INSTABILITY OF THE SELF-SIMILAR FLOW INTO A CAVITY

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Received June 24, 1969

(Communicated by Prof. P. L. Bhatnagar, F.A.SC.)

Abstract

In this paper we have investigated the instability of the self-similar flow behind the boundary of a collapsing cavity. The similarity solutions for the flow into a cavity in a fluid obeying a gas law $p = K\rho^{\gamma}$, K = constant and $7 \geq \gamma > 1$ has been solved by Hunter, who finds that for the same value of γ there are two self-similar flows, one with accelerating cavity boundary and other with constant velocity cavity boundary. We find here that the first of these two flows is unstable. We arrive at this result only by studying the propagation of disturbances in the neighbourhood of the singular point.

1. Introduction

For self-similar flows the partial differential equations of fluid dynamics reduce to a system of ordinary differential equations in one independent variable ξ which is some combination of the spatial co-ordinate r and time t. Zel'dovich and Raizer (1967) have classified such flows into two classes: (i) self-similar flows of first kind where the dimensional constants, appearing in the initial and boundary conditions and the differential equations, uniquely determine the exponent δ in the similarity variable

$$\xi = \frac{r}{|t|^{\delta}}$$

and (ii) self-similar flows of second kind where δ is uniquely determined only from a necessary condition that the correct integral curve representing the flow must pass through a singular point of the ordinary differential equations. An example of a self-similar flow of second kind is the limiting flow into a cavity in a fluid when the radius of the cavity tends to zero. This problem has been discussed by Hunter (1963) and Zel'dovich and Raizer (1967). The integral curve of the problem passes through a singular point on the critical parabola.

On the assumption that the pressure-density relation is $p = K\rho^{\gamma}$ where K and γ are constants. Hunter finds that for a given value of $\gamma > 3/2$ there exists two solutions: (i) Similarity solutions with accelerating front, *i.e.*, with $0 < \delta < 1$ and (ii) Similarity solutions with constant velocity front *i.e.*, with $\delta = 1$. These two solutions satisfy the same initial and boundary conditions and hence represent two possible flows for the same problem. According to Hunter, it is possible that one of the types of similarity solutions is unstable whereas the other is not, so that only one of them occurs in nature. Birkhoff (1954) has shown that in the case of an incompressible fluid, radially symmetric similarity solution is unstable with respect to non-radially symmetric disturbances. In this paper we have shown that Hunter's similarity solutions with accelerating front are unstable with respect to radially symmetric disturbances in the neighbourhood of the singular point and hence they do not occur in nature. In this investigation we have followed Bhatnagar and Prasad (1969), who have shown that the investigations of Kulikovskii and Slobodkina (1967) for the propagation of disturbances in a steady flow in the neighbourhood of a sonic point can be extended to self-similar flows.

2. Basic Equations

On the assumption that the pressure p and density ρ are connected by gas equation of the type

$$p = K\rho^{\gamma} \tag{2.1}$$

where K and γ are constant, we can write momentum and continuity equations for a spherically symmetric flow in the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\gamma - 1}\frac{\partial}{\partial r}(a_s^2) = 0 \tag{2.2}$$

and

$$\frac{\partial}{\partial t}(a_s^2) + u\frac{\partial}{\partial r}(a_s^2) + (\gamma - 1)\, a_s^2\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) = 0. \tag{2.3}$$

Here u represents the particle velocity in the radial direction,

$$a_s = \sqrt{\frac{\gamma p}{\rho}}$$

the sound speed, t the time measured from the instant of collapse of the cavity and r the radial distance from the point of symmetry. In terms of the new dependent variables Z, V and new independent variables τ and η defined by

$$a_s^2 = \frac{r^2}{t^2} Z(\eta, \tau), \; u = \frac{r}{t} V(\eta, \tau) \tag{2.4}$$

and

$$\eta = ln \left\{\frac{r}{A(-t)^\delta}\right\}, \; \tau = \frac{1}{\delta} ln \{A(-t)^\delta\} \tag{2.5}$$

where A and δ are two constants, the equations (2.2) and (2.3) reduce to

$$\frac{\partial V}{\partial \tau} + (V - \delta)\frac{\partial V}{\partial \eta} + \frac{1}{(\gamma - 1)}\frac{\partial Z}{\partial \eta} + \left\{\frac{2Z}{\gamma - 1} + V(V - 1)\right\} = 0 \tag{2.6}$$

and

$$\frac{\partial Z}{\partial \tau} + (V - \delta)\frac{\partial Z}{\partial \eta} + (\gamma - 1) Z \frac{\partial V}{\partial \eta} + Z\{2(V - 1) + 3(\gamma - 1)V\} = 0 \tag{2.7}$$

The characteristic curves of (2.6) and (2.7) in (η, τ)-plane are

$$\frac{d\eta}{d\tau} = V - \delta \pm \sqrt{Z}. \tag{2.8}$$

If a suffix zero represents the value of the flow variables in a self-similar flow, then V_0 and Z_0 are functions of η only and they satisfy a system of ordinary differential equations. As discussed by Hunter the integral curve, representing the flow behind an accelerating cavity boundary, passes through the singular point (V_0^*, Z_0^*) given by

$$f(V_0^*) \equiv 2(\gamma - 1)(V_0^*)^2 + (5\delta + \gamma - 3 - 3\gamma\delta) V_0^* - 2\delta(\delta - 1) = 0 \tag{2.9}$$

and

$$\sqrt{Z_0^*} = (\delta - V_0^*). \tag{2.10}$$

The characteristic velocity

$$C_0 = V_0 - \delta - \sqrt{Z_0} \tag{2.11}$$

which vanishes at (V_0^*, Z_0^*) satisfies

$$\frac{dC_0}{d\eta} = \frac{2(V_0 - \delta - 1)}{\gamma - 1} - \frac{3\gamma - 1}{\gamma - 1} V_0 - \sqrt{Z_0}$$

$$+ \frac{\left\{V_0 - \delta - \frac{\gamma - 1}{2}\sqrt{Z_0}\right\} f(V_0)}{Z_0 - (\delta - V_0)^2} \tag{2.12}$$

We can also show that at the point (V_0^*, Z_0^*)

$$\left(\frac{dV_0}{d\eta}\right)^* = \frac{2}{\gamma - 1}\left[\left(\frac{dC_0}{d\eta}\right)^* - 1 - \frac{3\gamma - 1}{2} V_0^*\right]. \tag{2.13}$$

In the neighbourhood of the singular point (V_0^*, Z_0^*)

$$\frac{f(V_0)}{Z_0 - (\delta - V_0)^2} = \frac{f'(V_0^*)\left(\frac{dV_0}{d\eta}\right)^*}{\{\sqrt{Z_0^*} - (\delta - V_0^*)\}\left(\frac{dC_0}{d\eta}\right)^*} + 0\,[(\eta - \eta_0^*)]$$

$$\tag{2.14}$$

where η^* is the value of η when $V_0 = V_0^*$. Therefore, it follows from (2.12) that

$$\left(\frac{dC_0}{d\eta}\right)^*$$

is given by

$$\left\{\left(\frac{dC_0}{d\eta}\right)^*\right\}^2 - \alpha\left(\frac{dC_0}{d\eta}\right)^* - \beta = 0 \tag{2.15}$$

where

$$\alpha = \frac{(3 - 5\gamma)\,\delta - 1 - \gamma}{2(\gamma - 1)}, \tag{2.16}$$

and

$$\beta = -\{(3\gamma - 1) V_0^* - 2\}\left\{V_0^* - \frac{5\delta + \gamma - 3 - 3\gamma\delta}{4(\gamma - 1)}\right\}. \tag{2.17}$$

The propagation of a perturbation of the self-similar flow and the self-similar flow in the neighbourhood of the critial point are governed by the equations

$$\frac{\partial C}{\delta \tau} + C\frac{\partial C}{\partial \eta} = \alpha C + \beta(\eta - \eta^*) \tag{2.18}$$

and

$$\frac{d\eta}{d\tau} = C_0, \; \frac{dC_0}{d\tau} = \alpha C_0 + \beta(\eta - \eta^*). \tag{2.19}$$

3. Numerical Results and Discussion

The values of δ has been uniquely determined by Hunter for various values of γ. Table I gives the values of α and β calculated from (2.16) and (2.17). We find that in all cases $\alpha < 0$ and the two roots λ_1 and λ_2 of

$$\lambda^2 - \alpha\lambda - \beta = 0 \tag{3.1}$$

Table I

γ	δ	α	β
7	0·5552	−0·814	−0·038
5	0·6009	−0·902	−0·120
3	0·7086	−1·126	−0·303
2·5	0·7641	−1·253	−0·391
2·45	0·7710	−1·26966	−0·402990
2·4	0·7782	−1·28707	−0·413956
2·3	0·7937	−1·326	−0·438
2·0	0·8502	−1·476	−0·524
1·9	0·8735	−1·538	−0·483
1·8	0·8996	−1·623	−0·554
1·7	0·9290	−1·721	−0·625
1·6	0·9623	−1·892	−0·695
1·55	0·9806	−1·916	−0·721
1·50	1·0000	The analysis of this paper is not valid in this case	

are real and negative. Therefore, the singular point ($C_0 = 0$, $\eta = \eta^*$) of (2.19) is a node. As in the reference (1), the new time τ varies from $+\infty$ to $-\infty$ as t varies from $-\infty$ to 0. Since $\alpha < 0$, it follows from the discussion in [1] and [3] that the above self-similar flows are unstable in neighbourhood of the critical point. Therefore, these self-similar flows, passing through the critical point, cannot occur in reality. But the self-similar flows with accelerating cavity boundary can exist only if the integral curve passes through the critical point.

The above discussion of the instability of the flows with accelerating cavity boundary, gives a definite answer that such self-similar flows do not occur in reality provided we assume that the equations of motion used by Hunter do represent the physical phenomena. As Hunter points out, there is another self-similar flow with a constant velocity of the cavity boundary for the same value of γ. We have not studied the stability of these flows.

ACKNOWLEDGEMENT

The authors are grateful to Prof. P. L. Bhatnagar for encouragement, kind help and valuable discussion.

REFERENCES

1. Bhatnagar, P. L. and Prasad, P. .. "Study of self-similar and steady flows near singularities," 1969 (To be published).
2. Birkhoff, G. .. *Quart. Appl. Math.*, 1954, **12**, 306.
3. Hunter, C. .. *J. Fluid Mech.*, 1960, **8**, 241.
4. ——— .. *Ibid.*, 1963, **15**, 289.
5. Kulikovskii, A. G. and Slobodkina, F. A. *PMM*, 1967, **31**, 623.
6. Zel'dovich, Y. B. and Raizer, Y. P. .. *Physics of Shock Waves and High Temperature Hydrodyanamic Phenomena*, Academic Press, 1967.