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Poisson formulae of Hecke type

By

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1. Introduction

In his lectures on "Forms of higher degree", Igusa [10] gave a simple proof of a Poisson formula due to Yamazaki [19] associated with coefficients of Dirichlet series with functional equations involving a single Γ-factor, by introducing an operator of order 2 in a function space, *via* the Mellin transform. Our object here is to show that this Poisson formula can be generalised a little, so as to bring within its ambit, functional equations of Dirichlet series involving multiple Γ-factors (and, in particular, those associated with Mellin transforms of non-analytic automorphic functions). We also indicate an adelic interpretation of certain Poisson formulae above just to highlight the fact that such a formula in adelic form constitutes an important step in the proof of theorem 5 in § 3 of [7] on a global representation of $(GL_2)_A$.

2. Mellin transforms

Let $\mu_{0,1}, \ldots, \mu_{0,a}, \mu_{1,1}, \ldots, \mu_{1,a}, \mu_{2,1}, \ldots, \mu_{2,a}$

be a sequence of a-tuples of complex numbers such that $\mathrm{Re}\,(\mu_{k,j}) = \lambda_k$ for every j with $1 \leqslant j \leqslant a$ and for every $k \geqslant 0$ and further let $0 \leqslant \lambda_0 < \lambda_1 < \lambda_2 \ldots \to \infty$. For every $k \geqslant 0$, let $m_k \geqslant 1$ be a fixed natural number.

Following Igusa [10], we define corresponding function spaces $\mathscr{F}$ and $\mathscr{Z}$ by the conditions:

(A) $\mathscr{F}$ consists of all complex valued C^∞ functions F on the space $\mathbf{R}_+^\times$ of positive real numbers behaving like Schwartz functions at infinity such that with well-determined complex constants $a_{k,j,m}$ the function F has a termwise differentiable asymptotic expansion

$$F(x) \approx \sum_{0 \leqslant k < \infty} \sum_{1 \leqslant j \leqslant a} \sum_{1 \leqslant m \leqslant m_k} a_{k,j,m} x^{\mu_{k,j}} (\log x)^{m-1}, \tag{1}$$

as x tends to 0, and

(B) $\mathscr{Z}$ consists of all complex valued functions Z meromorphic on $\mathbf{C}$ with poles at most at the points $-\mu_{k,j}$ $(k \geqslant 0;\ 1 \leqslant j \leqslant a)$ and having principal part at $-\mu_{k,j}$ of the form.

$$\sum_{1 \leqslant m \leqslant m_k} b_{k,j,m}(s + \mu_{k,j})^{-m}$$

and further such that for every polynomial P in s and any vertical strip $B_{\sigma_1,\sigma_2} = \{s = \sigma + ti \in \mathbf{C}\ ;\ \sigma_1 \leqslant \sigma \leqslant \sigma_2\}$ with arbitrary σ_1, σ_2, the function $P(s)\,Z(s)$ is bounded in B_{σ_1,σ_2} with small neighbourhoods of the poles of Z deleted therefrom.

For $k \geqslant 0$ and any j, m with $1 \leqslant j \leqslant a$ and $1 \leqslant m \leqslant m_k$, let us define

$$\varphi_{k,j,m}(x) = x^{\mu_{k,j}} (\log x)^{m-1} \text{ for } x \in \mathbf{R}_+^\times\ ;$$

then $\quad \varphi_{k,j,m} = o(\varphi_{k+1,l,n})$ as x tends to 0,

for every j, l, m and n. Further if we define, for $k \geqslant 0$,

$$R_k(x) = F(x) - \sum_{0 \leqslant l \leqslant k} \sum_{1 \leqslant j \leqslant a} \sum_{1 \leqslant m \leqslant m_l} a_{l,j,m} x^{\mu_{l,j}} (\log x)^{m-1} \tag{2}$$

then the asymptotic expansion (1) for F in $\mathscr{F}$ simply means that $R_k(x) = O(\varphi_{k+1,p,q})$ for every $k \geqslant 0$ and any p, q as x tends to 0.

The space $\mathscr{F}$ is non-empty, since it contains the constant function $F = 0$ on $\mathbf{R}_+^\times$. Further it is stable under homothety-invariant differential operators; this property of $\mathscr{F}$ corresponds to $\mathscr{Z}$ being stable under multiplication by polynomials in s.

Theorem 1. *There exists a bijective correspondence*

$M : \mathscr{F} \xrightarrow{\sim} \mathscr{Z}$ given by $F \to MF$ where

$$(MF)(s) = \int_0^\infty F(x)\, x^s\, d(\log x), \tag{3}$$

defined for $s = \sigma + ti$ in $\mathbf{C}$ *with* $\sigma = \mathrm{Re}(s) > 0$ *has a meromorphic continuation to the whole of* $\mathbf{C}$. *Conversely, for every Z in $\mathscr{Z}$, the inverse transform*

$$(M^{-1}Z)(x) = \frac{1}{2\pi i} \int_{\sigma - \infty i}^{\sigma + \infty i} Z(s)\, x^{-s}\, ds \tag{4}$$

gives rise to an element of $\mathscr{F}$, independently of σ for $\sigma > 0$. Moreover, we have the relation

$$b_{l,j,m} = (-1)^{m-1}(m-1)!\ a_{l,j,m}, \tag{5}$$

for every l, j, m.

The proof of this theorem is exactly the same as in [10] and for the sake of completeness, we shall quickly sketch its proof.

(i) $F \in \mathscr{F} \Rightarrow Z = MF \in \mathfrak{Z}$:

Let $s = \sigma + ti$ with $0 < \sigma_1 \leqslant \sigma \leqslant \sigma_2 < \infty$. It is easy to see that $x^s F(x)$ is dominated by a φ in $L^1(\mathbf{R}_+^\times, d(\log x))$; therefore $Z(s)$ is holomorphic for $\mathrm{Re}\, s > 0$, in view of its being defined by an absolutely convergent integral and the integrand being holomorphic. It is clear, for a similar reason that $Z_2(s) = \int_1^\infty F(x)\, x^s\, d(\log x)$ is an entire function of s and further bounded in vertical strips. For every $\rho < \lambda_{k+1}$, $R_k(x) = o(x^\rho)$ as x tends to 0 and therefore, for every σ_1 with $-\lambda_{k+1} < \sigma_1 \leqslant \sigma$ and $-\sigma_1 \leqslant \rho < \lambda_{k+1}$, the function $x^{\sigma_1} R_k(x)$ is dominated by an element of $L^1((0,1], d(\log x))$. As a result, $Z_1(s) = \int_0^1 R_k(x)\, x^s\, d(\log x)$ is holomorphic in s for $\mathrm{Re}\,(s) > -\lambda_{k+1}$ and bounded in vertical strips. Splitting up the integral in (3) over $(0, 1]$ and $[1, \infty)$, it is easy to see from (2) that for $\mathrm{Re}\, s > 0$,

$$Z(s) = \sum_{0 \leqslant l \leqslant k} \sum_{1 \leqslant j \leqslant a} \sum_{1 \leqslant m \leqslant m_l} b_{l,j,m}(s + \mu_{l,j})^{-m} + Z_1(s) + Z_2(s)$$

with $b_{l,j,m}$ satisfying (5). It is now immediate that Z satisfies the first half of condition (B) and also the second half with $P \equiv 1$. For the case of arbitrary P, one has only to invoke the homothety-invariance of $\mathscr{F}$ and it now follows that Z is in $\mathfrak{Z}$.

(ii) $Z \in \mathfrak{Z} \Rightarrow F = M^{-1} Z \in \mathscr{F}$:

For $0 < \sigma_1 < \sigma_2$, let L denote the boundary (covered anticlockwise) of the rectangle in the complex s-plane with vertices at $\sigma_1 \pm t_0 i, \sigma_2 \pm t_0 i$ for some $t_0 > 0$. By Cauchy's theorem, $\int_L Z(s)\, x^{-s}\, ds = 0$ for any $x > 0$. As t_0 tends to ∞, the contribution to this integral from the "horizontal" sides of L tends to 0, in view of the second part of condition (B) for Z. Thus the integral (4) is defined independently of σ and moreover converges absolutely in view of condition (B). Now, for any $k \geqslant 0$,

$$x^k \frac{d^k F(x)}{dx^k} = \frac{1}{2\pi i} \int_{\sigma - \infty i}^{\sigma + \infty i} (-1)^k s(s+1)\ldots(s+k-1)\, Z(s)\, x^{-s}\, ds$$

and this integral converges absolutely by condition (B). Thus F is in $C^\infty(\mathbf{R}_+^\times)$ and behaves like a Schwartz function at infiniity. We then see that $R_k(x)$ defined by (2) satisfies the condition $R_k(x) = o(x^\rho)$ for any ρ with $\lambda_k < \rho < \lambda_{k+1}$ and $k \geqslant 0$. If L_1 is the boundary traversed anticlcokwise of the rectangle in the s-plane with vertices at $\sigma \pm t_0 i$, $-\rho \pm t_0 i$ for a large enough $t_0 > 0$, one shows by choosing a proper contour and using condition (B) that

$$F(x) = S + \frac{1}{2\pi i} \int_{-\rho - \infty i}^{-\rho + \infty i} Z(s)\, x^{-s}\, ds,$$

where S is the sum of the residues of the integrand at the poles $-\mu_{0,1},\ldots,-\mu_{0,a}$, $-\mu_{1,1},\ldots,-\mu_{1,a},\ldots$, lying inside L_1. But $F(x)-S$ is precisely $R_k(x)$ and the required o-estimate for $R_k(x)$ follows, in view of (B), by majorising the integral. Now $-s\,Z(s)=(\mu_{l,p}-(s+\mu_{l,p}))\,Z(s)$ is in $\mathscr{Z}$ with

$$b'_{l,j,p}=\begin{cases}\mu_{l,j}\,b_{l,j,p}-b_{l,j,p+1} & 1\leqslant p<m_l\\ \mu_{l,j}\,b_{l,j,p} & p=m_l\end{cases}$$

in place of the earlier $b_{l,j,p}$. Working with $-sZ(s)$ in place of $Z(s)$, we get $xF'(x)$ instead of $F(x)$ and further

$$xF'(x)\approx\sum_{0\leqslant\cdot<\infty}\ \sum_{1\leqslant j\leqslant a}\ \sum_{1\leqslant m\leqslant m_k}\tilde{a}_{k,j,m}x^{\mu_{k,j}}(\log x)^{m-1}$$

with $\quad\tilde{a}_{k,j,m}=(-1)^{m-1}\,b'_{k,j,m}\,/\,(m-1)\,!$

$$=\begin{cases}\mu_{k,j}\,a_{k,j,m}+ma_{k,j,m+1}, & 1\leqslant m<m_k\\ \mu_{k,j}\,a_{k,j,m}, & m=m_k.\end{cases}$$

Thus the asymptotic expansion of F is once termwise differentiable; interaction shows that F is in $\mathscr{F}$.

Remarks : (i) If the condition $\lambda_k\geqslant 0$ for every k is replaced by a condition of the form $\lambda_k\geqslant\lambda$ for every k for some fixed real $\lambda\leqslant 0$, the corresponding $\mathscr{Z}$-space is obtained from the earlier one by a mere translation of the varible s by λ; similarly, the corresponding space $\mathscr{F}$ is obtained by multiplying the elements of the original $\mathscr{F}$-space by x^{λ}.

(ii) One can also consider more general sequences of the form

$$\mu_{0,1},\ldots,\mu_{0,a_0},\mu_{1,1},\ldots,\mu_{1,a_1},\mu_{2,1},\ldots,\mu_{2,a_2},\ldots$$

with $\mathrm{Re}\,(\mu_{k,j})=\lambda_k$ for every fixed $k\geqslant 0$ and for $1\leqslant j\leqslant a_k$. However, arithmetically interesting situations arise when, for example, the sequence of $\mu_{k,j}$'s coincides with the set of poles of a product of Γ-factors, say

$$G(s)=\prod_{1\leqslant j\leqslant a}\Gamma(a_js+\beta_j)^{m_j}\ \text{with}\ a_j>0,\ \mathrm{Re}\,\beta_j\geqslant 0\ \text{and}$$

$a_i\beta_j-a_j\beta_i$ not of the form ma_j-na_i for $m,n\geqslant 0$ in $\mathbf{Z}$ if $i\neq j$. In such a situation, $\mathscr{Z}$ can be characterised as the space of meromorphic functions Z on $\mathbf{C}$ such that $Z(s)/G(s)$ is entire and for every polynomial P in s, $P(s)\,Z(s)$ is bounded in vertical strips. In the *corresponding space* $\mathscr{F}$, we have for every $\kappa>0$, an involution $F\mapsto\mathbf{W}F$ defined by

$$\frac{(M(\mathbf{W}F))(s)}{G(s)}=\frac{(MF)(\kappa-s)}{G(\kappa-s)}. \tag{6}$$

(As we shall see later, an analogue of this operator $\mathbf{W}$ exists already in the non-archimedean case as well). We shall be interested in obtaining Poisson formulae involved with functional equations containing multiple Γ-factors.

3. Poisson formula of Hecke type

Let $\{\varphi_j(s) = \sum_{n\neq 0} a_n^{(j)} |n|^{-s} \,;\; 1 \leqslant j \leqslant N\}$ and

$$\{\psi_j^*(s) = \sum_{n\neq 0} b_n^{(j)} |n|^{-s} \,;\; 1 \leqslant j \leqslant N\}$$

be two sets of N Dirichlet series each converging in some (right) s-halfplane such that if we set, for $1 \leqslant j \leqslant N$ and a fixed $A > 0$,

$$\xi_j(s) = A^s G(s)\varphi_j(s), \quad \eta_j^*(s) = A^s G(s)\psi_j^*(s), \tag{7}$$

then we have the functional equations

$$\xi_j(\kappa - s) = \sum_{1\leqslant k\leqslant N} c_{jk}\,\eta_k^*(s), \quad 1 \leqslant j \leqslant N \tag{8}$$

with real c_{jk}. We may assume $\xi_1, \dots, \xi_N$ to be linearly independent over $\mathbf{C}$; then, $(c_{jk})^2$ is the N-rowed identity matrix. We also assume that ξ_k, η_l^* have only finitely many poles in $\mathbf{C}$.

From (6), (7) and (8), we obtain

$$(MWF)(s)\,A^s \varphi_k(s) = \sum_{1\leqslant l\leqslant N} c_{kl}(MF)(\kappa - s)\,A^{\kappa-s}\psi_l^*(\kappa - s). \tag{9}$$

In view of the absolute convergence of the Dirichlet series $\varphi_j(s)$ in some right half plane, we see that the integral of the left hand side of (9) from $\sigma - \infty i$ to $\sigma + \infty i$ for any fixed sufficiently large σ is simply

$$2\pi i \sum_{n\neq 0} a_n^{(k)}\,(WF)(|n|/A).$$

Thus, for large σ_1, we have

$$\sum_{n\neq 0} a_n^{(k)}\,(WF)(|n|/A)$$

$$= \frac{1}{2\pi i}\int_{\kappa-\sigma-\infty i}^{\kappa-\sigma+\infty i} (M(WF))(\kappa - s)\,A^{\kappa-s}\varphi_k(\kappa - s)ds$$

$$= \frac{1}{2\pi i}\sum_{1\leqslant l\leqslant N} c_{kl}\int_{\kappa-\sigma-\infty i}^{\kappa-\sigma+\infty i} (MF)\,(s)\,A^s\psi_l^*(s)ds$$

$$= \frac{1}{2\pi i}\sum_{1\leqslant l\leqslant N} c_{kl}\int_{\sigma_1-\infty i}^{\sigma_1+\infty i} \frac{(MF)(s)}{G(s)}\,\eta_l^*(s)\,ds - S^*$$

where $S^* = \sum \operatorname{Res}\left(\sum_{1 \leqslant l \leqslant N} c_{kl} \frac{(MF)(s)}{G(s)} \eta_l^*(s) \right)$,

the sum of the residues of the sum inside at all the poles. Using the absolute convergence of the Dirichlet series $\psi_l^*(s)$, we have

$$\sum_{n \neq 0} a_n^{(k)} (\mathbf{W}F)(|n|/A) = \sum_{1 \leqslant l \leqslant N} c_{kl} \sum_{n \neq 0} b_n^{(l)} F(|n|/A) - S^*$$

which, on replacing F by $\mathbf{W}F$ becomes

$$\sum_{n \neq 0} a_n^{(k)} F(|n|/A) = \sum_{1 \leqslant l \leqslant N} c_{kl} \sum_{n \neq 0} b_n^{(l)} (\mathbf{W}F)(|n|/A) -$$

$$- \sum \operatorname{Res}\left(\frac{(M(\mathbf{W}F))(s)}{G(s)} \xi_k(\kappa - s) \right) \tag{10}$$

where Σ Res is the sum of the residues at all poles, as before.

For fixed $y > 0$, let us replace F in (10) by F_y where $F_y(x) = F(xy)$. Then

$$\mathbf{W}(F_y) = y^{-\kappa} (\mathbf{W}F)_{y^{-1}}, \tag{11}$$

where $(\mathbf{W}F)_{y^{-1}}(x) = (\mathbf{W}F)(xy^{-1})$

for every $x > 0$. Therefore

$$M(\mathbf{W}(F_y))(s) = y^{-\kappa} M((\mathbf{W}F)_{y^{-1}})(s) = y^{s-\kappa} (M(\mathbf{W}F))(s)$$

and (10) now becomes

$$\sum_{n \neq 0} a_n^{(k)} F(|n| y/A) =$$

$$= y^{-\kappa} \sum_{1 \leqslant l \leqslant N} c_{kl} \sum_{n \neq 0} b_n^{(l)} (\mathbf{W}F)(|n|/(Ay)) -$$

$$- \sum \operatorname{Res}\left(\frac{(M(\mathbf{W}F))(s)}{G(s)} y^{s-\kappa} \xi_k(\kappa - s) \right) \tag{10$'$}$$

$$= y^{-\kappa} \sum_{1 \leqslant l \leqslant N} c_{kl} \sum_{n \neq 0} b_n^{(l)} (\mathbf{W}F)(|n|/(Ay)) +$$

$$+ \sum \operatorname{Res}\left(y^{-s} \frac{(MF)(s)}{G(s)} \xi_k(s) \right). \tag{10$''$}$$

The second term in (10)$''$ is a 'residual function' in the sense of Bochner [3] and (10)$''$ gives rise to a "generalised modular relation".

For the sake of simplicity, we assume that no $\xi_k(s)$ has a pole on the line $\mathrm{Re}(s) = \kappa/2$. Let $u_1, \ldots, u_p$ be the poles of $\xi_k(s)$. Then we may rewrite (10) as

$$\sum_{n\neq 0} a_n^{(k)} F(|n|/A) + \sum_{\mathrm{Re}\, u_j > \kappa/2} \operatorname*{Res}_{s=u_j} \frac{(M(\mathbf{W}F))(s)}{G(s)} \xi_k(\kappa - s) \tag{12}$$

$$= \sum_{1\leq l\leq N} c_{kl} \sum_{n\neq 0} b_n^{(l)} (\mathbf{W}F)(|n|/A)\Big) - \sum_{\mathrm{Re}\, u_j < \kappa/2} \operatorname*{Res}_{s=u_j} \frac{(M(\mathbf{W}F))(s)}{G(s)} \xi_k(k-s).$$

The second term on the left hand side of (12) is the same as

$$- \sum_{\mathrm{Re}\, u_j > \kappa/2} \operatorname*{Res}_{s=\kappa-u_j} \frac{MF(s)}{G(s)} \xi_k(s) = - \sum_{\mathrm{Re}\, u_j < \kappa/2} \operatorname*{Res}_{s=u_j} \frac{(MF)(s)}{G(s)} \xi_k(s)$$

while the one on its right hand side is equal to

$$- \sum_{1\leq l\leq N} c_{kl} \sum_{\mathrm{Re}\, u_j < \kappa/2} \operatorname*{Res}_{s=u_j} \frac{(M(\mathbf{W}F))(s)}{G(s)} \eta_l^*(s)$$

Thus we have a Poisson formula of Hecke type given by the following *Theorem* 2. *For any F whose Mellin transform MF is such that $(MF)(s)/G(s)$ is entire and $P(s)\, MF(s)$ is bounded in vertical strips for every polynomial P and for $\xi_1(s), \ldots, \xi_N(s)$, $\eta_1^*(s), \ldots, \eta_N^*(s)$ satisfying functional equations* (8), we have

$$\sum_{n\neq 0} a_n^{(k)} F(|n|/A) - \sum_{\mathrm{Re}\, u_j < \kappa/2} \operatorname*{Res}_{s=u_j} \frac{MF(s)}{G(s)} \xi_k(s).$$

$$= \sum_{1\leq l\leq N} c_{kl} \Big(\sum_{n\neq 0} b_n^{(l)} (\mathbf{W}F)(|n|/A)$$

$$- \sum_{\mathrm{Re}\, u_j < \kappa/2} \operatorname*{Res}_{s=u_j} \frac{(M(\mathbf{W}F))(s)}{G(s)} \eta_l^*(s) \Big). \tag{13}$$

4. Known cases of Poisson formulae above

Formula (13) generalises some well-known relations of a similar nature due to Hecke, Maass and Yamazaki and may be called a "generalized modular relation".

(1) First let $a = 1$, $N = 1$, $\mu_{k,1} = k$ for $k \geqslant 0$, $G(s) = \Gamma(s)$,

$$\xi_1(s) = \xi(s) = \gamma \xi(\kappa - s) = \eta_1^*(s), \quad c_{11} = \gamma, \quad \varphi_1(s) = \varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

$A = \lambda/2\pi$; let $s = 0$, κ be the only poles (and indeed of order 1) of $\xi(s)$. Then

$$\operatorname*{Res}_{s=0} \frac{MF(s)}{G(s)} \xi_1(s) = b_{0,1,1} \operatorname*{Res}_{s=0} \xi(s) \qquad \text{by Theorem 1}$$

$$= a_{0,1,1} \operatorname*{Res}_{s=0} \xi(s) \qquad \text{by Theorem 1}$$

$$= F(0) \times (-\gamma \operatorname*{Res}_{s=\kappa} A^s \Gamma(s) \varphi(s))$$

$$= -\gamma F(0) A^\kappa \Gamma(\kappa) \operatorname*{Res}_{s=\kappa} \varphi(s)$$

$$= -CF(0), \text{ say.}$$

Replacing F by $\mathbf{W}F$ in the argument above, we obtain

$$\operatorname*{Res}_{s=0} \frac{(M(\mathbf{W}F))(s)}{G(s)} \eta_1^*(s) = \operatorname*{Res}_{s=0} \frac{(M(\mathbf{W}F))(s)}{G(s)} \xi(s)$$

$$= -C(WF)(0).$$

Thus we get from (13) the Poisson formula due to Yamazaki [19] in the form derived by Igusa [10]: namely, for every F in $\mathscr{F}$ (with $G(s) = \Gamma(s)$)

$$CF(0) + \sum_{n\geqslant 1} a_n F(2\pi n/\lambda) = \gamma\,(C(WF)(0) + \sum_{n\geqslant 1} a_n (\mathbf{W}F)(2\pi n/\lambda)). \qquad (14)$$

(2) Let us take the same situation in (1) above and in particular,

$$\varphi(s) = \sum_{1\leqslant n<\infty} \tau(n)\, n^{-s}$$

where $\tau(n)$ is Ramanujan's function; then $\kappa = 12$, $A = 1/2\pi$, $\xi(s)$ is entire and for every F in the space $\mathscr{F}$, we have

$$\sum_{1\leqslant n<\infty} \tau(n)\, F(2\pi n) = \sum_{1\leqslant n<\infty} \tau(n)\, (\mathbf{W}F)(2\pi n). \qquad (15)$$

For $F(x) = \exp(-xy/2\pi)$ with fixed $y > 0$, we obtain in view of (11) and (15), formula (4) of [4]. On the other hand, if we take $F(x) = \exp(-s\sqrt{x})/\sqrt{x}$ for fixed $s > 0$, then we obtain *formally* from (15), formula (7) of [4]. One has to note that in the situation of (1) above, we can define for any F in $\mathscr{F}$, the function $\mathbf{W}F$ also by

$$(\mathbf{W}F)(x) = \int_0^\infty (xt)^{-\nu/2} J_\nu(2\sqrt{xt})\, F(t)\, t^\nu\, dt, \qquad (16)$$

with $\nu = \kappa - 1$. (See [10]), where J_ν is the usual Bessel function of order ν. However, the function $x \mapsto [\exp(-s\sqrt{x})]/\sqrt{x}$ is *not* in the space $\mathscr{F}$ above. If we approximate to it by a sequence $\{F_n\}$ from $C_c^\infty(\mathbf{R}_+^\times)$ (and indeed therefore with F_n in $\mathscr{F}$) and note that $\{MF_n\}$ converges to the (usual) Mellin transform MF

of F while $\{WF_n\}$ converges to WF (defined in the same way as in (16)), the validity of formula (7) of [4] can be deduced from above, in view of the two sides of this formula being absolutely convergent series. Again, for *integral* $\rho > 0$, formula (5) of [4] can be derived from (15), by taking for F, the function in $\mathscr{F}$ given by

$$F(t) = \begin{cases} \dfrac{1}{\Gamma(\rho+1)}(1-tx)^{\rho} & \text{for } \ 0 < t \leqslant x^{-1} \\ 0 & \text{for } \ t > x^{-1}, \end{cases}$$

with *fixed* $x > 0$. On the other hand, for getting the same formula for non-integral $\rho > 0$, a more refined argument as above has to be used; the case $\rho = 0$ also needs a more careful argument.

(3) When $\varphi(s)$ is the Dedekind zeta function associated with an imaginary quadratic field of discriminant d over the field of rational numbers and $A = \sqrt{|d|}/2\pi$ in (1) above, then we get the Poisson formula due to Hecke [8].

(4) The relation in Kubota [11] corresponding to our relation (9) above is, in his own notation,

$$M\Psi_b(s)\, Z_{a,b}(s) = M\tilde{\Psi}_{-b}(2n-2-s)\; Z_{-a,-b}(2n-2-s),$$

and proceeding exactly as above, one can derive his Poisson formula

$$\sum_{(m)} c'_m\, \Phi_b(m^{1/n}) = \sum_{(m)} c'_m\, \Phi^*_b(m^{1/n})$$

and also his relation

$$-AM(\Psi_0, 2n-2) + \sum_{(m)} c'_m\, \Phi_0(m^{1/n})$$

$$= -AM(\tilde{\Psi}_0, 2n-2) + \sum_{(m)} c'_m\, \Phi^*_0(m^{1/n})$$

(see pages 187–188 in [11]).

(5) Let now $a = 2$, $\mu_{k,1} = 2k + ir$, $\mu_{k,2} = 2k - ir$ for $k = 0, 1, 2, \ldots$ and a *fixed* $r \geqslant 0$, $m_k = 1$ or 2 according as $r \neq 0$ or $r = 0$,

$$A = \lambda/\pi,\ \varphi_k(s) = \psi^*_k(s) = \sum_{n\neq 0} a_n^{(k)}\, |n|^{-s}\ (1 \leqslant k \leqslant N) \ \text{with} \ a_n^{(k)} = 0$$

unless $n \equiv b_k \pmod q$ for fixed integers $q \geqslant 1$ and

$$b_1, \ldots, b_N,\ \kappa = 1,\ G(s) = \Gamma\left(\frac{s-ir}{2}\right)\Gamma\left(\frac{s+ir}{2}\right)$$

and $\quad \xi_k(s) = \eta^*_k(s) = A^s\, G(s)\, \varphi_k(s)$ for $1 \leqslant k \leqslant N$.

$$\text{Let} \quad \varphi_k(s) - \begin{cases} \dfrac{\alpha_k}{s-1-ir} + \dfrac{\beta_k}{s-1+ir} & (r \neq 0) \\[2mm] \dfrac{\alpha_k}{s-1} + \dfrac{\beta_k}{(s-1)^2} & (r = 0) \end{cases}$$

be entire (and of finite genus) for $1 \leqslant k \leqslant N$. *Equivalently* ([13]), if now C denotes Euler's constant and

$$M_0 = \frac{\sqrt{\lambda}}{4} \Gamma(\tfrac{1}{2} + ir)(\lambda/\pi)^{\frac{1}{2}+ir},$$

Let
$$\xi_k(s) - \begin{cases} \dfrac{4M_0 a_k}{s-1-ir} + \dfrac{4\bar{M}_0 \beta_k}{s-1+ir} - \dfrac{4M_0\rho_k}{s+ir} - \dfrac{4\bar{M}_0\sigma_k}{s-ir} \; (r \neq 0) \\ \dfrac{4M_0}{s-1}(a_k + \beta_k(\log(\lambda/4\pi) - C)) + \dfrac{4M_0\beta_k}{(s-1)^2} - \dfrac{4M_0}{s} \times \\ \times (\rho_k + \sigma_k)(\log(\lambda/4\pi) - C)) + \dfrac{4M_0\sigma_k}{s_2} \; (r = 0) \end{cases}$$

be entire in s (and of finite genus) with

$$\rho_k = \sum_{1 \leqslant l \leqslant N} c_{kl} a_l$$

and
$$\sigma_k = \sum_{1 \leqslant l \leqslant N} c_{kl}\beta_l, \text{ for } 1 \leqslant k \leqslant N.$$

Then for F in the corresponding $\mathscr{F}$-space, we have as t tends to 0, the asymptotic expansion

$$F(t) \approx \begin{cases} a_{0,1,1}t^{ir} + a_{0,2,1}t^{-ir} + \sum\limits_{\leqslant k < \infty} (a_{k,1,1}\, t^{2k+ir} + a_{k,2,1}t^{2k-ir}) & (r \neq 0) \\ a_{0,1,1} + a_{0,1,2}\log t + \sum\limits_{1 \leqslant k < \infty} (a_{k,1,1} + a_{k,1,2}\log t)\, t^{2k} & (r = 0). \end{cases}$$

By Theorem 1 above,

$$(MF)(s) - \begin{cases} \dfrac{a_{0,1,1}}{s+ir} + \dfrac{a_{0,2,1}}{s-ir} & (r \neq 0) \\ \dfrac{a_{0,1,1}}{s} - \dfrac{a_{0,1,2}}{s^2} & (r = 0). \end{cases} \quad \text{is regular at } s = \pm ir. \tag{17}$$

If $r \neq 0$, the only poles of $\xi_k(s)$ to the left of the line $\operatorname{Re} s = \frac{1}{2}$ are at $s = \pm ir$ and of order 1. Then

$$-\operatorname*{Res}_{s=ir} \frac{(MF)(s)}{G(s)} \xi_k(s) = - \left. \frac{(MF)(s)}{\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right)} \right|_{s=ir} \times \operatorname*{Res}_{s=ir} \xi_k(s)$$

$$= 2\bar{M}_0 a_{0,2,1}(F)\sigma_k/\Gamma(ir),$$

writing $a_{0,j,m}(F)$ instead of $a_{s,j,m}$ to emphasise the dependence on F. Similarly we have

$$-\operatorname*{Res}_{s=-ir} \frac{(MF)(s)}{G(s)} \xi_k(s) = 2M_0 a_{0,1,1}(F)\rho_k/\Gamma(-ir).$$

In order to find

$$-\operatorname*{Res}_{s=\pm ir} \frac{(M(\mathbf{W}F))(s)}{G(s)} \eta_l^*(s),$$

we have only to work with $\mathbf{W}F$ instead of F in the arguments above.

If $r=0$, the only pole of $\xi_k(s)$ to the left of the line $\operatorname{Re} s = 1$ is at $s=0$ and indeed it is of order 2. Using (17) and the expansion $1/\Gamma^2(s/2) = (s^2/4)(1 + Cs + \ldots)$ at $s=0$, we obtain

$$\operatorname*{Res}_{s=0} \frac{(MF)(s)}{\Gamma^2(s/2)} \xi_k(s) = a_{0,1,1}(F) M_0 \sigma_k + a_{0,1,2}(F) M_0 (\rho_k + \sigma_k (\log(\lambda/4\pi) - 2C)).$$

The residue of $(M(\mathbf{W}F))(s)\, \xi_k(s)/\Gamma^2(s/2)$ at $s=0$ is obtained by arguing with $\mathbf{W}F$ in lieu of F above. Thus, from (13), we have, for any F in the space $\mathscr{F}$, the Poisson formula

$$\sum_{n\neq 0} a_n^{(k)} F(|n|/A) + \begin{cases} 2M_0\rho_k a_{0,1,1}(F)/\Gamma(-ir) + 2\bar{M}_0 \sigma_k a_{0,2,1}(F)/\Gamma(ir) \\ -M_0\sigma_k a_{0,1,1}(F) - M_0(\rho_k + \sigma_k(\log(\lambda/4\pi) - 2C))\, a_{0,1,2}(F) \end{cases} \tag{18}$$

$$= \sum_{1\leqslant l\leqslant N} c_{kl} \Bigg[\sum_{n\neq 0} a_n^{(l)} (\mathbf{W}F)(|n|/A)$$

$$+ \begin{cases} 2M_0\rho_l a_{0,1,1}(\mathbf{W}F)/\Gamma(-ir) + 2\bar{M}_0\sigma_l a_{0,2,1}(\mathbf{W}F)/\Gamma(ir) \\ -M_0\sigma_l a_{0,1,1}(\mathbf{W}F) - M_0(\rho_l + \sigma_l(\log(\lambda/4\pi) - 2C))\, a_{0,1,2}(\mathbf{W}F) \end{cases} \Bigg]$$

according as $r\neq 0$ or $r=0$. The roles of $F(0)$ and $(WF)(0)$ in (14) are played now by the coefficients $a_{0,j,m}(F)$ and $a_{0,j,m}(\mathbf{W}F)$ in their asymptotic expansions. If we take $F(t) = 4\sqrt{y}\, K_{ir}(2ty)$ for $t>0$ with a fixed $y>0$, then

$$a_{0,1,1}(F) = 2y^{\frac{1}{2}+ir}\Gamma(-ir),\quad a_{0,2,1}(F) = 2y^{\frac{1}{2}-ir}\Gamma(ir), \qquad (r\neq 0)$$

$$a_{0,1,1}(F) = -4\sqrt{y}(C + \log y),\quad a_{0,1,2}(F) = -4\sqrt{y}, \qquad (r=0)$$

$$MF(s) = y^{\frac{1}{2}-s}\, \Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right).$$

Similar formulae for $\mathbf{W}F$ are valid; we have only to replace y by $1/y$ and F by $\mathbf{W}F$ in the formulae above. With this specialisation, formula (18) is the same as the "automorphic" relation

$$F_k^*(1/y) = \sum_{1\leqslant i\leqslant N} c_{kl} F_i^*(y)$$

of Maass ([13], p. 152).

(6) If $k = \mathbf{Q}(\sqrt{d})$ is a real quadratic field of discriminant d over the field $\mathbf{Q}$ of rational numbers, then the Dedekind zeta function $\zeta_k(s)$ satisfies the functional equation $(\pi/\sqrt{d})^{-s}\,\Gamma^2(s/2)\,\zeta_k(s) = \xi(s) = \xi(1-s)$. The function $\zeta_k(s)$ has a pole only at $s = 1$ and the residue at this (simple) pole is $2h$ $(\log \epsilon)/\sqrt{d}$ where h is the class number of K and ϵ is the fundamental unit in K. Moreover, $\zeta_k(0) = 0$ and $\zeta_k'(0) = -h(\log \epsilon)/2$. If we take

$$\varphi(s) = (\pi/\sqrt{d})^{-2s}\,\zeta_k(2s) = \sum_{n\geqslant 1} a(n)\,\lambda_n^{-s}$$

with $\lambda_n = \pi^2 n^2/d$, then $\varphi(s)\,\Gamma^2(s) = \varphi(1/2 - s)\,\Gamma^2(1/2 - s)$. From Theorem 1 of Berndt [2], we get (on noting that his $E_2(y)$ is just $2K_0(2\sqrt{y})$) the following Poisson formula,

$$2\sum_{1\leqslant n<\infty} a(n)\,K_0(2\pi ny/d) = 2y^{-1}\sum_{1\leqslant n<\infty} a(n)\,K_0(2\pi\, n/(y\sqrt{d})) + P(y^2)$$

where K_0 is the usual Bessel function. Here $P(y^2)$ = sum of the residues of $y^{-2s}\,\Gamma^2(s)\,\varphi(s)$ is seen to be equal to h $(\log \epsilon)/y + 2\,\zeta_k'(0) = h\,(\log \epsilon)\,(y^{-1} - 1)$. Now this formula reduces to a special case of the Poisson formula of Maass mentioned in (5) above.

5. A Poisson formula associated with a generalised Γ-function

In this section, we derive a Poisson formula for a situation involving Dirichlet series with functional equations containing a generalised Γ-function $\Gamma(s;\alpha,\beta)$ introduced by Maass ([14], [15]). This function $\Gamma(s;\alpha;\beta)$ is not, in general, a product of usual Γ-functions although, however, $\Gamma(s;\alpha,\beta)/(\Gamma(s)\,\Gamma(s+1-\alpha-\beta))$ is an entire function of s (with finite genus). The question is one of defining correctly the W-operator geared to the Poisson formula in this case; one has been guided here by the definition of a "Hankel transform" through the two-component Mellin transform for L^2-functions on $\mathbf{R}\setminus\{0\}$ defined in ([16], Theorem 10) (see also [17]).

First we recall the definition of the Whittaker functions $W_{l,m}(y)$ for $l, m \in \mathbf{C}$ and $y > 0$ as the "unique" solution $W(y)$ of the differential equation

$$4y^2\frac{d^2W}{dy^2} + (1 - 4m^2 + 4ly - y^2)\,W(y) = 0,$$

with the asymptotic behaviour

$$W(y) \sim \exp(-y/2)\,y^l\left\{1 + \sum_{1\leqslant n<\infty}\frac{1}{y^n\,n!}\prod_{r=1}^{n}(m^2 - (l + \tfrac{1}{2} - r)^2)\right\}$$

s y tends to ∞. Let

$$W(y;\alpha,\beta) = y^{-(\alpha+\beta)/2}\,W_{(\alpha-\beta)/2,\,(\alpha+\beta-1)/2}(2y),$$

for $y > 0$ and

$$\Gamma(s;\ \alpha, \beta) = \int_0^\infty W(y; \alpha, \beta) y^{s-1}\, dy,$$

for Re $s > K_0 = \max\ (0,\ |\text{Re}\ (\alpha+\beta)| - 1)$. Then $\Gamma(s; \alpha, \beta)$ is regular for Re $s > K_0$, noting that $W(y; \alpha, \beta) = O(y^{-K})$ for $K > K_0$ as y tends to 0 and further it has a meromorphic continuation to the entire s-plane satisfying the condition

$$\Gamma(s; \alpha, \beta) = 2^{(\alpha-\beta)/2} \frac{\Gamma(s)\,\Gamma(s+1-\alpha-\beta)}{\Gamma(s+1-\alpha)} F(\beta, 1-\alpha, s+1-\alpha; \tfrac{1}{2})$$

where $$F(a, b, c; z) = 1 + \sum_{1 \leqslant n < \infty} \frac{a(a+1)\ldots(a+n-1)\, b(b+1)\ldots(b+n-1)}{c(c+1)\ldots(c+n-1)\,.\,1\,.\,2\ldots n} z^n$$

is the hypergeometric function. It is also known that for every polynomial P $P(s)\,\Gamma(s; \alpha, \beta)$ is bounded in vertical strips in the s-plane. If we set

$$M(s; \alpha, \beta) = \begin{pmatrix} \Gamma(s; \alpha, \beta) & \Gamma(s; \beta, \alpha) \\ \Gamma(s+1;\ \alpha, \beta) & -\Gamma(s+1;\ \beta, \alpha) \end{pmatrix}$$

then we know from Maass [15] that its determinant $D(s)$ is $-2\,\Gamma(s) \times \Gamma(s+1-\alpha-\beta)$; further, the entries of the inverse matrix are entire functions of s. Let us write q for $\alpha+\beta$ in the sequel and assume that $q \neq 1$.

Before we go on to the Poisson formula, let us introduce the Dirichlet series whose functional equations involve the function $\Gamma(s; \alpha, \beta)$.

Let $$\varphi(s) = \sum_{1 \leqslant n < \infty} a_n n^{-s} \text{ and } \psi(s) = \sum_{-\infty < n < 0} a_n |n|^{-s},$$

be two Dirichlet series converging absolutely in some s-half plane and let, for some $\lambda > 0$,

$$\begin{aligned} &\xi(s) = (\lambda/2\pi)^s\,(\Gamma(s; \alpha, \beta)\,\varphi(s) + \Gamma(s; \beta, \alpha)\,\psi(s)), \\ &\eta(s) + \lambda((\alpha-\beta)/4\pi)\,\xi(s) = (\lambda/2\pi)^{s+1}\,(\Gamma(s+1; \alpha, \beta)\,\varphi(s) - \\ &\qquad - \Gamma(s+1;\ \beta, \alpha)\,\psi(s)) \end{aligned} \tag{19}$$

satisfy the functional equations

$$\xi(q-s) = \gamma\xi(s), \eta(q-s) = -\gamma\,\eta(s), \tag{20}$$

with fixed $\gamma = \pm 1$. Further, let us assume that

$$\xi(s),\ \eta_1(s) = \eta(s) + \lambda((\alpha-\beta)/4\pi)\,\xi(s)$$

have poles at most at $s = 0, 1, q-1, q$ with principal parts as given by the conditions:

$$\xi(s) - \frac{a_0}{s(s+1-q)} - \frac{\gamma a_0}{(q-s)(1-s)} + \frac{b_0}{s} + \frac{b_0}{q-s}$$

is entire, and

$$\eta_1(s) - \lambda \frac{a-\beta}{2\pi}\left(\frac{b_0}{s-q} + \gamma \frac{a_0}{(s-q)(s-1)} \right) \tag{21}$$

is entire with suitable constants a_0, b_0. The relations (19) can be inverted to read

$$\varphi(s) = -\frac{\Gamma(s+1;\beta,a)}{D(s)}(2\pi/\lambda)^s \xi(s) - \frac{\Gamma(s;\beta,a)}{D(s)}(2\pi/\lambda)^{s+1}\eta_1(s)$$

$$\psi(s) = -\frac{\Gamma(s+1;\ a,\beta)}{D(s)}(2\pi/\lambda)^s \xi(s) + \frac{\Gamma(s;a,\beta)}{D(s)}(2\pi/\lambda)^{s+1}\eta_1(s). \tag{19$'$}$$

The only poles of φ and ψ are at $s = 1, q$. Now functional equations (20) go over into

$$(\lambda/2\pi)^s M(s;a,\beta)\begin{pmatrix}\varphi(s)\\ \psi(s)\end{pmatrix} = (\lambda/2\pi)^{q-s}\begin{pmatrix}1 & 0\\ a- & \beta-1\end{pmatrix} M(q-s;a,\beta)$$

$$\begin{pmatrix}\varphi(q-s)\\ \psi(q-s)\end{pmatrix}. \tag{20$'$}$$

From (19)′ and (21), we obtain

$$\varphi(0) = \frac{1-q}{2} w(1;\beta,a)\left(\frac{a_0}{1-q} - b_0\right) + \frac{\pi}{\lambda} w(0;\beta,a)\,\eta_1(0),$$

$$\varphi(q-1) = -\left(\frac{2\pi}{\lambda}\right)^{q-1} w(q;\beta,a)\frac{a_0}{2} + \tfrac{1}{2}\left(\frac{2\pi}{\lambda}\right)^q w(q-1;\ \beta,a)\,\eta_1(q-1),$$

$$\psi(0) = \frac{1-q}{2} w(1;a,\beta)\left(\frac{a_0}{1-q} - b_0\right) - \frac{\pi}{\lambda} w(0;a,\beta)\,\eta_1(0) \times$$

$$\psi(q-1) = -\left(\frac{2\pi}{\lambda}\right)^{q-1} w(q;\ a,\beta)\frac{a_0}{2} - \tfrac{1}{2}\left(\frac{2\pi}{\lambda}\right)^q w(q-1;\ a,\beta)$$

$$\times\ \eta_1(q-1) \tag{22}$$

where $w(s;a,\beta) = -2\,\Gamma(s;a,\beta)/D(s)$ and $w(s;\beta,a)$ is defined similarly.

Let us now consider the space $\mathcal{F}$ of C^∞ functions F on $\mathbf{R}_+^\times$ which behave like Schwartz functions at infinity and which have the termwise differentiable asymptotic expansion

$$F(x) \approx \sum_{0\leqslant n<\infty} a_n(F)\,x^n + x^{1-q}\sum_{0\leqslant n<\infty} \beta_n(F)\,x^n,$$

as x tends to 0. From Theorem 1, we, know that $MF(s)/(\Gamma(s)\Gamma(s+1-q))$ is entire and further, for every polynomial P in s, $P(s)\ MF(s)$ is bounded in vertical strips. We assume that $0 < \mathrm{Re}\ q < 1$ in the sequel.

Let F_1, F_2 in $\mathcal{F}$ above satisfy the conditions:

(i) for every $\epsilon > 0$, $MF_j(s) = O(\exp(-\epsilon\,|\,t\,|))$ as $s = \sigma + ti$ tends to infinity in vertical strips B_{σ_1,σ_2}, for $j = 1, 2$, and (23)

(ii) $(MF_1(s)\ MF_2(s))\ M(s;a,\beta)^{-1} = (G_1(s)\ G_2(s))$ with entire G_1, G_2. For such a pair F_1, F_2, we define $\mathbf{W}F_1, \mathbf{W}F_2$ through the functional equation

$$((M(\mathbf{W}F_1))(s)\, M((\mathbf{W}F_2))(s))\, M(s;\alpha,\beta)^{-1}$$

$$= (G_1(q-s)\, G_2(q-s)\begin{pmatrix} 1 & 0 \\ \alpha- & \beta-1 \end{pmatrix}. \tag{24}$$

From (23) and (24), it is clear that the left hand side of (24) is entire. Further for every polynomial P in s, $P(s)\,(M(\mathbf{W}F_j))(s)$ is bounded in vertical strips, for $= 1, 2$ (condition (i) in (23) has been imposed in order to ensure this and if perhaps

$$M(q-s;\alpha,\beta)^{-1}\begin{pmatrix} 1 & 0 \\ \alpha- & \beta & -1 \end{pmatrix} M(s;\alpha,\beta)$$

consists of functions bounded at infinity in vertical strips, then condition (i) can be waived). Thus, as x tends to $0, \mathbf{W}F_j$ has the termwise differentiable asymptotic expansion

$$(\mathbf{W}F_j)(x) \approx \sum_{0\leqslant n<\infty} \alpha_n(WF_j)\, x^n + x^{1-q} \sum_{0\leqslant n<\infty} \beta_n(WF_j)\, x^n.$$

Replacing s by $q-s$ in (20)′ and then multiplying out relations (20)′ and (24) we obtain

$$\lambda/(2\pi)^{q-s}\,((M(\mathbf{W}F_1))(q-s)\,\varphi(q-s) + (M(\mathbf{W}F_2))(q-s)\,\psi(q-s))$$

$$= \gamma\,(\lambda/2\pi)^s\,(MF_1(s)\,\varphi(s) + MF_2(s)\psi(s)). \tag{25}$$

In order to get the Poisson formula, we proceed as in the proof of Theorem 2 In view of the absolute convergence of φ, ψ for sufficiently large Re s, we see that the integral of the right hand side of (25) from $\sigma - \infty i$ to $\sigma + \infty i$ for σ large enough, is simply

$$2\pi i\gamma\left(\sum_{n>0} a_n F_1(2\pi n/\lambda)\right) + \sum_{n<0}' a_n F_2(2\pi \mid n \mid/\lambda),$$

which, by (25), is therefore equal to

$$\int_{\sigma-\infty i}^{\sigma+\infty i} (\lambda/2\pi)^{q-s}\,((M(\mathbf{W}F_1))(q-s)\,\varphi(q-s) + (M(\mathbf{W}F_2))(q-s)\psi(q-s))\, ds$$

$$= \int_{q-\sigma-\infty i}^{q-\sigma+\infty i} (\lambda/2\pi)^s\,[(M(\mathbf{W}F_1))(s)\,\varphi(s) + (M(\mathbf{W}F_2)(s)\psi(s))]\, ds$$

$$= \int_{\sigma_1-\infty i}^{\sigma_1+\infty i} (\lambda/2\pi)^s\,(M(\mathbf{W}F_1))(s)\,\varphi(s) + (M(\mathbf{W}F_2)(s)\psi(s))\, ds - 2\pi i\, S^*,$$

for σ_1 large enough, with S denoting the sum of the residues of the integrand at all the poles encountered when the line of integration is shifted from

Re $(s) = q - \sigma$ to Re $(s) = \sigma_1$ far to the right. Because of the absolute convergence of the Dirichlet series $\varphi(s), \psi(s)$ again, we obtain the relation

$$\Big(\sum_{n>0} a_n F_1(2\pi n/\lambda) + \sum_{n<0} a_n F_2(2\pi\,|\,n\,|/\lambda)\Big) = \sum_{n>0} a_n (WF_1)(2\pi n/\lambda)$$

$$+ \sum_{n<0} a_n (WF_2)(2\pi\,|\,n\,|/\lambda) - S^*.$$

This can be rewritten as in the proof of Theorem 2 as

$$\sum_{n>0} a_n F_1(2\pi n/\lambda) + \sum_{n<0} a_n F_2(2\pi\,|\,n\,|/\lambda) - \sum_{\operatorname{Re} u_j < a/2} \operatorname{Res}\,((\lambda/2\pi)^s MF_1(s)\,\varphi(s)$$

$$+ MF_2(s)\,\varphi(s))$$

$$= \gamma \sum_{n>0} a_n (WF_1)(2\pi n/\lambda) + \gamma \sum_{n<0} a_n (WF_2)(2\pi\,|\,n\,|/\lambda) - \sum_{\operatorname{Re} u_j < a/2} \operatorname{Res}\,((\lambda/2\pi)^s \times$$

$$\times ((M(WF_1))(s)\,\varphi(s) + (M(WF_2))(s)\,\psi(s)), \tag{26}$$

where the third summations on both sides are over the residues at all the poles u_j satisfying the condition stated. In view of our assumption that $0 < \operatorname{Re} q < 1$, $s = 0$, $q - 1$ are the only poles involved in these summations. The residue of $(\lambda/2\pi)^s\,(MF_1(s)\,\varphi(s) + MF_2(s)\psi(s))$ at 0 is seen to be equal to

$$((1-q)/2)\;(a_0/(1-q) - b_0)\;(a_0(F_1)\,w(1, \beta, a) + a_0(F_2)\,w(1, a, \beta)) -$$

$$-(\pi/\lambda)\;\{w(0, \beta, a)\,a_0(F_1) - w(0, a, \beta)\,a_0(F_2)\}\,\eta_1(0),$$

in view of (22). The expression inside the curly brackets is 0, since it is essentially the residue at $s = 0$ of the right hand side of (25) while the left hand side of (25) is regular at $s = 0$. Similarly, the residue at $q - 1$ of

$$(\lambda/2\pi)^s\,(MF_1(s)\,\varphi(s) + MF_2(s)\psi(s))$$

is seen to be just

$$-\,a_0\,(\beta_0(F_1)\,w(q; \beta, a) + \beta_0(F_2)\,w(q; a, \beta))/2.$$

The residues involved in the third summation on the right hand side of (25) are computed just as above, replacing F_i by WF_i everywhere. Thus formula (26) leads to

Theorem 3. *For C^∞ functions F_1, F_2 on $\mathbf{R}_+^\times$ behaving like Schwartz functions at infinity and satisfying* (23) *and Dirichlet series $\varphi(s)$, $\psi(s)$ for which the functional equations* (20) *hold, we have, for* $0 < \operatorname{Re} q < 1$, *the Poisson formula*

$$\sum_{n>0} a_n F_1(2\pi n/\lambda) + \sum_{n<0} a_n F_2(2\pi\,|\,n\,|/\lambda) - \tfrac{1}{2}\,a_0\,(a_0(F_1)\,w(1; \beta, a)$$

$$+ a_0(F_2)\,w(1; a, \beta) - \beta_0(F_1)\,w(q; \beta, a) - \beta_0(F_2)\,w(q; a, \beta)) -$$

$$-\,b_0\,(q-1)\,(a_0(F_1)\,w(1, \beta, a) + a_0(F_2)\,w(1; a, \beta,))$$

$$= \gamma \Big[\sum_{n>0} a_n (\mathbf{W}F_1)(2\pi n/\lambda) + \sum_{n<0} a_n (\mathbf{W}F_2)(2\pi \mid n \mid /\lambda) -$$

$$- \tfrac{1}{2} a_0 (a_0 (\mathbf{W}F_1) w(1; \beta, \alpha) + a_0 (\mathbf{W}F_2) w(1; \alpha, \beta)$$

$$- \beta_0 (\mathbf{W}F_1) w(q; \beta, \alpha) - \beta_0 (\mathbf{W}F_2) w(q; \alpha, \beta)$$

$$- \tfrac{1}{2} b_0 (q-1)(a_0(\mathbf{W}F_1) w(q; \beta, \alpha) + a_0 (\mathbf{W}F_2) w(1; \alpha, \beta)) \Big],$$

where $a_0(F_i)$, $\beta_0(F_i)$, *etc., are coefficients in the asymptotic expansions at* 0.

Note. The values of $w(0; \alpha, \beta)$, etc., are given explicitly as follows:

$$w(0; \alpha, \beta) = 2^{q/2}/\Gamma(1-\alpha), \; w(1; \alpha, \beta) = \Gamma(1; \alpha, \beta)/\Gamma(2-q),$$

$$w(0; \beta, \alpha) = 2^{q/2}/\Gamma(1-\beta), \; w(1; \beta, \alpha) = \Gamma(1; \beta, \alpha)/\Gamma(2-q),$$

$$w(q-1; \alpha, \beta) = 2^{1-q/2}/\Gamma(\beta), \; w(q; \alpha, \beta) = \Gamma(q; \alpha, \beta)/\Gamma(q),$$

$$w(q-1; \beta, \alpha) = 2^{1-q/2}/\Gamma(\alpha), \; w(q; \beta, \alpha) = \Gamma(q; \beta, \alpha)/\Gamma(q).$$

Let now $F_1(x) = W(xy; \alpha, \beta)$, $F_2(x) = W(xy; \beta, \alpha)$ for fixed $y > 0$. Then by (11), $(\mathbf{W}F_1)(x) = W(xy^{-1}; \alpha, \beta) y^{-q}$, $(\mathbf{W}F_2)(x) = W(xy^{-1}; \beta, \alpha) y^{-q}$. Further

$$a_0(F_1)\Gamma(1-\alpha) = a_0(F_2)\Gamma(1-\beta) = 2^{q/2}\Gamma(1-q),$$

$$\beta_0(F_1)\Gamma(\beta) = \beta_0(F_2)\Gamma(\alpha) = 2^{1-q/2} y^{1-q} \Gamma(q-1),$$

$$a_0(\mathbf{W}F_1)\Gamma(1-\alpha) = a_0(\mathbf{W}F_2)\Gamma(1-\beta) = 2^{q/2} y^{-q}\Gamma(1-q),$$

$$\beta_0(\mathbf{W}F_1)\Gamma(\beta) = \beta_0(\mathbf{W}F_2)\Gamma(\alpha) = 2^{1-q/2} y^{q-1} . y^{-q}\Gamma(q-1),$$

$$\Gamma(\alpha)\Gamma(q; \beta, \alpha) + \Gamma(\beta)\Gamma(q; \alpha, \beta)$$

$$= 2^{q/2}\Gamma(q)\{B_{1/2}(\beta, \alpha) + B_{1/2}(\alpha, \beta)\} = 2^{q/2}\Gamma(q) B_1(\alpha, \beta)$$

$$= 2^{q/2}\Gamma(\alpha)\Gamma(\beta) \text{ where } B_x(\alpha, \beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt,$$

and $\quad \Gamma(1-\beta)\Gamma(1; \beta, \alpha) + \Gamma(1-\alpha)\Gamma(1; \alpha, \beta) = 2^{1-q/2}\Gamma(1-\alpha)\Gamma(1-\beta)$

(See [1]). This gives us in particular, the first of the two formulae (6) of ([15], p. 230) with $0 < \mathrm{Re}(\alpha + \beta) < 1$. It seems likely from [16], that there are quite a few of pairs F_1, F_2 which satisfy the conditions of Theorem 3 and for which therefore a Poisson formula holds.

6. A p-adic analogue of the W-operator

Let $\mathscr{F} = \mathscr{F}(\mathbf{Q}_p^\times)$ be now the space of locally constant complex-valued functions F on $\mathbf{Q}_p^\times = \mathbf{Q}_p \setminus \{0\}$ with $F(x) = 0$ for all x with $\mid x \mid_p$ sufficiently large and $F(x) = a\mu_1(x) \mid x \mid_p^{\frac{1}{2}} + b\mu_2(x) \mid x \mid_p^{\frac{1}{2}}$ for all x with $\mid x \mid_p$ sufficiently small, where μ_1, μ_2 are quasicharacters of $\mathbf{Q}_p^\times$, $\mid x \mid_p$ is a 'normalised' valuation of $\mathbf{Q}_p$ and a, b are complex constants. Such spaces $\mathscr{F}$ occur as "Kirillov models" $\mathscr{K}(\pi)$ for irreducible admissible representations π_p of $GL_2(\mathbf{Q}_p)$; associated with π_p, we have the L-function $L(s,\pi_p) = \{(1-\mu_1(p)p^{-(s-\frac{1}{2})})(1-\mu_2(p)p^{-(s-\frac{1}{2})})\}^{-1}$ where $s \in \mathbf{C}$ and more generally,

$$L(s,\chi,\pi_p) = \{(1-(\mu_1\chi^{-1})(p)p^{\frac{1}{2}-s})(1-(\mu_2\chi^{-1})(p)p^{\frac{1}{2}-s})\}^{-1},$$

for any character χ of $\mathbf{Z}_p^\times = \mathbf{Z}_p \setminus p\mathbf{Z}_p$. If $(MF_\chi)(s) = \int_{\mathbf{Q}_p^\times} F(x)\chi(x)\,|x|_p^s\, d^\times x$ for any F in $\mathscr{F}$, then $(MF_\chi)(s)/L(s,\chi,\pi_p)$ is entire in s and in particular, for the identity character χ_0, $(MF_{\chi_0})(s)$ is in the space $\mathfrak{Z}$ defined by Igusa ([10] chapter I, § 5.2). There exists F_0 in $\mathscr{F}$ such that $(M(F_0)_\chi)(s) = L(s,\chi,\pi_p)$ (see [7], § 1.14–1.16) and for $\chi = \chi_0$, F_0 is given by

$$F_0(x) = \begin{cases} 0 \text{ if } x \notin p^{-1}Z_p \\ |x|_p^{\frac{1}{2}} \sum\limits_{i+j=v_p(x)} \mu_1(p^i)\mu_2(p^j) \text{ if } x \in p^{-1}\mathbf{Z}_p \text{ and } x \neq 0, \end{cases}$$

where $v_p(x)$ is defined by $|x|_p = |p|_p^{v_p(x)}$ for $x \neq 0$. This relation can also be proved by using Theorem 5.3 of Igusa ([10], chapter I) taking $\Lambda = \{s_1 + \frac{1}{2}, s_2 + \frac{1}{2}\}$ where $\mu_i(x) = |x|_p^{s_i}$, $i = 1, 2$.

Corresponding to F in $\mathcal{G}$, let $\mathcal{W} = \mathcal{W}_F$ be the Whittaker function on $GL_2(\mathbf{Q}_p)$; then if

$$L_{\mathcal{W}}(g,\chi,s) = \int_{\mathbf{Q}_p^\times} \mathcal{W}\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right) \chi^{-1}(x)\,|x|_p^{2s-1}\, d^\times x,$$

or $g \in GL_2(\mathbf{Q}_p)$, we have the functional equation

$$L_{\mathcal{W}}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \chi^{-1}, 1-s\right) = \epsilon(s,\chi,\pi_p)\,\frac{L(1-s,\chi^{-1},\pi_p)}{L(s,\chi,\pi_p)} \times$$
$$\times L_{\mathcal{W}}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \chi, s\right) \tag{27}$$

where $\epsilon(s,\chi,\pi_p)$ is of the form $cp^{-\nu s}$ for suitable constants c, ν. By using the inverse M^{-1} of the Mellin transform M in Theorem 5.3 (Chapter I) of Igusa [10], the W-operator in the space $\mathcal{G}(\mathbf{Q}_p^\times)$ can be defined by rewriting the functional equation above as

$$\frac{(M(\mathrm{W}F)_{\chi^{-1}})(1-s)}{L(1-s,\chi^{-1},\pi_p)} = \epsilon(s,\chi,\pi_p)\,\frac{(MF_\chi)(s)}{L(s,\chi,\pi_p)},$$

(see also § 1.3, [7]). The function $\epsilon(s,\chi,\pi_p)$ satisfies of course the condition $\epsilon(s,\chi,\pi_p)\,\epsilon(1-s,\chi^{-1},\pi_p) = 1$ (see [7]) and therefore the **W**-operator is of order 2

7. An adelic version of the Poisson formula

Let $F_\infty \in \mathcal{G}(\mathbf{R}_+^\times)$ such that for its Mellin transform $(MF_\infty)(s)$, the quotient $(MF_\infty)(s)/L(s,\pi_\infty)$ is an entire function of s, where [5]

$$L(s, \pi_\infty) = \begin{cases} (2\pi)^{-s-(\;+1)/2}\, \Gamma(s + (p+1)/2), p \geqslant 0 \text{ in } \mathbf{Z} \\ \pi^{-s-\nu}\, \Gamma((s+\nu)/2)\, \Gamma((s-\nu)/2), \nu \in \mathbf{C}. \end{cases}$$

Define $\mathbf{W}F_\infty$ by

$$\frac{(M(\mathbf{W}F_\infty))(s)}{L(s, \pi_\infty)} = \frac{(MF_\infty)\ (1-s)}{L(1-s, \pi_\infty)}.$$

Let $\mathbf{Q}_A$ be the ring of $\mathbf{Q}$-adeles and $\mathbf{Q}_A^\times$, the group of $\mathbf{Q}$-ideles; denote elements x of $\mathbf{Q}_A^\times$ by $(x_\infty, \ldots, x_p, \ldots,)$ with $x_\infty \in \mathbf{R}\setminus\{0\}$ and $x_p \in \mathbf{Q}_p\setminus\{0\}$ and write $\mathbf{Q}^\times$ for $\mathbf{Q}\setminus\{0\}$. Let π_p be irreducible unitary representations of $GL_2(\mathbf{Q}_p)$ for primes p such that the tensor product $\pi = \pi_\infty \otimes_p \pi_p$ gives an irreducible unitary representation of $GL_2(\mathbf{Q}_A)$ and further

$$\prod_p L(s, \pi_p) = \sum_{n \in Z\setminus\{0\}} a_n\, |n|^{-s}$$

is a Dirichlet series converging in a right s-half-plane and for every $\chi \in \widehat{\prod_p \mathbf{Z}_p^\times}$,

$$L(s, \chi, \pi) := L(s, \pi_\infty) \prod_p L(s, \chi, \pi_p) = \prod_p \epsilon(s, \chi, \pi_p)\, L(1-s, \chi^{-1}, \pi). \tag{27}$$

In particular, the functional equation implies that $\prod_p L(s, \pi_p)$ is at most of order $|s|^r$ for some constant $r = r(B)$ in vertical strips B. On the other hand, $MF_\infty(s)$ and $L(s, \pi_\infty)$ are rapidly decreasing at infinity in vertical strips. Let W_p^0 be the Whittaker function [7] on $GL_2(\mathbf{Q}_p)$ whose Mellin transform (with respect to $\mathbf{Q}_p^\times$) is precisely $L(s, \pi_p)$ for every prime p. Define for $x = (x_\infty, \ldots, x_p, \ldots) = (x_\infty, x_f)$ in $\mathbf{Q}_A^\times$, the function F on $\mathbf{Q}_A^\times$ by

$$F(x) = F_\infty(|x_\infty|) \prod_p \mathbf{W}_p^0\left(\begin{pmatrix} x_p & 0 \\ 0 & 1 \end{pmatrix}\right) = F_\infty(|x_\infty|)\, F_f(x_f)$$

and the function $\mathbf{W}F$ on $\mathbf{Q}_A^\times$ by

$$(\mathbf{W}F)(x) = \mathbf{W}F_\infty(|x_\infty|) \prod_p W_p^0\left(\begin{pmatrix} x_p & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right).$$

From the properties of F, the series

$$\sum_{\xi \in \mathbf{Q}^\times} F(x\xi)$$

converges absolutely to a function $\varphi_F(x)$ and further $\varphi_F(x)$ is rapidly decreasing as $|x_\infty|$ tends to ∞ or 0 (see [5]). Thus for every character χ on $\mathbf{Q}^\times\setminus\mathbf{Q}_A^\times$ and for $\mathrm{Re}(s)$ sufficiently large,

$$L_F\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}, \chi, s\right) = \int_{\mathbf{Q}^\times\backslash\mathbf{Q}_A^\times} \varphi_F(x)\,\chi^{-1}(x)\,|x|^{2s-1}\,d^\times x$$

$$= \int_{\mathbf{Q}_A^\times} F(x)\,\chi^{-1}(x)\,|x|^{2s-1}\,d^\times x,$$

is holomorphic in s and admits of a meromorphic continuation to the whole plane; for $\chi \neq \chi_0$ (the identity) it represents an entire function of s. For large enough Re (s), it is just

$$(MF_\infty)(s)\prod_p L(s, \chi, \pi_p) = (MF_\infty)(s)\sum_{n\neq 0}\chi(n)\,a_n\,|n|^{-s};$$

in vertical strips the first factor is rapidly decreasing at infinity (by Theorem 1) while the second factor is at most of order $|s|^a$ for some a (in view of the functional equation and Stirling's formula). Thus $L_F\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}, \chi, s\right)$ is rapidly decreasing at infinity in vertical strips.

From the local functional equations (27) and (27)′, we get

$$L_{WF}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}, \chi^{-1}, 1-s\right) = L_F\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}, \chi, s\right). \tag{28}$$

Let us define for $x \in \mathbf{Q}_A^\times$.

$$F'(x) = \frac{1}{2\pi i}\sum_{\chi'}\int_{\mathrm{Re}\, s=\sigma} L_F\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}, \chi', s\right)|x|^{1-2s}\,ds$$

and $$F''(x) = \frac{1}{2\pi i}\sum_{\chi'}\int_{\mathrm{Re}\, s=\sigma_1} L_{WF}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}, \chi'^{-1}, 1-s\right)|x|^{1-2s}\,ds$$

where σ and $-\sigma_1$ are sufficiently large and χ' runs over characters of $\prod_p \mathbf{Z}_p^\times$.

Because of the properties of the integrand mentioned above and the functional equation (28), we obtain by the usual argument

$$F' = (x) - F''(x) = \text{Sum of the residues of } L_F\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}, \chi_0, s\right)|x|^{1-2s}$$

(at the poles encountered while shifting the integration from the far right to the far left, i.e. at all the poles).

Since $$L_F\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}, \chi_0, s\right) = \frac{MF_\infty(s)}{L(s,\pi_\infty)}\left(L(s,\pi_\infty)\sum_{n\neq 0} a_n\,|n|^{-s}\right),$$

the poles arise from the function inside the simple brackets; the residue can be computed and seen to be of the form c_0 or $c_1\mu_1(x_\infty) + c_2\mu_2(x_\infty)$ with constants c_0, c_1, c_2, depending on the nature of π_∞. The last relation involving $F'(x) - F''(x)$ may thus be viewed as the adelic formulation of the Poisson formula. When no poles are encountered, it reduces to an 'automorphic' relation $F'(x) = F''(x)$.

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