)$, and to reducibility questions for $U(n, n)$. Thus the following theorem, proved using global methods, is a key ingredient towards a proof of the above theorems.

Theorem 1.6. Let π be a square-integrable representation of $\mathrm{GL}_n(E)$. Then $L_1(s, \mathrm{As}(\pi)) = L_2(s, \mathrm{As}(\pi))$. $\square$

2 Asai L-functions

2.1 Langlands parameters

Let F be a non-Archimedean local field and let E be a quadratic extension of F. The Weil-Deligne group W'_E of E is of index two in the Weil-Deligne group W'_F of F. Choose $\sigma \in W'_F \backslash W'_E$ of order 2. Given a continuous, Φ-semisimple representation ρ of W'_E of dimension n, the representation $\mathrm{As}(\rho) : W'_F \to \mathrm{GL}_{n^2}(\mathbb{C})$ given by tensor induction of ρ is defined as

$$\mathrm{As}(\rho)(x) = \begin{cases} \rho(x) \otimes \rho(\sigma^{-1} x \sigma) & \text{if } x \in W'_E, \\ [\rho(\sigma x) \otimes \rho(x \sigma)] \circ I & \text{if } x \notin W'_E, \end{cases} \tag{2.1}$$

where $I(v_1 \otimes v_2) = v_2 \otimes v_1$ is the switching operator. Let π be an irreducible, admissible representation of $GL_n(E)$ with Langlands parameter ρ_π. The Asai L-function $L(s, As(\pi))$ is defined to be the L-function $L(s, As(\rho_\pi))$.

2.2 Rankin-Selberg method

2.2.1 Local theory. We recall the Rankin-Selberg theory of the Asai L-function [3, 5, 12]. Let F be a non-Archimedean local field and let E be either a quadratic extension of F or $F \oplus F$. Let π be an irreducible admissible generic representation of $GL_n(E)$. We take an additive character ψ of E which restricts trivially to F. There exists an additive character ψ_0 of F such that $\psi(x) = \psi_0(\Delta(x - \bar{x}))$, where Δ is a trace zero element of E^*. Let $\mathcal{W}(\pi, \psi)$ denote the Whittaker model of π with respect to ψ. Let $N_n(F)$ be the unipotent radical of the Borel subgroup of $GL_n(F)$. Consider the integral (see [3])

$$\Psi(s, W, \Phi) = \int_{N_n(F)\backslash GL_n(F)} W(g)\Phi\big((0, 0, \dots, 1)g\big)|\det g|_F^s dg, \tag{2.2}$$

where $\Phi \in \mathcal{S}(F^n)$, the space of locally constant compactly supported functions on F^n, and dg is a $GL_n(F)$-invariant measure on $N_n(F)\backslash GL_n(F)$.

In [5], Flicker proves that the above integral converges absolutely in some right half-plane to a rational function in $X = q^{-s}$, where $q = q_F$ is the cardinality of the residue field of F. The space spanned by $\Psi(s, W, \Phi)$ (as W and Φ vary) is a fractional ideal in $\mathbb{C}[X, X^{-1}]$ containing the constant function 1. We can choose a unique generator of this ideal of the form $P_1(X)^{-1}$, $P_1(X) \in \mathbb{C}[X]$ such that $P_1(0) = 1$. Define the Asai L-function $L_1(s, As(\pi))$ as

$$L_1\big(s, As(\pi)\big) = P_1\big(q^{-s}\big)^{-1}. \tag{2.3}$$

This does not depend on the choice of the additive character ψ. Moreover, $\Psi(s, W, \Phi)$ satisfies the functional equation

$$\Psi(1 - s, \widetilde{W}, \widehat{\Phi}) = \gamma_1\big(s, As(\pi), \psi\big)\Psi(s, W, \Phi), \tag{2.4}$$

where $\widetilde{W}(g) = W(w\,{}^t g^{-1})$, w is the longest element of the Weyl group, and $\widehat{\Phi}$ is the Fourier transform of Φ with respect to ψ_0. The epsilon factor

$$\epsilon_1(s, \mathrm{As}(\pi), \psi) = \gamma_1(s, \mathrm{As}(\pi), \psi)\frac{L_1(s, \mathrm{As}(\pi))}{L_1(1-s, \mathrm{As}(\pi^\vee))} \tag{2.5}$$

is a monomial in q_F^{-s}.

If $E = F \oplus F$, write $\pi = \pi_1 \times \pi_2$ considered as a representation of $GL_n(F) \times GL_n(F)$. Then

$$L_1(s, \mathrm{As}(\pi)) = L(s, \pi_1 \times \pi_2), \tag{2.6}$$

where the right-hand side is the Rankin-Selberg L-factor of $\pi_1 \times \pi_2$.

We have the following proposition [3, proposition in Section 3].

Proposition 2.1. Suppose E/F is an unramified quadratic extension. Let $\pi = \mathrm{Ps}(\mu_1, \ldots, \mu_n)$ be an unramified unitary representation induced from the character $(t_1, \ldots, t_n) \to \prod \mu_i(t_i)$ of the diagonal torus in $GL_n(E)$. Let W_π^0 be the spherical Whittaker function, and let Φ_F^0 be the characteristic function of $\mathcal{O}_F^n$. Then

$$\Psi(s, W_\pi^0, \Phi_F^0) = \prod_{j=1}^{n}(1 - \mu_j(\varpi_F)q_F^{-s})^{-1} \cdot \prod_{i<j}(1 - \mu_i(\varpi_F)\mu_j(\varpi_F)q_F^{-2s})^{-1}, \tag{2.7}$$

where ϖ_F is a uniformizing parameter of F. □

The following proposition is proved in [12, Theorem 4].

Proposition 2.2. Let π be a square-integrable representation of $GL_n(E)$. Then $L_1(s, \mathrm{As}(\pi))$ is regular in the region $\mathrm{Re}(s) > 0$. □

We remark that for the proof of Theorem 1.6 all that we require is that $L_1(s, \mathrm{As}(\pi))$ be regular in the region $\mathrm{Re}(s) \geq 1/2$.

2.2.2 Global theory. Now let L/K be a quadratic extension of number fields. We assume that the Archimedean places of K split in L. Let ψ_0 be a nontrivial character of $\mathbb{A}_K/K$, and let $\psi = \psi_0(\Delta(x - \bar{x}))$, where Δ is an element of trace 0 in L. For a global field K, let Σ_K denote the set of places of K. Let $\Pi = \bigotimes_{w \in \Sigma_L} \Pi_w$ be a representation of $GL_n(\mathbb{A}_L)$. Let T be a finite set of places of K containing the following places:

(i) the Archimedean places of K,
(ii) the ramified places of the extension L/K,
(iii) the places v of K dividing a place w of L, where either $\psi_{0,v}$, ψ_{L_w}, or Π_w is ramified.

Define

$$L'_{1,v}(s, \mathrm{As}(\Pi)) = \begin{cases} L_1(s, \mathrm{As}(\Pi_w)), & w|v, v \in T, v \text{ inert}, \\ \Psi_v(s, W^0_{\Pi_w}, \Phi^0_{F_v}), & v \text{ inert}, v \notin T, \\ L(s, \Pi_{w_1} \times \Pi_{w_2}), & v \text{ splits}, v = w_1 w_2. \end{cases} \tag{2.8}$$

Remark 2.3. Let v be a place of K not in T, inert in L, and w the place of L dividing v. It is not known that $L_1(s, \mathrm{As}(\Pi_w)) = \Psi(s, W^0_{\Pi_w}, \Phi^0_{K_v})$. In the notation of **Proposition 2.1**, the right-hand side is the L-factor associated by Langlands functoriality.

Following Kable [12], we define the Rankin-Selberg Asai L-function $L_1(s, \mathrm{As}(\Pi))$ as follows:

$$L_1(s, \mathrm{As}(\Pi), T) = \prod_{v \in \Sigma_K} L'_{1,v}(s, \mathrm{As}(\Pi)). \tag{2.9}$$

We have the following functional equation.

Proposition 2.4 (see [12, Theorem 5]). Let Π be a cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$. Then $L_1(s, \mathrm{As}(\Pi), T)$ admits a meromorphic continuation to the entire plane and satisfies the functional equation

$$L_1(s, \mathrm{As}(\Pi), T) = \epsilon_1(s, \mathrm{As}(\Pi), T) L_1(1 - s, \mathrm{As}(\Pi^\vee), T), \tag{2.10}$$

where the function $\epsilon_1(s, \mathrm{As}(\Pi), T)$ is entire and nonvanishing, where T is a finite set of places of K chosen as above. □

2.3 Langlands-Shahidi method

2.3.1 Local theory. We now recall the Langlands-Shahidi approach to the Asai L-function [6, 18]. Let $G = U(n, n)$ be the quasisplit unitary group in $2n$ variables with respect to E/F. The group $M = R_{E/F} GL_n$ can be embedded as a Levi component of a maximal parabolic subgroup P of G with unipotent radical N. Let r be the adjoint representation of the L-group of M on the Lie algebra of the L-group of N. Fix an additive character ψ_0 of F. The Langlands-Shahidi gamma factor $\gamma_2(s, \pi, r, \psi_0)$ defined in [18] is a rational function of q^{-s}. Let $P_2(X)$ be the unique polynomial satisfying $P_2(0) = 1$ such that $P_2(q^{-s})$ is the numerator of $\gamma_2(s, \pi, r, \psi_0)$. For a tempered π, the Langlands-Shahidi Asai L-function is defined as

$$L_2(s, \mathrm{As}(\pi)) = \frac{1}{P_2(q^{-s})}. \tag{2.11}$$

The L-function is independent of the additive character. The quantity

$$\epsilon_2(s, \mathrm{As}(\pi), \psi_0) = \gamma_2(s, \pi, r, \psi_0) \frac{L_2(s, \mathrm{As}(\pi))}{L_2(1-s, \mathrm{As}(\pi^\vee))} \tag{2.12}$$

is the Langlands-Shahidi epsilon factor, and is a monomial in q^{-s}.

The analytical properties of $L_2(s, \mathrm{As}(\pi))$ are proved in [18, Theorem 3.5, Proposition 7.2].

Proposition 2.5. Let π be an irreducible admissible representation of $\mathrm{GL}_n(E)$. Then the following hold.

(1) If E is an unramified extension of F and $\pi = \mathrm{Ps}(\mu_1, \ldots, \mu_n)$ is a unitary unramified representation of $\mathrm{GL}_n(E)$, as in the hypothesis of Proposition 2.1, then

$$L_2(s, \mathrm{As}(\pi)) = \prod_{j=1}^{n} \left(1 - \mu_j(\varpi_F) q_F^{-s}\right)^{-1} \cdot \prod_{i<j} \left(1 - \mu_i(\varpi_F)\mu_j(\varpi_F) q_F^{-2s}\right)^{-1}. \tag{2.13}$$

(2) Let π be a tempered representation of $\mathrm{GL}_n(E)$. Then $L_2(s, \mathrm{As}(\pi))$ is regular in the region $\mathrm{Re}(s) > 0$. □

2.3.2 Global theory. Let L/K be a quadratic extension of number fields, and let $\Pi = \bigotimes_w \Pi_w$ be a representation of $\mathrm{GL}_n(\mathbb{A}_L)$. Define for a place v of K,

$$L_{2,v}(s, \mathrm{As}(\Pi)) = \begin{cases} L_2(s, \mathrm{As}(\Pi_w)), & w|v, v \text{ inert}, \\ L(s, \Pi_{w_1} \times \Pi_{w_2}), & v \text{ splits}, v = w_1 w_2. \end{cases} \tag{2.14}$$

Define the global L-function

$$L_2(s, \mathrm{As}(\Pi)) = \prod_{v \in \Sigma_K} L_{2,v}(s, \mathrm{As}(\Pi)). \tag{2.15}$$

Then we have the following functional equation [18].

Proposition 2.6. Let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_L)$. Then $L_2(s, \mathrm{As}(\Pi))$ admits a meromorphic continuation to the entire plane and satisfies the functional equation

$$L_2(s, \mathrm{As}(\Pi)) = \epsilon_2(s, \mathrm{As}(\Pi)) L_2(1-s, \mathrm{As}(\Pi^\vee)), \tag{2.16}$$

where the function $\epsilon_2(s, \mathrm{As}(\Pi))$ is entire and nonvanishing. □

3 Proof of Theorem 1.6

The proof of Theorem 1.6 is via global methods. The following proposition embedding a square-integrable representation π as the local component of a cuspidal automorphic representation is well known [12, Lemma 5] and [2, Chapter 1, Lemma 6.5].

Proposition 3.1. Let E/F be a quadratic extension of non-Archimedean local fields of characteristic zero and residue characteristic p. Let π be a square-integrable representation of $GL_n(E)$. Then the following hold.

(1) There exist a number field K, a quadratic extension L of K, and a place v_0 of K inert in L such that $K_{v_0} \simeq F$ and $L_{w_0} \simeq E$, where w_0 is the unique place of L dividing v_0. Further, v_0 is the unique place of K lying over the rational prime p, and the real places of K are split in L.

(2) There exists a cuspidal automorphic representation Π of $GL_n(\mathbb{A}_L)$ such that $\Pi_{w_0} \simeq \pi$. □

Let Π be a cuspidal representation of $GL_n(\mathbb{A}_L)$ satisfying the properties of the above proposition. Choose a finite set T of places of K as in Proposition 2.4. Consider the ratio

$$F(s, \Pi) = \frac{L_2\big(s, As(\Pi)\big)}{L_1\big(s, As(\Pi), T\big)}. \tag{3.1}$$

If $v = w_1 w_2$ is a place of K which splits into two places w_1 and w_2 of L, then

$$L'_{1,v}\big(s, As(\Pi)\big) = L_{2,v}\big(s, As(\Pi)\big) = L\big(s, \Pi_{w_1} \times \Pi_{w_2}\big). \tag{3.2}$$

By Propositions 2.1 and 2.5, if v is a place of K which is inert and not in T, then

$$L'_{1,v}\big(s, As(\Pi)\big) = L_{2,v}\big(s, As(\Pi)\big). \tag{3.3}$$

Hence,

$$F(s, \Pi) = \prod_{v \in T} \frac{L_{2,v}\big(s, As(\Pi)\big)}{L'_{1,v}\big(s, As(\Pi)\big)}. \tag{3.4}$$

Write

$$F(s,\Pi) = G(s,\Pi)Q(s,\Pi)P_0(s,\Pi), \tag{3.5}$$

where

(i) the function $G(s,\Pi)$ is the ratio of the L-factors at the Archimedean places; it is a ratio of products of Gamma functions of the form $\Gamma(as+b)$ for some suitable constants a, b;

(ii) the function

$$Q(s,\Pi) = \frac{\prod_{i=1}^{n}\left(1-\alpha_i q_{v_i}^{-s}\right)}{\prod_{j=1}^{m}\left(1-\beta_j q_{v_j}^{-s}\right)}, \quad v_i, v_j \in T' := T\backslash\{v_0\} \tag{3.6}$$

is a ratio of the L-factors at the finite set of places of T not equal to v_0; it is a ratio of products of distinct functions of the form $(1-\beta q_v^{-s})$, $\beta \neq 0$, where $v \in T' := T\backslash\{v_0\}$, and q_v is the number of elements of the residue field; by our assumption on K, $(p, q_v) = 1$;

(iii) the function

$$P_0(s,\Pi) = \frac{L_2\left(s, \mathrm{As}(\pi)\right)}{L_1\left(s, \mathrm{As}(\pi)\right)} \tag{3.7}$$

is a ratio of products of functions of the form $(1-\alpha q_{v_0}^{-s})$.

By Propositions 2.2 and 2.5, the functions $P_0(s,\Pi)$ and $P_0(s,\Pi^\vee)$ are regular and nonvanishing in the region $\mathrm{Re}(s) \geq 1/2$.

We claim the following.

Claim 3.2. Let γ_0 be a pole (resp., zero) of $P_0(s,\Pi)$. The function $F(s,\Pi)$ has a pole (resp., zero) at all but finitely many elements of the form $\gamma_0 + 2\pi i k/\log q_{v_0}$, $k \in \mathbb{Z}$. □

Proof. Suppose that the function $F(s,\Pi)$ is regular at points of the form $\gamma_0 + 2\pi i l/\log q_{v_0}$ for integers $l \in C$, where C is an infinite subset of the integers. Since $G(s)$ can contribute only finitely many zeros on any line with real part constant, these poles have to be cancelled by zeros of $Q(s,\Pi)$. Since T is finite, and the local L-factors are polynomial functions in q_v^{-s}, there are a $v \in T'$, $\gamma \in \mathbb{C}$, and a function $f : C \to \mathbb{Z}$ such that

$$\gamma_0 + \frac{2\pi i l}{\log q_{v_0}} = \gamma + \frac{2\pi i f(l)}{\log q_v} \tag{3.8}$$

for infinitely many $l \in C$. Taking the difference of any two elements, we get $\log q_{v_0}/\log q_v \in \mathbb{Q}$. This is not possible as q_{v_0} and q_v are coprime integers. Hence, all but finitely many poles of the form $\gamma_0 + 2\pi i k/\log q_{v_0}$, $k \in \mathbb{Z}$, are poles of $F(s, \Pi)$. ■

Since $P_0(s, \Pi)$ is regular in the region $\mathrm{Re}(s) \geq 1/2$, we obtain $\mathrm{Re}(\gamma_0) < 1/2$. From the global functional equations given by Propositions 2.4 and 2.6, $F(s, \Pi)$ satisfies a functional equation

$$F(s, \Pi) = \eta(s, \Pi) F\big(1 - s, \Pi^{\vee}\big), \tag{3.9}$$

where $\eta(s, \Pi)$ is an entire nonvanishing function. Hence, $F(s, \Pi^{\vee})$ has infinitely many poles of the form $1 - \gamma_0 + 2\pi i k/\log q_{v_0}$ with $k \in \mathbb{Z}$. Since $P_0(s, \Pi^{\vee})$ is regular in the region $\mathrm{Re}(s) \geq 1/2$, these poles have to be poles of $G(s, \Pi^{\vee}) Q(s, \Pi^{\vee})$. Arguing as in proof of the above claim, we obtain a contradiction. Arguing similarly with the zeros instead of poles, we obtain that $P_0(s, \Pi)$ is an entire nonvanishing function, and hence it is a constant. Since the L-factors are normalised, we obtain a proof of Theorem 1.6.

Remark 3.3. The method of proof of Theorem 1.6 is a general method allowing us to establish an equality for two possibly different definitions of L-factors at "bad" places. This requires a global functional equation, equality of the L-factors at all good places, and regularity in the region $\mathrm{Re}(s) \geq 1/2$ for the "bad" L-factors. The method is illustrated in [16] in the context of functoriality, but allowing the use of cyclic base change. It is used by Kable in [12] to prove, for a square-integrable representation, that the Rankin-Selberg L-factor $L(s, \pi \times \bar{\pi})$ factorizes as a product of $L_1(s, \mathrm{As}(\pi))$ times $L_1(s, \mathrm{As}(\pi \otimes \widetilde{\omega}))$, where $\widetilde{\omega}$ is an extension of $\omega_{E/F}$, the quadratic character corresponding to the extension E/F. A proof of strong multiplicity one in the Selberg class using similar arguments is given in [13].

Remark 3.4. It has been shown by Henniart [10] using similar global methods, that for any irreducible, admissible representation π of $GL_n(E)$, the equality $L(s, \mathrm{As}(\pi)) = L_2(s, \mathrm{As}(\pi))$. Henniart's proof uses cyclic base change and the inductivity of γ-factors to go from square-integrable to all irreducible, admissible representations. Since we do not know inductivity of the Rankin-Selberg γ-factors $\gamma_1(s, \mathrm{As}(\pi), \psi)$, we cannot derive a similar statement for the Rankin-Selberg L-factors.

Remark 3.5. Using cyclic base change as in [16] or [10], it is possible to show that the ϵ-factors $\epsilon_1(s, \mathrm{As}(\pi), \psi)$ and $\epsilon_2(s, \mathrm{As}(\pi), \psi_0)$ are equal up to a root of unity, when π is square-integrable.

4 Applications

4.1 Analytic characterisation of distinguished representations

The proofs of Theorems 1.2 and 1.3 use the following proposition relating the concept of distinguishedness with the analytical properties of the (Rankin-Selberg) Asai L-function [1, Corollary 1.5].

Proposition 4.1. Let π be a square-integrable representation of $GL_n(E)$. Then π is distinguished with respect to $GL_n(F)$ if and only if $L_1(s, As(\pi))$ has a pole at $s = 0$. □

4.2 Base change for $U(3)$ and proof of Theorem 1.2

Let $W_{E/F}$ be the relative Weil group of E/F defined as the semidirect product of $E^* \rtimes \mathrm{Gal}(E/F)$ for the natural action of $\mathrm{Gal}(E/F)$ on E^*. The Langlands dual group of $U(n)$ is given by ${}^L U(n) = GL_n(\mathbb{C}) \rtimes W_{E/F}$, where $W_{E/F}$ acts via the projection to $\mathrm{Gal}(E/F)$, and the nontrivial element $\sigma \in \mathrm{Gal}(E/F)$ acts by $\sigma(g) = J({}^t g^{-1})J^{-1}$ on $GL_n(\mathbb{C})$. The Langlands dual group of $R_{E/F}(GL_n)$ is given by

$$ {}^L R_{E/F}\big(U(n)\big) = \big[\, GL_n(\mathbb{C}) \times GL_n(\mathbb{C})\big] \rtimes W_{E/F}. \tag{4.1} $$

Here again the action of $W_{E/F}$ is via the projection to $\mathrm{Gal}(E/F)$, and σ acts by $(g, h) \mapsto (J\,{}^t h^{-1} J^{-1}, J\,{}^t g^{-1} J^{-1})$.

There are two natural mappings from the L-group of $U(n)$ to the L-group of $GL_n(E)$, called the stable and the unstable base change maps. At the L-group level, the stable base change map, which corresponds to the restriction of parameters from the Weil group W_F of F to the Weil group W_E of E, is given by the diagonal embedding ψ : ${}^L U(n) \to {}^L R_{E/F}(U(n))$. The unstable base change map is defined by first choosing a character $\widetilde{\omega}$ of E^* extending the quadratic character $\omega_{E/F}$ of F^* associated to the quadratic extension E/F. At the level of L-groups, the unstable base change corresponds to the homomorphism $\psi' : {}^L U(n) \to {}^L R_{E/F}(U(n))$ given by $\psi'(g \times w) = (\widetilde{\omega}(w)g, \widetilde{\omega}(w)^{-1}g) \times w$ for $w \in E^*$, $g \in GL_n(\mathbb{C})$, and $\psi'(1, \sigma) = (1, -1) \times \sigma$. The base change lift for $n = 3$ has been established by Rogawski [17].

Proof of Theorem 1.2. By [6, Corollary 4.6], a supercuspidal representation π of $GL_3(E)$ is a stable base change lift from $U(3)$ if and only if the Langlands-Shahidi Asai L-function $L_2(s, As(\pi))$ has a pole at $s = 0$. By Theorem 1.6, this amounts to saying that the Rankin-Selberg Asai L-function $L_1(s, As(\pi))$ has a pole at $s = 0$. Now Theorem 1.2 follows by appealing to Proposition 4.1. ■

Remark 4.2. If π is a square-integrable representation such that $\pi^\vee \cong \bar{\pi}$, and the central character of π has trivial restriction to F^*, then Kable [12] has proved that π is distinguished or distinguished with respect to $\omega_{E/F}$, the quadratic character associated to the extension E/F (see [9, 15] for earlier results in this direction). The given conditions on π are expected to be necessary for π to be in the image of the base change map from $U(n)$. Thus Kable's result can be thought of as a weaker version of the conjecture stated in the introduction. On the other hand, it is expected that $U(n)$-distinguished representations of $GL_n(E)$ are base change lifts from $GL_n(F)$. This has been proved in several cases [8, 15].

4.3 Proof of Theorem 1.3

We now prove Theorem 1.3 regarding the reducibility of representations of $U(n,n)$ parabolically induced from $GL_n(E)$. In [6, 7], Goldberg proves that for a discrete series representation π with $\pi^\vee \cong \bar{\pi}$, $I(\pi)$ is irreducible if and only if $L_2(s, As(\pi))$ has a pole at $s = 0$ (see also [11]). By [7, Theorem 3.4], $R(\pi) \simeq (\mathbb{Z}/2\mathbb{Z})^r$, where r is the number of inequivalent representations π_i satisfying $\pi_i^\vee \simeq \bar{\pi}_i$, and the Plancherel measure $\mu(s, \pi_i)$ does not have zero at $s = 0$. By [18, Corollary 3.6], the latter condition amounts to knowing that the Asai L-functions $L_2(s, As(\pi_i))$ are regular at $s = 0$.

Theorem 1.3 follows from the following claim.

Claim 4.3. An irreducible, square-integrable representation π of $GL_n(E)$ is $\omega_{E/F}$ distinguished if and only if $\pi^\vee \simeq \bar{\pi}$ and $L_2(s, As(\pi))$ is regular at $s = 0$. □

Proof. By [6, Corollary 5.7],

$$L(s, \pi \times \bar{\pi}) = L_2\big(s, As(\pi)\big) L_2\big(s, As(\pi \otimes \widetilde{\omega})\big), \tag{4.2}$$

where $\widetilde{\omega}$ is a character of E^* which restricts to $\omega_{E/F}$ on F^*. Now $L(s, \pi \times \bar{\pi})$ has a pole at $s = 0$ if and only if $\pi^\vee \simeq \bar{\pi}$. Hence, $\pi^\vee \simeq \bar{\pi}$ and $L_2(s, As(\pi))$ is regular at $s = 0$ is equivalent to saying that $L_2(s, As(\pi \otimes \widetilde{\omega}))$ has a pole at $s = 0$. By Theorem 1.6 this is the same as saying that $L_1(s, As(\pi \otimes \widetilde{\omega}))$ has a pole at $s = 0$. By Proposition 4.1, the latter condition is equivalent to saying that π is $\omega_{E/F}$ distinguished. This proves the claim and hence Theorem 1.3. ■

Remark 4.4. The R-group in this context is also computed in terms of the Langlands parameters by Prasad [14, Proposition 2.1]. According to this computation, $R(\pi)$ is a product of r copies of $\mathbb{Z}/2\mathbb{Z}$'s, where r is the number of π_i's such that $\pi_i^\vee \cong \bar{\pi}_i$, and $c(\sigma_i) = -1$, where σ_i is the Langlands parameter of π_i. Here $c(\sigma_i) \in \{\pm 1\}$ denotes the constant introduced by Rogawski [17, Lemma 15.1.1]. Also $c(\sigma_i) = (-1)^{n_i - 1}$ if and only if σ_i can be extended to a parameter for $U(n_i)$. Together with Theorem 1.3, this gives an evidence for the conjecture stated in the introduction.

4.4 Distinguishedness of Steinberg representation of $GL(n)$

We now prove Theorem 1.5. Let $G = GL(n)$. For a representation π of $GL_n(E)$, let $I(\pi)$ be the parabolically induced representation of $U(n,n)$. If π is a discrete series representation such that $\pi^\vee \ncong \bar{\pi}$, then $I(\pi)$ is known to be irreducible [6]. Suppose $\pi^\vee \cong \bar{\pi}$. Let a and b be integers such that $ab = n$, such that π is the unique square-integrable constituent of the representation induced from $\pi_1 \otimes \cdots \otimes \pi_b$, where $\pi_i = \pi_0 \otimes |\,|_E^{(b+1-2i)/2}$, and π_0 a supercuspidal representation of $GL_a(E)$. Then $\pi_0^\vee \cong \bar{\pi}_0$. We have the following result of Goldberg [6, Section 7].

Proposition 4.5. The representation $I(\pi)$ of $U(n,n)$ is irreducible if and only if $L_2(s, As(\pi_0))$ (resp., $L_2(s, As(\pi_0 \otimes \widetilde{\omega})))$ has a pole at $s = 0$ if b is odd (resp., even). Here $\widetilde{\omega}$ is a character of E^* that restricts to $\omega_{E/F}$. □

Now if π is the Steinberg representation of $GL_n(E)$, then $a = 1$, $b = n$, and π_0 is the trivial character. Thus $I(\pi)$ is irreducible when n is odd and reducible when n is even. By the corollary to Theorem 1.3, π is $\omega_{E/F}$-distinguished when n is even, and π is not $\omega_{E/F}$-distinguished when n is odd.

Since $\pi^\vee \cong \bar{\pi}$ and $\omega_\pi = 1$, we know that π is either distinguished or $\omega_{E/F}$-distinguished, but not both (see [12, Theorem 7] and [1, Corollary 1.6]). Therefore, it follows that when n is odd (resp., even), π is distinguished (resp., $\omega_{E/F}$-distinguished), and that π is not distinguished with respect to any other character. This finishes the proof of Theorem 1.5.

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