ON A BERRY-ESSEEN TYPE BOUND FOR THE MAXIMUM LIKELIHOOD ESTIMATOR OF A PARAMETER FOR SOME STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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This paper is concerned with the study of the rate of convergence of the distribution of the maximum likelihood estimator of a parameter appearing linearly in the drift coefficients of two types of stochastic partial differential equations (SPDEs).

1. Introduction

Maximum likelihood estimation of a parameter appearing linearly in some stochastic partial differential equations (SPDEs) has been considered by Hübner et al. [3]. Detailed discussion of these SPDEs and some interesting phenomena arising out of the parameter estimation have been considered by them in two examples. In this paper, we study the rate of convergence of the distribution of the maximum likelihood estimator (MLE) $\hat{\theta}_{N,\epsilon}$ of the parameter θ occurring linearly in such SPDEs. Bounds on the difference $|\hat{\theta}_{N,\epsilon} - \theta_0|$, where θ_0 is the true value of the parameter, can be obtained using these results as in Mishra and Prakasa Rao [6]. In Section 2, we describe a SPDE with parameter θ such that the corresponding stochastic process u_ϵ generates measures $\{P_\theta^\epsilon,\ \theta \in \Theta\}$ which are mutually absolutely continuous, and the main results pertaining to this section have been described in Section 3. In Section 4, we describe a SPDE with parameter θ such that the corresponding stochastic process u_ϵ generates measures which form a family of probability measures $\{P_\theta^\epsilon,\ \theta \in \Theta\}$ which are singular with respect to each other, and this section also contains the main results connected to this problem. Comprehensive surveys on statistical inference for such classes of SPDEs are given by Prakasa Rao [7, 8]. Throughout the paper, we will denote by C a positive constant different at different places of occurrence, possibly dependent on the initial conditions of the SPDEs.

2. SPDE with linear drift (absolutely continuous case): estimation

Let $(\Omega, \mathcal{F}, P)$ be a probability space and consider the process $u_\epsilon(t,x)$, $0 \le x \le 1$, $0 \le t \le T$, governed by the SPDE

$$du_\epsilon(t,x) = \big(\Delta u_\epsilon(t,x) + \theta u_\epsilon(t,x)\big)dt + \epsilon dW_Q(t,x) \tag{2.1}$$

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with the initial and boundary conditions given by

$$u_\epsilon(0,x) = f(x), \quad f \in L_2[0,1], \tag{2.2}$$

$$u_\epsilon(t,0) = u_\epsilon(t,1) = 0, \quad 0 \le t \le T, \tag{2.3}$$

where $\Delta = \partial^2/\partial x^2$. Let $\epsilon \to 0$ and $\theta \subset \mathbb{R}$. Here, Q is a nuclear covariance operator for the Wiener process $W_Q(t,x)$ taking values in $L_2[0,1]$ so that $W_Q(t,x) = Q^{1/2}W(t,x)$ and $W(t,x)$ is a cylindrical Brownian motion in $L_2[0,1]$. Then it is known that (cf. Rozovskiĭ [9])

$$W_Q(t,x) = \sum_{i=1}^{\infty} q_i^{1/2} e_i(x) W_i(t) \quad \text{a.s.}, \tag{2.4}$$

where $\{W_i(t),\ 0 \le t \le T\}$, $i \ge 1$, are independent one-dimensional standard Wiener processes, $\{e_i\}$ is a complete orthonormal system (CONS) in $L_2[0,1]$ consisting of the eigenvectors of Q, and $\{q_i\}$ are the corresponding eigenvalues of Q. We consider a special covariance operator Q with $e_k = \sin k\pi x$, $k \ge 1$, and $\lambda_k = (\pi k)^2$, $k \ge 1$. Then $\{e_n\}$ is a CONS with the eigenvalues $q_i = (1+\lambda_i)^{-1}$, $i \ge 1$, for the operator Q, where $Q = (I-\Delta)^{-1}$. Furthermore, $dW_Q = Q^{1/2}dW$. We define a solution $u_\epsilon(t,x)$ of (2.1) as a formal sum:

$$u_\epsilon(t,x) = \sum_{i=1}^{\infty} u_{i\epsilon}(t) e_i(x) \tag{2.5}$$

(cf. Rozovskiĭ [9]). It is known that the Fourier coefficients $u_{i\epsilon}(t)$ satisfy the stochastic differential equation

$$du_{i\epsilon}(t) = (\theta - \lambda_i) u_{i\epsilon}(t)dt + \frac{\epsilon}{\sqrt{\lambda_i + 1}} dW_i(t), \quad 0 < t \le T, \tag{2.6}$$

with the initial conditions

$$u_{i\epsilon}(0) = v_i, \quad v_i = \int_0^1 f(x) e_i(x)dx. \tag{2.7}$$

It is further known that the function $u_\epsilon(t,x)$ as defined above belongs to $L_2([0,T] \times \Omega; L_2[0,1])$ together with its derivative in t. Furthermore, $u_\epsilon(t,x)$ is the only solution of (2.1) under the boundary conditions (2.2) and (2.3). Let P_θ^ϵ be the measure generated by u_ϵ on $C[0,T]$ when θ is the true parameter. It has been shown by Hübner et al. [3] that the family of measures $\{P_\theta^{(\epsilon)},\ \theta \in \Theta\}$ is mutually absolutely continuous and

$$\begin{aligned} \log \frac{dP_\theta^\epsilon}{dP_{\theta_0}}(u_\epsilon) = \sum_{i=1}^{\infty} \frac{\lambda_i + 1}{\epsilon^2} \bigg[(\theta - \theta_0) \int_0^T u_{i\epsilon}(t) du_{i\epsilon}(t) \\ - \frac{1}{2}\Big\{ (\theta - \lambda_i)^2 - (\theta_0 - \lambda_i)^2 \Big\} \int_0^T u_{i\epsilon}^2(t)dt \bigg]. \end{aligned} \tag{2.8}$$

The projection of the solution $u_\epsilon(t,x)$ onto the subspace π^N spanned by $\{e_1, e_2, \ldots, e_N\}$ (see Liptser and Shiryayev [4]) is given by $u_\epsilon^N(t,x) = \sum_{i=1}^{N} u_{i\epsilon}(t) e_i(x)$. Let $P_\theta^{\epsilon,N}$ be the probability measure generated by the process $u_\epsilon^N(t,x)$ on $C[0,T]$ when θ is the true parameter.

Then the measure $P_\theta^{\epsilon,N}$ is absolutely continuous with respect to the measure $P_{\theta_0}^{\epsilon,N}$ and

$$\log\frac{dP_\theta^{\epsilon,N}}{dP_{\theta_0}^{\epsilon,N}}=\sum_{i=1}^{N}\frac{\lambda_i+1}{\epsilon^2}\Big[(\theta-\theta_0)\int_0^T u_{i\epsilon}(t)du_{i\epsilon}(t)-\frac{1}{2}\Big\{(\theta-\lambda_i)^2-(\theta_0-\lambda_i)^2\Big\}\int_0^T u_{i\epsilon}(t)dt\Big]. \tag{2.9}$$

The MLE of the parameter θ is given by

$$\hat{\theta}_{N,\epsilon}=\frac{\sum_{i=1}^{N}(\lambda_i+1)\int_0^T u_{i\epsilon}(t)\big(du_{i\epsilon}(t)+\lambda_i u_{i\epsilon}(t)dt\big)}{\sum_{i=1}^{N}(\lambda_i+1)\int_0^T u_{i\epsilon}^2(t)dt} \tag{2.10}$$

(cf. [3, page 154]).

3. SPDE with linear drift (absolutely continuous case): Berry-Esseen type bound

We now prove two theorems leading to a Berry-Esseen type bound for the MLE $\hat{\theta}_{N,\epsilon}$. It can be checked that $E_{\theta_0}\int_0^T u_{i\epsilon}^2(t)dt<\infty$ for $i\geq 1$. We assume that $\theta_0<\pi^2$, where θ_0 is the true parameter. Let $\Phi(\cdot)$ denote the standard normal distribution function and define

$$Q_{N,T}^{(\epsilon)}=\sum_{i=1}^{N}\frac{\lambda_i+1}{2(\theta-\lambda_i)}\Big(v_i^2\big(e^{2(\theta-\lambda_i)T}-1\big)-T\frac{\epsilon^2}{\lambda_i+1}\Big). \tag{3.1}$$

Theorem 3.1. *For any* $0<\delta<1$,

$$\begin{aligned}\sup_y\Big|P_{\theta_0}^{\epsilon,N}\Big\{\sqrt{Q_{N,T}^{(\epsilon)}}\epsilon^{-1}(\hat{\theta}_{N,\epsilon}-\theta_0)\leq y\Big\}-\Phi(y)\Big|\\ \leq 2P_{\theta_0}^{\epsilon,N}\Big\{\Big|\sum_{i=1}^{N}\frac{\lambda_i+1}{Q_{N,T}^{(\epsilon)}}\int_0^T u_{i\epsilon}^2(t)dt-1\Big|\geq\delta\Big\}+3\sqrt{\delta}.\end{aligned} \tag{3.2}$$

Theorem 3.2. *Let $N\geq 1$ be fixed. Then there exists a constant C depending on θ_0, $\|f\|$, and T such that, for any $0<\delta\leq 1$ and $0<\epsilon<1$,*

$$P_{\theta_0}^{\epsilon,N}\Big\{\Big|\sum_{i=1}^{N}\frac{\lambda_i+1}{Q_{N,T}^{(\epsilon)}}\int_0^T u_{i\epsilon}^2(t)dt-1\Big|\geq\delta\Big\}\leq C\frac{\epsilon}{\delta}\Big(\frac{1+T^{1/2}}{Q_{N,T}^{(\epsilon)}}\Big). \tag{3.3}$$

We first state two lemmas needed in the sequel.

Lemma 3.3. *Let $(\Omega,\mathcal{F},P)$ be a probability space and let f and g be $\mathcal{F}$-measurable functions. Then, for any $\delta>0$,*

$$\begin{aligned}\sup_x\Big|P\Big\{\omega:\frac{f(\omega)}{g(\omega)}\leq x\Big\}-\Phi(x)\Big|\\ \leq\sup_y|P\{\omega:f(\omega)\leq y\}-\Phi(y)|+P\{\omega:|g(\omega)-1|\geq\delta\}+\delta.\end{aligned} \tag{3.4}$$

Proof. See Michel and Pfanzagl [5]. □

Lemma 3.4. *Let $\{W(t),\ t \geq 0\}$ be a standard Wiener process and let Z be a nonnegative random variable. Then, for every $x \in \mathbb{R}$ and $\delta > 0$,*

$$\left| P\{W(Z) \leq x\} - \Phi(x) \right| \leq (2\delta)^{1/2} + P\{|Z-1| \geq \delta\}. \tag{3.5}$$

Proof. See Hall and Heyde [2, page 85]. □

Proof of Theorem 3.1. It follows from (2.10) that

$$\sqrt{Q_{N,T}^{(\epsilon)}}\epsilon^{-1}(\hat{\theta}_{N,\epsilon} - \theta_0) = \frac{\{\sum_{i=1}^{N}\sqrt{\lambda_i+1}\int_0^1 u_{i\epsilon}(t)dW_i(t)\}/\sqrt{Q_{N,T}^{(\epsilon)}}}{\{\sum_{i=1}^{N}(\lambda_i+1)\int_0^T u_{i\epsilon}^2(t)dt\}/Q_{N,T}^{(\epsilon)}}. \tag{3.6}$$

Now, for any $y \in \mathbb{R}$,

$$\begin{aligned}
&\left| P_{\theta_0}^{\epsilon,N}\left\{\sqrt{Q_{N,T}^{(\epsilon)}}\epsilon^{-1}(\hat{\theta}_{N,\epsilon} - \theta_0) \leq y\right\} - \Phi(y)\right| \\
&\quad \leq \left| P_{\theta_0}^{\epsilon,N}\left\{\frac{\sum_{i=1}^{N}\sqrt{\lambda_i+1}\int_0^T u_{i\epsilon}(t)dW_i(t)/\sqrt{Q_{N,\epsilon}^{(\epsilon)}}}{\sum_{i=1}^{N}(\lambda_i+1)\int_0^T u_{i\epsilon}^2(t)dt/Q_{N,T}^{(\epsilon)}} \leq y\right\} - \Phi(y)\right| \\
&\quad \leq \sup_x \left| P_{\theta_0}^{\epsilon,N}\left\{\frac{\sum_{i=1}^{N}\sqrt{\lambda_i+1}\int_0^T u_{i\epsilon}(t)dW_i(t)}{\sqrt{Q_{N,T}^{(\epsilon)}}} \leq x\right\} - \Phi(x)\right| \\
&\qquad + P_{\theta_0}^{\epsilon,N}\left\{\left|\frac{\sum_{i=1}^{N}(\lambda_i+1)\int_0^T u_{i,\epsilon}^2(t)dt}{Q_{N,T}^{(\epsilon)}} - 1\right| \geq \delta\right\} + \delta \quad (\text{by Lemma 3.3}) \\
&\quad = \sup_x \left| P_{\theta_0}^{\epsilon,N}\left\{\widetilde{W}\left(\sum_{i=1}^{N}\frac{\lambda_i+1}{Q_{N,T}^{(\epsilon)}}\int_0^T u_{i\epsilon}^2(t)dt\right) \leq x\right\} - \Phi(x)\right| \\
&\qquad + P_{\theta_0}^{\epsilon,N}\left\{\left|\sum_{i=1}^{N}\frac{\lambda_i+1}{Q_{N,T}^{(\epsilon)}}\int_0^T u_{i\epsilon}^2(t)dt - 1\right| \geq \delta\right\} + \delta,
\end{aligned} \tag{3.7}$$

where $\widetilde{W}(\cdot)$ is an independent standard Wiener process by using Theorem 2.3 in Feigin [1] (due to Kunita-Watanabe) and the fact that $\int_0^T u_{i\epsilon}^2(t)dW_i(t)$, $1 \leq i \leq n$, are independent square-integrable martingales.

Hence

$$\begin{aligned}
&\left| P_{\theta_0}^{\epsilon,N}\left\{\sqrt{Q_{N,T}^{(\epsilon)}}\epsilon^{-1}(\hat{\theta}_{N,\epsilon} - \theta_0) \leq y\right\} - \Phi(y)\right| \\
&\quad \leq \sqrt{2\delta} + 2P_{\theta_0}^{\epsilon,N}\left\{\left|\sum_{i=1}^{N}\frac{\lambda_i+1}{Q_{N,T}^{(\epsilon)}}\int_0^T u_{i\epsilon}^2(t)dt - 1\right| \geq \delta\right\} + \delta \quad (\text{by Lemma 3.4}) \\
&\quad \leq 2\left[P_{\theta_0}^{\epsilon,N}\left\{\left|\sum_{i=1}^{N}\frac{\lambda_i+1}{Q_{N,T}^{(\epsilon)}}\int_0^T u_{i\epsilon}^2(t)dt - 1\right| \geq \delta\right\}\right] + 3\sqrt{\delta}
\end{aligned} \tag{3.8}$$

for $0 < \delta \leq 1$. □

Proof of Theorem 3.2. From (2.6), we obtain that

$$du_{i\epsilon}(s) = (\theta - \lambda_i)u_{i\epsilon}(s)ds + \frac{\epsilon}{\sqrt{\lambda_1 + 1}}dW_i(s), \quad 0 < t \le T, \\ u_{i\epsilon}(0) = v_i. \tag{3.9}$$

By the Îto formula, we have

$$d\big(u_{i\epsilon}(s)\bar{e}^{(\theta-\lambda_i)s}\big) = \frac{\epsilon}{\sqrt{\lambda_i+1}}\bar{e}^{(\theta-\lambda_i)s}dW_i(s) \tag{3.10}$$

or

$$u_{i\epsilon}(t)\bar{e}^{(\theta-\lambda_i)t} - v_i = \int_0^t \frac{\epsilon}{\sqrt{\lambda_i+1}}\bar{e}^{(\theta-\lambda_i)s}dW_i(s). \tag{3.11}$$

Furthermore,

$$d\big(u_{i\epsilon}^2(t)\big) = 2(\theta-\lambda_i)u_{i\epsilon}^2(t)dt + \frac{\epsilon}{\sqrt{\lambda_i+1}}u_{i\epsilon}(t)dW_i(t) + \frac{\epsilon^2}{\lambda_i+1}dt \tag{3.12}$$

or, equivalently,

$$\frac{\lambda_i+1}{2(\theta-\lambda_i)}u_{i\epsilon}^2(T) - \frac{\lambda_i+1}{2(\theta-\lambda_i)}v_i^2 \\ = \int_0^T(\lambda_i+1)u_{i\epsilon}^2(t)dt + \frac{\epsilon\sqrt{\lambda_i+1}}{2(\theta-\lambda_i)}\int_0^T u_{i\epsilon}(t)dW_i(t) + \frac{\epsilon^2}{2(\theta-\lambda_i)}T. \tag{3.13}$$

We know, from (3.11), that

$$u_{i\epsilon}^2(T) = v_i^2e^{2(\theta-\lambda_i)T} + e^{2(\theta-\lambda_i)T}\left(\frac{\epsilon}{\sqrt{\lambda_i+1}}\int_0^T\bar{e}^{(\theta-\lambda_i)s}dW_i(s)\right)^2 \\ + 2v_ie^{2(\theta-\lambda_i)T}\int_0^T\frac{\epsilon}{\sqrt{\lambda_i+1}}\bar{e}^{(\theta-\lambda_i)s}dW_i(s). \tag{3.14}$$

From (3.11) and (3.13), we obtain that

$$\sum_{i=1}^N\frac{\lambda_i+1}{2(\theta-\lambda_i)}\left\{v_i^2\big(e^{2(\theta-\lambda_i)T}-1\big) - \frac{\epsilon^2}{\lambda_i+1}T\right\} - \sum_{i=1}^N\int_0^T(\lambda_i+1)u_{i\epsilon}^2(t)dt \\ = \sum_{i=1}^N\frac{\epsilon\sqrt{\lambda_i+1}}{2(\theta-\lambda_i)}\int_0^T u_{i\epsilon}(t)dW_i(t) \\ - \sum_{i=1}^N\frac{\lambda_i+1}{2(\theta-\lambda_i)}\left[e^{2(\theta-\lambda_i)T}\left(\frac{\epsilon}{\sqrt{\lambda_i+1}}\int_0^T\bar{e}^{(\theta-\lambda_i)s}dW_i(s)\right)^2 \right. \\ \left. - \frac{2v_i\epsilon}{\sqrt{\lambda_i+1}}e^{2(\theta-\lambda_i)T}\int_0^T\bar{e}^{(\theta-\lambda_i)s}dW_i(s)\right]. \tag{3.15}$$

Since

$$Q_{N,T}^{(\epsilon)} = \sum_{i=1}^{N} \frac{\lambda_i + 1}{2(\theta - \lambda_i)} \left(\nu_i^2 \left(e^{2(\theta-\lambda_i)T} - 1 \right) - T \frac{\epsilon^2}{\lambda_i + 1} \right), \tag{3.16}$$

we have

$$\begin{aligned}
&P_\theta^{\epsilon,N}\left\{ \left| \frac{\int_0^T \left(\sum_{i=1}^N (\lambda_i+1) u_{i\epsilon}^2(t) \right) dt}{Q_{N,T}^{(\epsilon)}} - 1 \right| \geq \delta \right\} \\
&\quad \leq P_\theta^{\epsilon,N}\left\{ \left| \sum_{i=1}^{N} \frac{\left(\epsilon\sqrt{\lambda_i+1}/2(\theta-\lambda_i)\right) \int_0^T u_{i\epsilon}(t) dW_i(t)}{Q_{N,T}^{(\epsilon)}} \right| \geq \frac{\delta}{3} \right\} \\
&\qquad + P_{\theta_0}^{\epsilon,N}\left\{ \left| \frac{\sum_{i=1}^N \left(2\nu_i\epsilon/\sqrt{\lambda_i+1}\right) e^{2(\theta-\lambda_i)T} \int_0^T \bar{e}^{(\theta-\lambda_i)s} dW_i(s)}{Q_{N,T}^{(\epsilon)}} \right| \geq \frac{\delta}{3} \right\} \\
&\qquad + P_\theta^{\epsilon,N}\left\{ \left| \frac{\sum_{i=1}^N \left(\epsilon^2/2(\theta-\lambda_i)\right) e^{2(\theta-\lambda_i)T} \left(\int_0^T \bar{e}^{(\theta-\lambda_i)s} dw_i(s) \right)^2}{Q_{N,T}^{(\epsilon)}} \right| \geq \frac{\delta}{3} \right\} \\
&\quad = I_1 + I_2 + I_3 \text{ (say)}.
\end{aligned} \tag{3.17}$$

Now

$$\begin{aligned}
I_1 &\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} \left\{ \sum_{k=1}^{N} \frac{\lambda_k+1}{(\theta-\lambda_k)^2} E_{\theta_0} \int_0^T u_{k\epsilon}^2(t) dt \right\}^{1/2} \\
&\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} \left\{ \sum_{k=1}^{N} \frac{\lambda_k+1}{(\theta-\lambda_k)^2} \left(\frac{1}{2(\theta-\lambda_k)} \nu_k^2 \left(1 - e^{2(\theta-\lambda_k)T}\right) \right) \right. \\
&\qquad\qquad \left. + \frac{\epsilon^2}{2} \frac{1}{(\lambda_k-\theta)^3} \left(T - \frac{1-\bar{e}^{2(\lambda_k-\theta)T}}{2(\lambda_k-\theta)} \right) \right\}^{1/2} \\
&\qquad \text{(following [3, page 154])} \\
&\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} \sum_{k=1}^{N} \left\{ \frac{\lambda_k+1}{(\lambda_k-\theta)^3} \nu_k^2 \left(1 - \bar{e}^{2(\lambda_k-\theta)T}\right) + \frac{\epsilon^2}{2} \frac{T}{(\lambda_k-\theta)^3} \right\}^{1/2} \\
&\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} \sum_{k=1}^{N} \left\{ \frac{\lambda_k+1}{(\lambda_k-\theta)^3} \nu_k^2 + \frac{\epsilon^2 T}{(\lambda_k-\theta)^3} \right\}^{1/2} \\
&\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} \left\{ \sum_{k=1}^{N} \frac{\|f\|}{k^2} + \epsilon T^{1/2} \sum_{k=1}^{N} \frac{1}{k^3} \right\} \\
&\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} + \frac{C\epsilon T^{1/2}}{\delta Q_{N,T}^{(\epsilon)}}.
\end{aligned} \tag{3.18}$$

Next,

$$\begin{aligned} I_2 &\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} \sum_{k=1}^{N} \left\{ \frac{\nu_k^2}{\lambda_k+1} \bar{e}^{2(\lambda_k-\theta)T} \int_0^T e^{2(\lambda_k-\theta)s} ds \right\}^{1/2} \\ &\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} \sum_{k=1}^{N} \left\{ \frac{\nu_k^2}{\lambda_k+1} \frac{1}{2(\lambda_k-\theta)} \left(1-\bar{e}^{2T(\lambda_k-\theta)}\right) \right\}^{1/2} \\ &\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} \sum_{k=1}^{N} \left\{ \frac{\nu_k^2}{(\lambda_k+1)(\lambda_k-\theta)} \right\}^{1/2} \\ &\leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}} \sum_{k=1}^{N} \frac{\|f\|}{k^2} \leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}}. \end{aligned} \tag{3.19}$$

In addition,

$$\begin{aligned} I_3 &\leq \frac{C\epsilon^2}{\delta Q_{N,T}^{(\epsilon)}} \sum_{k=1}^{N} \left\{ \frac{1}{(\lambda_k-\theta)^2} \bar{e}^{4(\lambda_k-\theta)T} E\left[\int_0^T e^{(\lambda_k-\theta)s} dW_k(s) \right]^4 \right\}^{1/2} \\ &\leq \frac{C\epsilon^2}{\delta Q_{N,T}^{(\epsilon)}} \sum_{k=1}^{N} \left\{ \frac{1}{(\lambda_k-\theta)^2} \frac{\bar{e}^{4(\lambda_k-\theta)T}}{4(\lambda_k-\theta)^2} \left[e^{2(\lambda_k-\theta)T}-1\right]^2 \right\}^{1/2} \\ &= \frac{C\epsilon^2}{\delta Q_{N,T}^{(\epsilon)}} \sum_{k=1}^{N} \left\{ \frac{1}{(\lambda_k-\theta)^2} \bar{e}^{2(\lambda_k-\theta)T} \left[e^{2(\lambda_k-\theta)T}-1\right] \right\} \\ &= \frac{C\epsilon^2}{\delta Q_{N,T}^{(\epsilon)}} \sum_{k=1}^{N} \left\{ \frac{1}{(\lambda_k-\theta)^2} \left(1-\bar{e}^{2(\lambda_k-\theta)T}\right) \right\} \\ &\leq \frac{C\epsilon^2}{\delta Q_{N,T}^{(\epsilon)}} \sum_{k=1}^{N} \frac{1}{(\lambda_k-\theta)^2} \leq \frac{C\epsilon^2}{\delta Q_{N,T}^{(\epsilon)}} \sum_{k=1}^{N} \frac{1}{k^4} \\ &\leq \frac{C\epsilon^2}{\delta Q_{N,T}^{(\epsilon)}}. \end{aligned} \tag{3.20}$$

Note that

$$\begin{aligned} Q_{N,T}^{(\epsilon)} &= \sum_{k=1}^{N} \frac{\lambda_k+1}{2(\theta-\lambda_k)} \left\{ \nu_k^2\left(e^{2(\theta-\lambda_k)T}-1\right) - \frac{\epsilon^2 T}{\lambda_k+1} \right\} \\ &= \sum_{k=1}^{N} \frac{\lambda_k+1}{2(\lambda_k-\theta)} \left\{ \nu_k^2\left(1-\bar{e}^{2(\lambda_k-\theta)T}\right) + \frac{\epsilon^2 T}{\lambda_k+1} \right\}. \end{aligned} \tag{3.21}$$

Using (3.18), (3.19), and (3.20), we get that

$$I_1+I_2+I_3 \leq \frac{C_1\epsilon}{\delta Q_{N,T}^{(\epsilon)}} + \frac{C_2\epsilon T^{1/2}}{\delta Q_{N,T}^{(\epsilon)}} + \frac{C_3\epsilon^2}{\delta Q_{N,T}^{(\epsilon)}} \leq \frac{C\epsilon}{\delta Q_{N,T}^{(\epsilon)}}\left(1+T^{1/2}\right). \tag{3.22}$$

This completes the proof of Theorem 3.2. □

Observe that

$$Q_{N,T}^{(\epsilon)} \geq C[\epsilon^2 T + \|f\|^2] \tag{3.23}$$

for large $N \geq N_0$ depending on θ and T and for all $0 < \epsilon < 1$. Choosing $\delta = \epsilon^{1-r}$, for some $0 < r < 1$, we get that the bound in Theorem 3.2 is of order

$$\frac{C\epsilon^r (1 + T^{1/2})}{(\epsilon^2 T + \|f\|^2)}. \tag{3.24}$$

As a consequence of Theorems 3.1 and 3.2, we have the following main result giving a Berry-Esseen type bound for the MLE $\hat{\theta}_{N,\epsilon}$.

Theorem 3.5. *Let $N \geq N_0$ be fixed, satisfying (3.23). Then there exists a constant C depending on θ_0, $\|f\|$, and T such that, for any $0 < \epsilon < 1$ and $0 < r < 1$,*

$$\sup_y \left| P_{\theta_0}^{\epsilon,N} \left\{ \sqrt{Q_{N,T}^{(\epsilon)}} \epsilon^{-1} (\hat{\theta}_{N,\epsilon} - \theta_0) \leq y \right\} - \Phi(y) \right| \leq \frac{C\epsilon^r (1 + T^{1/2})}{\epsilon^2 T + \|f\|^2} + 3\sqrt{\epsilon^{1-r}}. \tag{3.25}$$

Remarks 3.6. Observe that the bound in Theorem 3.5 is of order $O(\epsilon^r) + O(\epsilon^{(1-r)/2})$. Choosing $r = 1/3$, we note that the bound is of order $O(\epsilon^{1/3})$.

4. SPDE with linear drift (singular case): estimation and Berry-Esseen type bound

Let $(\Omega, \mathcal{F}, P)$ be a probability space and consider the process $u_{i\epsilon}(t,x)$, $0 \leq x \leq 1$, $0 \leq t \leq T$, governed by the SPDE

$$du_\epsilon(t,x) = \theta \Delta u_\epsilon(t,x)dt + \epsilon(I - \Delta)^{-1/2} dW(t,x), \tag{4.1}$$

where $\theta > 0$ satisfies the initial and boundary conditions

$$\begin{gathered} u_\epsilon(0,x) = f(x), \quad 0 < x < 1, \ f \in L_2[0,1], \\ u_\epsilon(t,0) = u_\epsilon(t,1) = 0, \quad 0 \leq t \leq T. \end{gathered} \tag{4.2}$$

Here, I is the identity operator, $\Delta = \partial^2/\partial x^2$ as defined in Section 3, and the process $W(t,x)$ is the cylindrical Brownian motion in $L_2[0,1]$. In analogy with the discussion following the stochastic differential equation given by (2.6), it can be checked that the Fourier coefficients $u_{i\epsilon}(t)$ satisfy the stochastic differential equation

$$du_{i\epsilon}(t) = -\theta \lambda_i u_{i\epsilon}(t)dt + \frac{\epsilon}{\sqrt{\lambda_1 + 1}} dW_i(t), \quad 0 < t \leq T, \tag{4.3}$$

where conditions (2.7) hold.

Let P_θ^ϵ be the measure generated by the process u_ϵ on $C[0,T]$ when θ is the true parameter. It can be shown that the family of measures $\{P_\theta^\epsilon, \theta \in \Theta\}$ does not form a family of equivalent probability measures. In fact, P_θ^ϵ is singular with respect to $P_{\theta'}^\epsilon$ when $\theta \neq \theta'$ in Θ (cf. Hübner et al. [3]). Let $u_\epsilon^{(N)}(t,x)$ be the projection of $u_\epsilon(t,x)$ onto the subspace spanned by $\{e_1, e_2, \dots, e_N\}$ in $L_2[0,1]$. In other words,

$$u_\epsilon^{(N)}(t,x) = \sum_{i=1}^{N} u_{i\epsilon}(t) e_i(x). \tag{4.4}$$

Let $P_\theta^{\epsilon,N}$ be the probability measure generated by the process $u_\epsilon^{(N)}$ on the subspace spanned by $\{e_1, \dots, e_N\}$ in $L_2[0,1]$. It can be shown that the measures $\{P_\theta^{\epsilon,N},\ \theta \in \Theta\}$ form an equivalent family and

$$\begin{aligned} \log \frac{dP_\theta^{\epsilon,N}}{dP_{\theta_0}^{(\epsilon,N)}}\left(u_\epsilon^{(N)}\right) = -\frac{1}{\epsilon^2} \sum_{i=1}^{N} \lambda_i(\lambda_i+1) \Big[(\theta-\theta_0) \int_0^T u_{i\epsilon}(t)\left(du_{i\epsilon}(t) + \theta_0 \lambda_i u_{i\epsilon}(t)dt\right) \\ + \frac{1}{2}(\theta-\theta_0)^2 \lambda_i \int_0^T u_{i\epsilon}^2(t)dt \Big]. \end{aligned} \tag{4.5}$$

It can be checked that the MLE $\hat{\theta}_{N,\epsilon}$ of θ based on $u_\epsilon^{(N)}$ satisfies the likelihood equation

$$\alpha_{\epsilon,N} = \epsilon^{-1}(\hat{\theta}_{N,\epsilon} - \theta_0)\beta_{\epsilon,N} \tag{4.6}$$

when θ_0 is the true parameter,

$$\alpha_{\epsilon,N} = \sum_{i=1}^{N} \lambda_i \sqrt{\lambda_i+1} \int_0^T u_{i\epsilon}(t)dW_i(t), \tag{4.7}$$

$$\beta_{\epsilon,N} = \sum_{i=1}^{N} (\lambda_i+1)\lambda_i^2 \int_0^T u_{i,\epsilon}^2(t)dt. \tag{4.8}$$

From (4.6), we obtain that

$$\sqrt{R_{N,T}^{(\epsilon)}}(\hat{\theta}_{N,\epsilon} - \theta_0) = \frac{\epsilon\left\{\sum_{i=1}^{N} \lambda_i \sqrt{\lambda_i+1} \int_0^T u_{i\epsilon}(t)dW_i(t)\right\} / \sqrt{R_{N,T}^{(\epsilon)}}}{\left\{\sum_{i=1}^{N} (\lambda_i+1)\lambda_i^2 \int_0^T u_{i\epsilon}^2(t)dt\right\} / R_{N,T}^{(\epsilon)}}, \tag{4.9}$$

where

$$R_{N,T}^{(\epsilon)} = \sum_{i=1}^{N} \frac{\lambda_i(\lambda_i+1)}{2\theta} \left\{ \nu_i^2 (1 - \bar{e}^{2\theta\lambda_i T}) + T \frac{\epsilon^2}{\lambda_i+1} \right\}. \tag{4.10}$$

It can be checked that

$$E_{\theta_0} \int_0^T u_{i\epsilon}^2(t)dt < \infty. \tag{4.11}$$

THEOREM 4.1. *For any* $0 < \delta < 1$,

$$\sup_y \left| P_{\theta_0}^{\epsilon,N} \left\{ \sqrt{R_{N,T}^{(\epsilon)}} (\hat{\theta}_{N,\epsilon} - \theta_0) \le y \right\} - \Phi(y) \right| \le 2P_{\theta_0}^{\epsilon,N} \left\{ \left| \frac{\epsilon^{-1} \sum_{i=1}^N (\lambda_i+1)\lambda_i^2 \int_0^T u_{i\epsilon}^2(t)dt}{R_{N,T}^{(\epsilon)}} - 1 \right| \ge \delta \right\} + 3\sqrt{\delta}. \tag{4.12}$$

We can prove Theorem 4.1 using Lemmas 3.3 and 3.4 and following the method in the proof of Theorem 3.1.

THEOREM 4.2. *Let* $0 < \epsilon < 1$ *be fixed. Then there exists a constant C depending on* θ_0, $\|f\|^2$, *and T such that, for any* $0 < \delta < 1$ *and* $N \ge 1$,

$$P_{\theta_0}^{\epsilon,N} \left\{ \left| \frac{\epsilon^{-1} \sum_{i=1}^N (\lambda_i+1)\lambda_i^2 \int_0^T u_{i\epsilon}^2(t)dt}{R_{N,T}^{(\epsilon)}} - 1 \right| \ge \delta \right\} \le \frac{CN^3(1+T^{1/2})}{\delta R_{N,T}^{(\epsilon)}}. \tag{4.13}$$

Proof. By the Îto formula, we get that

$$d(u_{i\epsilon}^2(t)) = -2\theta\lambda_i u_{i\epsilon}^2(t)dt + \frac{\epsilon}{\sqrt{\lambda_i+1}} u_{i\epsilon}(t)dW_i(t) + \frac{\epsilon^2}{\lambda_i+1}dt \tag{4.14}$$

or, equivalently,

$$d\left(\frac{\lambda_i(\lambda_i+1)}{2\theta} u_{i\epsilon}^2(t)\right) = -\lambda_i^2(\lambda_i+1)u_{i\epsilon}^2(t)dt + \frac{\epsilon\lambda_i\sqrt{\lambda_i+1}}{2\theta} u_{i\epsilon}(t)dW_i(t) + \frac{\epsilon^2\lambda_i}{2\theta}dt \tag{4.15}$$

or

$$\frac{\lambda_i(\lambda_i+1)}{2\theta} u_{i\epsilon}^2(T) - \frac{\lambda_i(\lambda_i+1)}{2\theta} v_i^2 = -\int_0^T \lambda_i^2(\lambda_i+1)u_{i\epsilon}^2(t)dt + \frac{\epsilon\lambda_i\sqrt{\lambda_i+1}}{2\theta} \int_0^T u_{i\epsilon}(t)dW_i(t) + \frac{\epsilon^2\lambda_i}{2\theta}T. \tag{4.16}$$

Again, by the Îto formula, it follows that

$$\begin{aligned} d(u_{i\epsilon}(t)e^{\theta\lambda_i t}) &= \frac{\epsilon}{\sqrt{\lambda_i+1}} e^{\theta\lambda_i t} dW_i(t), \\ u_{i\epsilon}(T)e^{\theta\lambda_i T} - v_i &= \int_0^T \frac{\epsilon}{\sqrt{\lambda_i+1}} e^{\theta\lambda_i t} dW_i(t), \\ u_{i\epsilon}(T) - v_i\bar{e}^{\theta\lambda_i T} &= \bar{e}^{\theta\lambda_i T} \int_0^T \frac{\epsilon}{\sqrt{\lambda_i+1}} e^{\theta\lambda_i t} dW_i(t), \\ u_{i\epsilon}^2(T) &= \bar{e}^{2\theta\lambda_i T} \left(\int_0^T \frac{\epsilon}{\sqrt{\lambda_i+1}} e^{\theta\lambda_i t} dW_i(t) \right)^2 \\ &\quad + v_i^2 \bar{e}^{2\theta\lambda_i T} + \frac{2\epsilon}{\sqrt{\lambda_i+1}} v_i \bar{e}^{2\theta\lambda_i T} \int_0^T e^{\theta\lambda_i t} dW_i(t), \end{aligned} \tag{4.17}$$

or

$$\frac{\lambda_i(\lambda_i+1)}{2\theta}u_{i\epsilon}^2(T)=\frac{\epsilon^2\lambda_i}{2\theta}\bar{e}^{2\theta\lambda_i T}\left(\int_0^T e^{\theta\lambda_i t}dW_i(t)\right)^2 + \frac{\lambda_i(\lambda_i+1)}{2\theta}\nu_i^2\bar{e}^{2\theta\lambda_i T}+\frac{\epsilon}{\theta}\lambda_i\sqrt{\lambda_i+1}\bar{e}^{2\theta\lambda_i T}\nu_i\int_0^T e^{\theta\lambda_i t}dW_i(t). \tag{4.18}$$

From (4.16) and (4.18), we get that

$$\begin{aligned}\sum_{i=1}^N\frac{\lambda_i(\lambda_i+1)}{2\theta}\left\{\nu_i^2(1-\bar{e}^{2\theta\lambda_i T})+\frac{\epsilon^2}{\lambda_i+1}T\right\}-\sum_{i=1}^N\int_0^T\lambda_i^2(\lambda_i+1)u_{i\epsilon}^2(t)dt\\ =\frac{\epsilon^2}{2\theta}\sum_{i-1}^N\lambda_i\bar{e}^{2\theta\lambda_i T}\left(\int_0^T e^{\theta\lambda_i t}dW_i(t)\right)^2+2\epsilon\sum_{i=1}^N\lambda_i\sqrt{\lambda_i+1}\nu_i\bar{e}^{2\theta\lambda_i T}\int_0^T e^{\theta\lambda_i t}dW_i(t)\\ -\epsilon\sum_{i=1}^N\frac{\lambda_i\sqrt{\lambda_i+1}}{2\theta}\int_0^T u_{i\epsilon}(t)dW_i(t).\end{aligned} \tag{4.19}$$

Hence

$$\begin{aligned}P_{\theta_0}^{\epsilon,N}&\left\{\left|\frac{\epsilon^{-1}\sum_{i=1}^N\int_0^T\lambda_i^2(\lambda_i+1)u_{i\epsilon}^2(t)dt}{R_{N,T}^{(\epsilon)}}-1\right|\geq\delta\right\}\\ &\leq P_{\theta_0}^{\epsilon,N}\left\{\left|\frac{(\epsilon/2\theta)\sum_{i=1}^N\lambda_i\bar{e}^{2\theta\lambda_i T}\left(\int_0^T e^{\theta\lambda_i t}dW_i(t)\right)^2}{R_{N,T}^{(\epsilon)}}\right|\geq\frac{\delta}{3}\right\}\\ &\quad+P_{\theta_0}^{\epsilon,N}\left\{\left|\frac{(1/\theta)\sum_{i=1}^N\lambda_i\sqrt{\lambda_i+1}\nu_i\bar{e}^{2\theta\lambda_i T}\int_0^T e^{\theta\lambda_i t}dW_i(t)}{R_{N,T}^{(\epsilon)}}\right|\geq\frac{\delta}{3}\right\}\\ &\quad+P_{\theta_0}^{\epsilon,N}\left\{\left|\frac{(1/2\theta)\sum_{i=1}^N\lambda_i\sqrt{\lambda_i+1}\int_0^T u_{i\epsilon}(t)dW_i(t)}{R_{N,T}^{(\epsilon)}}\right|\geq\frac{\delta}{3}\right\}\\ &=J_1+J_2+J_3\ (\text{say}),\end{aligned} \tag{4.20}$$

where

$$R_{N,T}^{(\epsilon)}=\sum_{i=1}^N\frac{\lambda_i(\lambda_i+1)}{2\theta}\left\{\nu_i^2(1-\bar{e}^{2\theta\lambda_i T})+\frac{\epsilon^2}{\lambda_i+1}T\right\}. \tag{4.21}$$

Therefore,

$$\begin{aligned}J_1&\leq\frac{C\epsilon}{\delta R_{N,T}^{(\epsilon)}}\sum_{i=1}^N\lambda_i\bar{e}^{2\theta\lambda_i T}\int_0^T e^{2\theta\lambda_i t}dt\\ &=\frac{C\epsilon}{\delta R_{N,T}^{(\epsilon)}}\sum_{i=1}^N\lambda_i\bar{e}^{2\theta\lambda_i T}\left\{\frac{(e^{2\theta\lambda_i T}-1)}{2\theta\lambda_i}\right\}\\ &\leq\frac{C\epsilon N}{\delta R_{N,T}^{(\epsilon)}},\end{aligned}$$

$$
\begin{aligned}
J_2 &\leq \frac{C}{\delta R_{N,T}^{(\epsilon)}} \sum_{i=1}^{N} \sqrt{\lambda_i}\sqrt{\lambda_i+1}\nu_i \\
&\leq \frac{CN^3}{\delta R_{N,T}^{(\epsilon)}}, \\
J_3 &\leq \frac{C}{\delta R_{N,T}^{(\epsilon)}} \left(\sum_{i=1}^{N} \lambda_i^2(\lambda_i+1) \int_0^T Eu_{i\epsilon}^2(t)dt \right)^{1/2} \\
&\leq \frac{C\sqrt{2\theta}}{\delta R_{N,T}^{(\epsilon)}} \left\{ \sum_{i=1}^{N} \lambda_i(\lambda_i+1)\nu_i^2\left(1-\bar{e}^{2\theta\lambda_i T}\right) + T\sum_{i=1}^{N}\lambda_i \right\}^{1/2} \\
&\quad \text{(following [3, page 158])} \\
&\leq \frac{C}{\delta R_{N,T}^{(\epsilon)}} \left\{ \sum_{i=1}^{N} \lambda_i(\lambda_i+1) + T\sum_{i=1}^{N}\lambda_i \right\}^{1/2} \\
&\leq \frac{C}{\delta R_{N,T}^{(\epsilon)}} \left\{ N^{5/2} + T^{1/2}N^{3/2} \right\}.
\end{aligned}
\tag{4.22}
$$

Hence

$$
J_1 + J_2 + J_3 \leq \frac{C}{\delta R_{N,T}^{(\epsilon)}}\left(N^3 + T^{1/2}N^{3/2}\right) \leq \frac{C}{\delta R_{N,T}^{(\epsilon)}} N^3\left(1+T^{1/2}\right). \tag{4.23}
$$

This completes the proof of Theorem 4.2. □

Observe that

$$
\begin{aligned}
R_{N,T}^{(\epsilon)} &= \sum_{k=1}^{N} \frac{\lambda_k(\lambda_k+1)}{2\theta} \left\{ \nu_k^2\left(1-e^{-2\theta\lambda_k T}\right) + \frac{T\epsilon^2}{\lambda_k+1} \right\} \\
&\geq C \sum_{k=k_1}^{N} k^4 \left\{ \nu_k^2 + \frac{T\epsilon^2}{k^2} \right\} \\
&\geq C \left(\sum_{k=1}^{N} k^4 \nu_k^2 + \epsilon^2 T N^3 \right)
\end{aligned}
\tag{4.24}
$$

for some k_1 depending on ϵ, θ, and T, and hence for $N \geq N_0$ depending on ϵ, θ, and T. Therefore,

$$
J_1 + J_2 + J_3 \leq \frac{CN^3\left(1+T^{1/2}\right)}{\delta\left(\epsilon^2 T N^3 + \sum_{k=1}^{N} k^4 \nu_k^2\right)} \tag{4.25}
$$

for $N \geq N_0$ depending on ϵ, θ, and T. Choosing $\delta = N^{-\gamma}$, for some $\gamma > 0$, we get that the bound is of order

$$\frac{CN^3(1+T^{1/2})}{N^{-\gamma}(\epsilon^2 TN^3 + \sum_{k=1}^{N} k^4 \nu_k^2)}. \tag{4.26}$$

As a consequence of Theorems 4.1 and 4.2, we have the following result which gives a Berry-Esseen type bound for the MLE $\hat{\theta}_{N,\epsilon}$ for any fixed $0 < \epsilon < 1$.

Theorem 4.3. *Let* $0 < \epsilon < 1$ *be fixed. Then there exists a constant* C *depending on* θ_0, $\|f\|^2$, *and* T *such that, for any* $\gamma > 0$ *and* $N \geq N_0$, *depending on* ϵ, θ_0, *and* T,

$$\begin{aligned} &\sup_y \left| P_{\theta_0}^{\epsilon,N}\left\{\sqrt{R_{N,T}^{(\epsilon)}}(\hat{\theta}_{N,\epsilon} - \theta_0) \leq y\right\} - \Phi(y)\right| \\ &\quad \leq \frac{CN^3}{N^{-\gamma}}\left(\frac{1+T^{1/2}}{\epsilon^2 TN^3 + \sum_{k=1}^{N} k^4 \nu_k^2}\right) + 3\sqrt{N^{-\gamma}}. \end{aligned} \tag{4.27}$$

Remarks 4.4. Observe that the bound in Theorem 4.3 is of order $O(N^{\gamma-2}) + O(N^{-\gamma/2})$ provided $\sum_{k=1}^{N} k^4 \nu_k^2 \geq g(N) = O(N^5)$. In such a case, the bound can be obtained to be of order $O(N^{-2/3})$ by choosing $\gamma = 4/3$. We can obtain the rate of convergence for the case when N is fixed but ϵ varies over the interval $(0,1)$ by arguments similar to those given above. We omit the details.

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