ON THE VISIBILITY OF ULTRASONIC WAVES.

BY P. RAMA PISHAROTY.

(From the Department of Physics, Indian Institute of Science, Bangalore.)

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1. *Introduction.*

RECENTLY Raman and Nath[1] have developed a theory of the phenomenon of the diffraction of light by high frequency sound waves, discovered by Debye and Sears[2] in America and Lucas and Biquard[3] in France. The essential idea of the theory is that the phenomenon depends on the corrugated form of the wavefront of the transmitted beam. The sinusoidally corrugated form of the wavefront is obtained by considering the phase changes accompanying the traversing beam and it is assumed that the amplitude is constant throughout. Now Hiedeman,[4] Bär,[5] and Lucas[6] have experimentally established the visibility of the progressive ultrasonic waves using a microscope and a Kerr cell. Raman and Nath have explained this by postulating amplitude changes as well on the emerging wavefront, their general wave-function being obtained as a general solution of the partial differential equation governing the propagation of light in a medium filled with sound waves.

We will consider in this paper how a beam of constant amplitude but having a simple harmonically corrugated wavefront builds up periodic amplitude changes as well, when propagated through distances small compared with the width of the wavefront,—small enough as not to allow the different plane components to separate out.

2. *Theory.*

The direct method to determine the form of the constant amplitude corrugated wavefront when propagated through a short distance would be to find the diffraction pattern on a plane parallel to the original mean wavefront and at a finite distance from it by the usual Fresnel procedure. But as the integration involved is complicated an alternative method is adopted. The corrugated wavefront is resolved into an infinite number of plane wavefronts having different inclinations to the mean wavefront. Then the effect of each of these plane waves at a point on the required plane is calculated and the effects summed up for all the plane components.

(*a*) *Resolution of the corrugated wavefront into plane waves.*—

Let us take the X axis along the direction in which the phase change periodically, the Y axis parallel to the lines of constant phase, and the Z axis along the direction of propagation of the light beam.

A simple harmonically corrugated wavefront is then represented by

$$Ae^{2\pi i\nu t}\, e^{\frac{2\pi}{\lambda} i \left(a \sin \frac{2\pi}{\lambda^*} x\right)}$$

where ν is the frequency of the light waves,
λ is its wave-length in the medium,
a is the amplitude of the phase variation,
and λ^* is the wave-length of the phase variation.

The factor

$$e^{\frac{2\pi i}{\lambda}\left(a \sin \frac{2\pi}{\lambda^*} x\right)}$$

may be written as

$$\cos (v \sin bx) + i \sin (v \sin bx)$$

where $v = \dfrac{2\pi a}{\lambda}$ and $b = \dfrac{2\pi}{\lambda^*}$.

Using the well-known Bessel expansions for

$$\cos (v \sin bx) \text{ and } \sin (v \sin bx)$$

we have,

$$\cos (v \sin bx) + i \sin (v \sin bx)$$

$$= 2 \sum_0^{\infty}{}' J_{2r}(v) \cos 2rbx + 2i \sum_0^{\infty} J_{2r+1}(v) \sin \overline{2r+1}\, bx \quad \ldots \quad (1)$$

$$= \Sigma' J_{2r}(v) \{e^{i2rbx} + e^{-i2rbx}\} + \sum_0^{\infty} J_{2r+1}(v) \{e^{i\overline{2r+1}\,bx} - e^{-i\overline{2r+1}\,bx}\}$$

Remembering that $J_n(v) = (-1)^n J_{(-n)} v$,
we may put the above as

$$\sum_{-\infty}^{+\infty} J_n(v)\, e^{inbx}$$

Then the corrugated wavefront is

$$\sum_{-\infty}^{+\infty} J_n(v)\, e^{inbx}\, e^{2\pi i\nu t}$$

$$= \sum_{-\infty}^{+\infty} J_n(v)\, e^{\frac{2\pi i}{\lambda}(\mathrm{V}t + npx)} \quad \ldots \quad \ldots \quad (2)$$

where $p = b/\frac{2\pi}{\lambda} = \dfrac{\lambda}{\lambda^*} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$

The interpretation of this expression is that a sinusoidally corrugated wavefront is equivalent to an infinite number of plane waves of amplitudes $J_n(v)$ and inclinations to the mean wavefront given by $\sin^{-1}(-np)$.

(*b*) *The evaluation of the diffraction effects on a parallel plane.*—

To evaluate the effect of the corrugated wavefront at a point P in a plane parallel to the *xy*-plane and at a distance Z from the *xy* plane, we propose to calculate the effect of each plane wave component separately and then to sum up the effect.

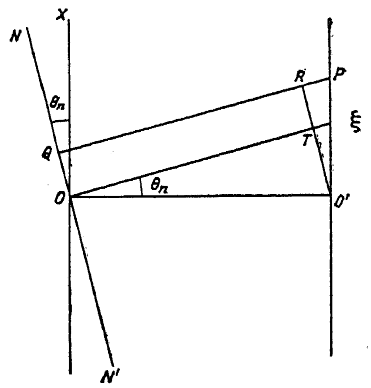

FIG. 1.

Consider a single plane wavefront NON′ having an inclination θ_n with the *xy*-plane. To consider its effect at P, a point on the required plane, drop the normal PQ on to NON′.

$$\text{Now PQ} = \text{QR} + \text{RP} = \text{OT} + \text{RP}$$
$$= Z \cos \theta_n + \xi \sin \theta_n.$$

Therefore the vibration at P due to the wavefront NON′

$$= J_n(v) \cos \frac{2\pi}{\lambda} (Vt - Pq)\cdot$$

$$= J_n(v) \cos \frac{2\pi}{\lambda} (Vt - Z \cos \theta_n - \xi \sin \theta_n).$$

Here $J_n(v)$ is the amplitude of the *n*th order plane wave, and θ_n its inclination given by the relation

$$\sin \theta_n = -np = -n \frac{\lambda}{\lambda^*}.$$

Hence the resultant vibration at P due to all the components

$$= \sum_{-\infty}^{+\infty} J_n(v) \cos \frac{2\pi}{\lambda} (Vt - Z \cos \theta_n - \xi \sin \theta_n) \quad \ldots \quad \ldots \quad (4)$$

$\cos \theta_n$ is always of the same sign, while $\sin \theta_n$ changes sign with *n*. $J_n(v)$ changes sign with *n* when *n* is odd. To take these into consideration the series is split up into four sub-series, two for positive and negative even values of n, and two for the corresponding odd values of n. Simplyfying on these lines and substituting for θ_n, the resultant vibration at P is given by

$$2\,\Sigma'\,J_{2r}(v)\cos\frac{2\pi}{\lambda}\left(Vt-Z\sqrt{1-\overline{2r}^2p^2}\right)\cos\frac{2\pi}{\lambda}\xi\cdot 2rp$$

$$+\,2\,\Sigma\,J_{2r+1}(v)\sin\frac{2\pi}{\lambda}\{Vt-Z\sqrt{1-\overline{2r+1}^2p^2}\}\sin\frac{2\pi}{\lambda}\xi\cdot\overline{2r+1}\,p.$$

$$= A\cos\frac{2\pi}{\lambda}Vt + B\sin\frac{2\pi}{\lambda}\cdot Vt \qquad \text{(say)} \qquad .. \qquad .. \qquad (5)$$

Then

$$A = 2\,\Sigma'\,J_{2r}(v)\cos\frac{2\pi}{\lambda}\cdot Z\sqrt{1-\overline{2r}^2p^2}.\ \cos\left(\frac{2\pi}{\lambda}\cdot\xi\cdot 2rp\right)$$

$$-\,2\,\Sigma\,J_{2r+1}(v)\sin\frac{2\pi}{\lambda}Z\sqrt{1-\overline{2r+1}^2p^2}.\ \sin\left(\frac{2\pi}{\lambda}\xi\cdot\overline{2r+1}\,p\right) \qquad (6)$$

and

$$B = 2\,\Sigma'\,J_{2r}(v)\sin\frac{2\pi}{\lambda}Z\sqrt{1-\overline{2r}^2p^2}.\ \cos\frac{2\pi}{\lambda}\cdot\xi\cdot 2rp.$$

$$+\,2\,\Sigma\,J_{2r+1}(v)\cos\frac{2\pi}{\lambda}Z\sqrt{1-\overline{2r+1}^2p^2}.\ \sin\frac{2\pi}{\lambda}\cdot\xi\cdot\overline{2r+1}\,p \qquad .. \qquad (7)$$

The intensity I at the point P is given by the equation

$$I = A^2 + B^2$$

when $Z = 0$, or a small multiple of λ,

$$A = 2\,\Sigma'\,J_{2r}(v)\cos\frac{2\pi}{\lambda}\cdot\xi\cdot 2rp = \cos(v\sin b\xi)$$

and $$B = 2\,\Sigma\,J_{2r+1}(v)\sin\frac{2\pi}{\lambda}\cdot\xi\cdot\overline{2r+1}\,p = \sin(v\sin b\xi)$$

since $$\frac{2\pi p}{\lambda} = \frac{2\pi}{\lambda}\cdot\frac{\lambda}{\lambda^*} = \frac{2\pi}{\lambda^*} = b.$$

Hence $I = A^2 + B^2 = 1$.

Therefore the expressions give a corrugated wavefront of constant amplitude when it is propagated only through a few wave-lengths of light.

But the expressions for A and B cease to be simple when Z is a large multiple of λ.

Let us put

$$\tfrac{1}{4}\cdot\frac{(\lambda^*)^2}{\lambda} = d.$$

This corresponds to a distance of a half-a-cm., when λ is taken as 5000 Å.U. and λ^*, the wave-length of the corrugations to be 0·01 cm., the value cor:esponding to the usual ultrasonic wave-lengths in liquids.

When $Z = d$, $\cos\frac{2\pi}{\lambda}Z\sqrt{1-n^2p^2}$ and $\sin\frac{2\pi}{\lambda}\sqrt{1-n^2p^2}$, assume simple values of ± 1 or 0.

The following table gives the values of A and B for definite values of Z, the distance of propagation :—

Z	A	B
0, $2d$, $4d$, etc.	$\cos(v \sin b\xi)$	$\pm \sin(v \sin b\xi)$
d, $5d$, $9d$, etc.	$\cos(v \sin b\xi) + \sin(v \sin b\xi)$	$\sin(v \sin b\xi)$
$3d$, $7d$, $11d$, etc.	$\cos(v \sin b\xi) - \sin(v \sin b\xi)$	$\sin(v \sin b\xi)$

Fig. 2 represents the values of $A^2 + B^2$, *i.e.*, the intensity for the various values of Z, for two different cases $v = 1$ and $v = 2$.

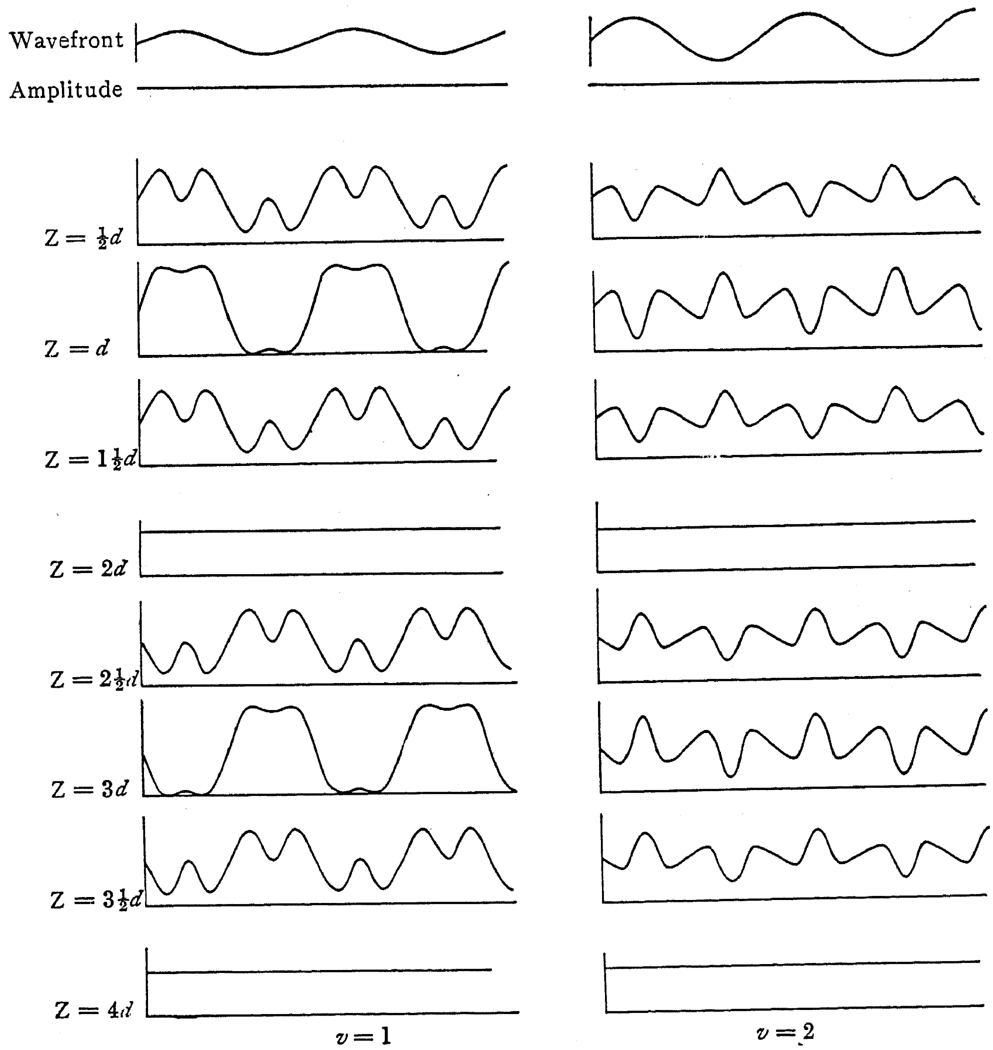

FIG. 2.

For values of Z equal to fractional values of d, there are amplitude changes as well as phase changes on the wavefront.

When $$Z = n\frac{d}{2},$$

$$A = \cos (v \sin b\xi) + \sin \left(\frac{n\pi}{4}\right). \sin (v \sin b\xi)$$

and $$B = \sin \left(\frac{n\pi}{4}\right). \sin (v \sin b\xi),$$

for values of v not greater than 3. These cases also are represented in Fig. 2.

We see therefore that a simple-harmonically corrugated wavefront develops amplitude changes on it when propagated. The number of amplitude changes is equal to that of the initial corrugations only when the depth of the corrugations is small. As the depth of the corrugations increase the number of the amplitude changes, corresponding to each wave-length of the corrugation, also increases. This is to be expected, for, when the depth of the corrugations increase more and more of the plane wave components assume importance and give rise to more interference bands.

3. Application of the Theory to Ultrasonics.

The visibility of the progressive ultrasonic waves observed through a microscope using a Kerr cell, may be explained by these deductions. The microscope is focussed on one of the planes at which the corrugated wavefront has developed into an amplitude grating. The fringes ought to disappear when the microscope is pushed in or out through a distance $d = \frac{1}{4}\frac{(\lambda^*)^2}{\lambda}$. (This corresponds to a distance of 0·5 cm. when λ^* is taken as 0·01 cm. and λ as 5000 Å.U.) But if it is moved through double this distance the fringes should appear again, but with a lateral displacement of $\frac{\lambda^*}{2}$ of the whole pattern. If, on the other hand, there are amplitude changes as well impressed on the wavefront on its emergence from the sound field, the pattern should continue to be visible throughout.

With a stationary ultrasonic field we have a pulsating wavefront and the above amplitude changes would be practically or completely washed out. The observers do report the visibility of a stationary sound field also. This shows that there are amplitude changes as well on the emerging wavefront before its further propagation. This may be due to the superposition of the Brillouin[7] effects on the phase change corrugations of the emerging wavefront. An experimental evidence for such a superposition of the two effects is given by Parthasarathy,[8] in a recent paper.

4. *Applicability of the 'Schlieren' Method.*

Lord Rayleigh[9] has given the theory of the Foucoult's test, *i.e.*, the Schlieren method, for the detection of small path differences. In Fig. 3, A represents the lens, with its rectangular aperture, which brings parallel rays to a focus. In the focal plane B there are two adjustable screens with horizontal edges and immediately behind is the objective of a small telescope.

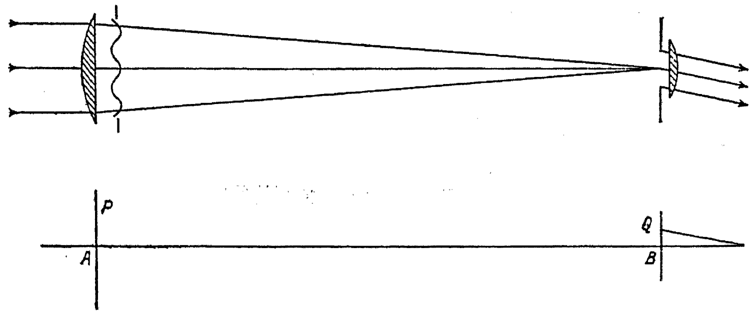

FIG. 3.

The rays from the various points Q of the second aperture which unite at a point in the focal plane of the telescope may be regarded as a parallel pencil inclined to the axis at an angle ψ. P is a point in the first aperture. AP $= x$, BQ $= \xi$, and AB $= f$. Let the additional path retardation operative at A, giving a corrugated form to the wavefront be denoted by R, a function of x. Thus if V be the velocity of propagation of light and $k = \frac{2\pi}{\lambda}$, the vibration at a point ξ of the second aperture will be represented by

$$\int dx \sin k\left(\mathrm{V}t - f - \mathrm{R} + \frac{x\xi}{f}\right)$$

Or, if $\frac{x}{f} = \theta$, by

$$\int d\theta. \sin k\ (\mathrm{V}t - f - \mathrm{R} + \xi\theta) \quad \ldots \quad \ldots \quad (8)$$

When we proceed to enquire what is to be observed at an angle ψ, we have to consider the integral

$$\int_{-\xi}^{+\xi} d\xi \int_{-\theta}^{+\theta} d\theta \sin k\ \{\mathrm{V}t - f - \mathrm{R} + \xi\ (\theta + \psi)\} \quad \ldots \quad \ldots \quad (9)$$

For a simple-harmonically corrugated wavefront, we have the function R given by the relation

$$\mathrm{R} = a \sin \frac{2\pi}{\lambda^*} x = a \sin \frac{2\pi f}{\lambda^*}\theta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

where λ^* is the wave-length of the corrugations, and a is the amplitude of the corrugations.

It is more convenient to carry out first the integration with respect to ξ.

Using the expansions for cos (v sin bx) and sin (v sin bx) given by the equation (1) and carrying out the second integration as well, we find the resultant vibration given by

$$A \sin (Vt - f) - B \cos (Vt - f) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

where

$$A = \Sigma' J_{2r}(ka) \cos 2rb\psi \{\mathrm{si}\,(k\xi + 2rb)(\theta + \psi) + \mathrm{si}\,(k\xi + 2rb)(\theta - \psi) + \mathrm{si}\,(k\xi - 2rb)(\theta + \psi) + \mathrm{si}\,(k\xi - 2rb)(\theta - \psi)\} \quad \ldots \quad \ldots \quad (11a)$$

and

$$B = \Sigma J_{2r+1}(ka) \sin (2r+1) b\psi \{\mathrm{si}\,(k\xi + \overline{2r+1}b)(\theta + \psi) + \mathrm{si}\,(k\xi + \overline{2r+1}\,b)(\theta - \psi) + \mathrm{si}\,(k\xi - \overline{2r+1}\,b)(\theta + \psi) + \mathrm{si}\,(k\xi - \overline{2r+1}\,b)(\theta - \psi)\} \quad \ldots \quad ((11b)$$

J_n (ka) represents the Bessel function of the nth order. $k = \frac{2\pi}{\lambda}$, $b = \frac{2\pi f}{\lambda^*}$ and 2ξ is the width of the second aperture which is symmetrically situated with respect to the geometrical focus.

$$\mathrm{si}\, x = \int_0^x \frac{\sin x}{x} dx.$$

The dash over the summation in A represents that the coefficient of the first term of the series is half of those of the rest.

As already shown in Section 3, the corrugated wavefront can be resolved into a number of plane waves having inclinations α to the mean wavefront given by

$$\sin \alpha = \pm n \frac{\lambda}{\lambda^*}.$$

The series in A and B continue only as long as $k\xi > nb$. For, when $k\xi < nb$ the corresponding terms in both the series vanish, since si (x) when x is large is $\frac{\pi}{2}$ and si $(-x) = -\frac{\pi}{2}$.

Now, $k\xi > nb$

$$i.e., \frac{2\pi}{\lambda}\xi > n.\frac{2\pi f}{\lambda^*}$$

$$i.e., \frac{\xi}{f} > n\frac{\lambda}{\lambda^*}$$

means that the nth order spectrum is admitted into the second aperture.

When a large number of spectra are admitted into the second aperture, A and B are to be continued to a large number of terms. Then

$$A = 2\pi \Sigma' J_{2r}(ka) \cos 2rb\psi = \pi \cos (ka \sin b\psi)$$

and $$B = 2\pi \Sigma J_{2r+1}(ka) \sin \overline{2r+1}\, b\psi = \pi \sin (ka \sin b\psi);$$

so that $A^2 + B^2 =$ a constant.

Therefore the intensity is uniform and the aperture at A appears uniformly illuminated. Uniform illumination is the result even when one half of the second aperture is cut out.

If on the other hand, the second aperture is just large enough to admit the direct and the first order of lateral spectra only, we have the intensity distribution with ψ given by

$$I \propto \{J_0 (ka)\}^2 + \{2J_1 (ka) \sin b\psi\}^2.$$

Here the intensity varies with ψ. Even in this case the visibility is in general poor because of the constant term $\{J_0 (ka)\}^2$. If the second orders are also admitted by slightly increasing the second aperture, a cosine term also comes in and the visibility practically ceases.

Thus we find that the ' Schlieren ' effects of a corrugated wavefront are poor. This is in accordance with Lord Rayleigh's statement that continuous phase changes are not easily detected by the Foucoult's test.

The author wishes to thank Sir C. V. Raman, KT., F.R.S., N.L., for the suggestion of the problem and the valuable help during its solution.

5. *Summary.*

The diffraction effects caused by a simple-harmonically corrugated wavefront when propagated through distances small compared with the extent of the wavefront—small enough as not to allow the various orders of spectra to separate out, are considered. Such a wavefront on propagation develops at a definite distance $d = \frac{1}{4} \frac{(\lambda^*)^2}{\lambda}$, into one of uniform phase but having periodic *amplitude* changes on it. On further propagation to twice this distance it becomes a corrugated wavefront again. At intermediate stages there are both amplitude and phase changes. At a distance $\frac{3}{4} \frac{(\lambda^*)^2}{\lambda}$ the pattern is similar to that at d but is displaced laterally through $\frac{\lambda^*}{2}$. It is suggested that the visibility of the progressive ultrasonic waves observed with a microscope and a Kerr cell is due to these amplitude changes brought about by the propagation of a purely corrugated wavefront. Lord Rayleigh's theory of the ' Foucoult's test ' is applied to such a wavefront and it is shown that its Schlieren effects would be poor.

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