SYMPLECTIC LOCAL ROOT NUMBERS, CENTRAL CRITICAL L-VALUES, AND RESTRICTION PROBLEMS IN THE REPRESENTATION THEORY OF CLASSICAL GROUPS

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1. Introduction

It has been almost 20 years since two of us proposed a rather speculative approach to the problem of restriction of irreducible representations from SO_n to SO_{n-1} [GP1, GP2]. Our predictions depended on the Langlands parametrization of irreducible representations, using L-packets and L-parameters. Since then, there has been considerable progress in the construction of local L-packets, as well as on both local and global aspects of the restriction problem. We thought it was a good time to review the precise conjectures which remain open, and to present them in a more general form, involving restriction problems for all of the classical groups.

Let k be a local field equipped with an automorphism σ with $\sigma^2 = 1$ and let k_0 be the fixed field of σ. Let V be a vector space over k with a non-degenerate sesquilinear form and let $G(V)$ be the identity component of the classical subgroup of $\mathrm{GL}(V)$ over k_0 which preserves this form. There are four distinct cases, depending on whether the space V is orthogonal, symplectic, hermitian, or skew-hermitian. In each case, for certain non-degenerate subspaces W of V, we define a subgroup H of the locally compact group $G = G(V) \times G(W)$ containing the diagonally embedded subgroup $G(W)$, and a unitary representation ν of H. The local restriction problem is to determine

$$d(\pi) = \dim_{\mathbb{C}} \mathrm{Hom}_H(\pi \otimes \overline{\nu}, \mathbb{C}),$$

where π is an irreducible complex representation of G.

The basic cases are when $\dim V - \dim W = 1$ or 0, where ν is the trivial representation or a Weil representation respectively. When $\dim V - \dim W \geq 2$, this restriction problem is also known as the existence and uniqueness of Bessel or Fourier-Jacobi models in the literature. As in [GP1] and [GP2], our predictions involve the Langlands parametrization, in a form suggested by Vogan [Vo], and the signs of symplectic root numbers.

We show that the Langlands parameters for irreducible representations of classical groups (and for genuine representations of the metaplectic group) are complex representations of the Weil-Deligne group of k, of specified dimension and with certain duality properties. We describe these parameters and their centralizers in detail, before using their symplectic root numbers to construct certain distinguished characters of the component group. Our local conjecture states that there is a unique representation π in each generic Vogan L-packet, such that the dimension $d(\pi)$ is equal to 1. Furthermore, this representation corresponds to a distinguished character χ of the component group. For all other representations π in the L-packet, we predict that $d(\pi)$ is equal to 0. The precise statements are contained in Conjectures 17.1 and 17.3.

Although this material is largely conjectural, we prove a number of new results in number theory and representation theory along the way:

(i) In Proposition 5.2, we give a generalization of a formula of Deligne on orthogonal root numbers to the root numbers of conjugate orthogonal representations.

(ii) We describe the L-parameters of classical groups, and unitary groups in particular, in a much simpler way than currently exists in the literature; this is contained in Theorem 8.1.

(iii) We show in Theorem 11.1 that the irreducible representations of the metaplectic group can be classified in terms of the irreducible representations of odd special orthogonal groups; this largely follows from fundamental results of Kudla-Rallis [KR], though the statement of the theorem did not appear explicitly in [KR].

(iv) We prove two theorems (cf. Theorems 15.1 and 16.1) that allow us to show the uniqueness of general Bessel and Fourier-Jacobi models over non-archimedean local fields. More precisely, we show that $d(\pi) \leq 1$ in almost all cases (cf. Corollaries 15.3, 16.2 and 16.3), reducing this to the basic cases when $\dim W^{\perp} = 0$ or 1, which were recently established by [AGRS], [S] and [Wa8]. The same theorems allow us to reduce our local conjectures to these basic cases, as shown in Theorem 19.1.

One subtle point about our local conjecture is its apparent dependence on the choice of an additive character ψ of k_0 or k/k_0. Indeed, the choice of such a character ψ is potentially used in 3 places:

(a) the Langlands-Vogan parametrization (which depends on fixing a quasi-split pure inner form G_0 of G, a Borel subgroup B_0 of G_0, and a non-degenerate character on the unipotent radical of B_0);

(b) the definition of the distinguished character χ of the component group;

(c) the representation ν of H in the restriction problem.

Typically, two of the above depend on the choice of ψ, whereas the third one doesn't. More precisely, we have:

- in the orthogonal case, none of (a), (b) or (c) above depends on ψ; this explains why this subtlety does not occur in [GP1] and [GP2].
- in the hermitian case, (a) and (b) depend on the choice of $\psi : k/k_0 \to \mathbb{S}^1$, but (c) doesn't.
- in the symplectic/metaplectic case, (a) and (c) depend on $\psi : k_0 \to \mathbb{S}^1$, but (b) doesn't.
- in the odd skew-hermitian case, (b) and (c) depend on $\psi : k_0 \to \mathbb{S}^1$, but (a) doesn't.
- in the even skew-hermitian case, (a) and (c) depend on $\psi : k_0 \to \mathbb{S}^1$ but (b) doesn't.

Given this, we check in §18 that the dependence on ψ cancels out in each case, so that our local conjecture is internally consistent with respect to changing ψ. There is, however, a variant of our local conjectures which is less sensitive to the choice of ψ, but is slightly weaker. This variant is given in Conjecture 20.1. Finally, when all the data involved are unramified, we state a more refined conjecture; this is contained in Conjecture 21.3.

After these local considerations, we study the global restriction problem, for cuspidal tempered representations of adelic groups. Here our predictions involve the central values of automorphic L-functions, associated to a distinguished symplectic representation R of the L-group. More precisely, let $G = G(V) \times G(W)$ and assume that π is an irreducible cuspidal representation of $G(\mathbb{A})$, where $\mathbb{A}$ is the ring of adèles of a global field F. If the vector space $\mathrm{Hom}_{H(\mathbb{A})}(\pi \otimes \bar{\nu}, \mathbb{C})$ is nonzero, our local conjecture implies that the global root number $\epsilon(\pi, R, \frac{1}{2})$ is equal to 1. If we assume π to be tempered, then our calculation of global root numbers and the general conjectures of Langlands and Arthur predict that π appears with multiplicity one in the discrete spectrum of $L^2(G(F)\backslash G(\mathbb{A}))$. We conjecture that the period integrals on the corresponding space of functions

$$f \mapsto \int_{H(k)\backslash H(\mathbb{A})} f(h) \cdot \overline{\nu(h)}\, dh$$

gives a nonzero element in $\mathrm{Hom}_{H(\mathbb{A})}(\pi \otimes \bar{\nu}, \mathbb{C})$ if and only if the central critical L-value $L(\pi, R, \frac{1}{2})$ is nonzero.

This first form of our global conjecture is given in §24, after which we examine the global restriction problem in the framework of Langlands-Arthur's conjecture on the automorphic discrete spectrum, and formulate a more refined global conjecture in §26. For this purpose, we formulate an extension of Langlands' multiplicity formula for metaplectic groups; see Conjecture 25.1.

One case in which all of these conjectures are known to be true is when $k = k_0 \times k_0$ is the split quadratic étale algebra over k_0, and V is a hermitian space over k of dimension n containing a codimension one nondegenerate subspace W. Then

$$G \cong \mathrm{GL}_n(k_0) \times \mathrm{GL}_{n-1}(k_0) \quad \text{and} \quad H \cong \mathrm{GL}_{n-1}(k_0).$$

Moreover, ν is the trivial representation. When k_0 is local, and π is a generic representation of $G = \mathrm{GL}_n(k_0) \times \mathrm{GL}_{n-1}(k_0)$, the local theory of Rankin-Selberg integrals [JPSS], together with the multiplicity one theorems of [AGRS], [AG], [SZ], [SZ2] and [Wa8], shows that

$$\dim \mathrm{Hom}_H(\pi, \mathbb{C}) = 1.$$

This agrees with our local conjecture, as the Vogan packets for $G = \mathrm{GL}_n(k_0) \times \mathrm{GL}_{n-1}(k_0)$ are singletons. If k_0 is global and π is a cuspidal representation of $G(\mathbb{A})$, then π appears with multiplicity one in the discrete spectrum. The global theory of

Rankin-Selberg integrals [JPSS] implies that the period integrals over $H(k)\backslash H(\mathbb{A})$ give a nonzero linear form on π if and only if

$$L(\pi, \mathrm{std}_n \otimes \mathrm{std}_{n-1}, 1/2) \neq 0,$$

where $L(\pi, \mathrm{std}_n \otimes \mathrm{std}_{n-1}, s)$ denotes the tensor product L-function. Again, this agrees with our global conjecture, since in this case, the local and global root numbers are all equal to 1, and

$$R = \mathrm{std}_n \otimes \mathrm{std}_{n-1} + \mathrm{std}_n^\vee \otimes \mathrm{std}_{n-1}^\vee.$$

In certain cases where the global root number $\epsilon = -1$, so that the central value is zero, we also make a prediction for the first derivative in §27. The cases we treat are certain orthogonal and hermitian cases, with $\dim W^\perp = 1$. We do not know if there is an analogous conjecture for the first derivative in the symplectic or skew-hermitian cases.

In a sequel to this paper, we will present some evidence for our conjectures, for groups of small rank and for certain discrete L-packets where one can calculate the distinguished character explicitly. We should mention that in a series of amazing papers [Wa4-7] and [MW], Waldspurger and Moeglin-Waldspurger have established the local conjectures for special orthogonal groups, assuming some natural properties of the characters of representations in tempered L-packets. There is no doubt that their methods will extend to the case of unitary groups.

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2. Classical groups and restriction of representations

Let k be a field, not of characteristic 2. Let σ be an involution of k having k_0 as the fixed field. If $\sigma = 1$, then $k_0 = k$. If $\sigma \neq 1$, k is a quadratic extension of k_0 and σ is the nontrivial element in the Galois group $\mathrm{Gal}(k/k_0)$.

Let V be a finite dimensional vector space over k. Let

$$\langle -, - \rangle : V \times V \to k$$

be a non-degenerate, σ-sesquilinear form on V, which is ϵ-symmetric (for $\epsilon = \pm 1$ in $k^\times$):

$$\begin{aligned} \langle \alpha v + \beta w, u \rangle &= \alpha \langle v, u \rangle + \beta \langle w, u \rangle \\ \langle u, v \rangle &= \epsilon \cdot \langle v, u \rangle^\sigma. \end{aligned}$$

Let $G(V) \subset \mathrm{GL}(V)$ be the algebraic subgroup of elements T in $\mathrm{GL}(V)$ which preserve the form $\langle -, - \rangle$:

$$\langle Tv, Tw \rangle = \langle v, w \rangle.$$

Then $G(V)$ is a classical group, defined over the field k_0. The different possibilities for $G(V)$ are given in the following table.

(k, ϵ)	$k = k_0$, $\epsilon = 1$	$k = k_0$, $\epsilon = -1$	k/k_0 quadratic, $\epsilon = \pm 1$
$G(V)$	orthogonal group $\mathrm{O}(V)$	symplectic group $\mathrm{Sp}(V)$	unitary group $\mathrm{U}(V)$

In our formulation, a classical group will always be associated to a space V, so the hermitian and skew-hermitian cases are distinct. Moreover, the group $G(V)$ is connected except in the orthogonal case. In that case, we let $\mathrm{SO}(V)$ denote the connected component, which consists of elements T of determinant $+1$, and shall refer to $\mathrm{SO}(V)$ as a connected classical group. We will only work with connected classical groups in this paper.

If one takes k to be the quadratic algebra $k_0 \times k_0$ with involution $\sigma(x, y) = (y, x)$ and V a free k-module, then a non-degenerate form $\langle -, - \rangle$ identifies the $k = k_0 \times k_0$ module V with the sum $V_0 + V_0^\vee$, where V_0 is a finite dimensional vector space over k_0 and $V_0^\vee$ is its dual. In this case $G(V)$ is isomorphic to the general linear group $\mathrm{GL}(V_0)$ over k_0.

If G is a connected, reductive group over k_0, the pure inner forms of G are the groups G' over k_0 which are obtained by inner twisting by elements in the pointed set $H^1(k_0, G)$. If $\{g_\sigma\}$ is a one cocycle on the Galois group of the separable closure k_0^s with values in $G(k_0^s)$, the corresponding pure inner form G' has points

$$G'(k_0) = \{a \in G(k_0^s) : a^\sigma = g_\sigma a g_\sigma^{-1}\}.$$

The group G' is well-defined up to inner automorphism over k_0 by the cohomology class of g_σ, so one can speak of a representation of $G'(k_0)$.

For connected, classical groups $G(V) \subset \mathrm{GL}(V)$, the pointed set $H^1(k_0, G)$ and the pure inner forms G' correspond bijectively to forms V' of the space V with its sesquilinear form $\langle , \rangle$ (cf. [KMRT, §29D and §29E]).

Lemma 2.1. *(1) If $G = \mathrm{GL}(V)$ or $G = \mathrm{Sp}(V)$, then the pointed set $H^1(k_0, G) = 1$ and there are no nontrivial pure inner forms of G.*

(2) If $G = \mathrm{U}(V)$, then elements of the pointed set $H^1(k_0, G)$ correspond bijectively to the isomorphism classes of hermitian (or skew-hermitian) spaces V' over k with $\dim(V') = \dim(V)$. The corresponding pure inner form G' of G is the unitary group $\mathrm{U}(V')$.

(3) *If* $G = \mathrm{SO}(V)$, *then elements of the pointed set* $H^1(k_0, G)$ *correspond bijectively to the isomorphism classes of orthogonal spaces* V' *over* k *with* $\dim(V') = \dim(V)$ *and* $\mathrm{disc}(V') = \mathrm{disc}(V)$. *The corresponding pure inner form* G' *of* G *is the special orthogonal group* $\mathrm{SO}(V')$.

Now let $W \subset V$ be a subspace, which is non-degenerate for the form $\langle -, - \rangle$. Then $V = W + W^\perp$. We assume that

$$\begin{array}{ll} 1) & \epsilon \cdot (-1)^{\dim W^\perp} = -1 \\ 2) & W^\perp \text{ is a split space.} \end{array}$$

When $\epsilon = -1$, so $\dim W^\perp = 2n$ is even, condition 2) means that $W^\perp$ contains an isotropic subspace X of dimension n. It follows that $W^\perp$ is a direct sum

$$W^\perp = X + Y,$$

with X and Y isotropic. The pairing $\langle -, - \rangle$ induces a natural map

$$Y \longrightarrow \mathrm{Hom}_k(X, k) = X^\vee$$

which is a k_0-linear isomorphism (and k-anti-linear if $k \neq k_0$). When $\epsilon = +1$, so dim $W^\perp = 2n+1$ is odd, condition 2) means that $W^\perp$ contains an isotropic subspace X of dimension n. It follows that

$$W^\perp = X + Y + E,$$

where E is a non-isotropic line orthogonal to $X + Y$, and X and Y are isotropic. As above, one has a k_0-linear isomorphism $Y \cong X^\vee$.

Let $G(W)$ be the subgroup of $G(V)$ which acts trivially on $W^\perp$. This is the classical group, of the same type as $G(V)$, associated to the space W. Choose an $X \subset W^\perp$ as above, and let P be the parabolic subgroup of $G(V)$ which stabilizes a complete flag of (isotropic) subspaces in X. Then $G(W)$, which acts trivially on both X and $X^\vee$, is contained in a Levi subgroup of P, and acts by conjugation on the unipotent radical N of P.

The semi-direct product $H = N \rtimes G(W)$ embeds as a subgroup of the product group $G = G(V) \times G(W)$ as follows. We use the defining inclusion $H \subset P \subset G(V)$ on the first factor, and the projection $H \to H/N = G(W)$ on the second factor. When $\epsilon = +1$, the dimension of H is equal to the dimension of the complete flag variety of G. When $\epsilon = -1$, the dimension of H is equal to the sum of the dimension of the complete flag variety of G and half of the dimension of the vector space W over k_0.

We call a pure inner form $G' = G(V') \times G(W')$ of the group G *relevant* if the space W' embeds as a non-degenerate subspace of V', with orthogonal complement isomorphic to $W^\perp$. We note:

Lemma 2.2. *Suppose* k *is non-archimedean.*

(i) In the orthogonal and hermitian cases, there are 4 pure inner forms of $G = G(V) \times G(W)$ *and among these, exactly two are relevant. Moreover, among the two relevant pure inner forms, exactly one is a quasi-split group.*

(ii) In the symplectic case, there is exactly one pure inner form of $G = G(V) \times G(W)$*, which is necessarily relevant.*

(iii) In the skew-hermitian case, there are 4 pure inner forms of $G = G(V) \times G(W)$*, exactly two of which are relevant. When* $\dim V$ *is odd, the two relevant pure inner forms are both quasi-split, and when* $\dim V$ *is even, exactly one of them is quasi-split.*

Proof. The statement (i) follows from the fact that an odd dimensional split quadratic space is determined by its discriminant and that there is a unique split hermitian space of a given even dimension. The statements (ii) and (iii) are similarly treated. □

Given a *relevant* pure inner form $G' = G(V') \times G(W')$ of G, one may define a subgroup $H' \subset G'$ as above. In this paper, we will study the restriction of irreducible complex representations of the groups $G' = G(V') \times G(W')$ to the subgroups H', when k is a local or a global field.

3. Selfdual and conjugate-dual representations

Let k be a local field, and let k^s be a separable closure of k. In this section, we will define selfdual and conjugate-dual representations of the Weil-Deligne group $WD(k)$ of k.

When $k = \mathbb{R}$ or $\mathbb{C}$, we define $WD(k)$ as the Weil group $W(k)$ of k, which is an extension of $\mathrm{Gal}(k^s/k)$ by $\mathbb{C}^\times$, and has abelianization isomorphic to $k^\times$. A representation of $WD(k)$ is, by definition, a completely reducible (or semisimple) continuous homomorphism

$$\varphi : WD(k) \to \mathrm{GL}(M),$$

where M is a finite dimensional complex vector space. When k is non-archimedean, the Weil group $W(k)$ is the dense subgroup $I \rtimes F^{\mathbb{Z}}$ of $\mathrm{Gal}(k^s/k)$, where I is the inertia group and F is a geometric Frobenius. We normalize the isomorphism

$$W(k)^{ab} \to k^\times$$

of local class field theory as in Deligne [D], taking F to a uniformizing element of $k^\times$. This defines the norm character

$$| - | : W(k) \to \mathbb{R}^\times, \quad \text{with } |F| = q^{-1}.$$

We define $WD(k)$ as the product of $W(k)$ with the group $\mathrm{SL}_2(\mathbb{C})$. A representation is a homomorphism

$$\varphi : WD(k) \to \mathrm{GL}(M)$$

with

(i) φ trivial on an open subgroup of I,
(ii) $\varphi(F)$ semi-simple,
(iii) $\varphi : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(M)$ algebraic.

The equivalence of this formulation of representations with that of Deligne [D], in which a representation is a homomorphism $\rho : W(k) \to \mathrm{GL}(M)$ and a nilpotent endomorphism N of M which satisfies $Ad\rho(w)(N) = |w| \cdot N$, is given in [GR, §2, Proposition 2.2]

We say two representations M and M' of $WD(k)$ are isomorphic if there is a linear isomorphism $f : M \to M'$ which commutes with the action of $WD(k)$. If M and M' are two representations of $WD(k)$, we have the direct sum representation $M \oplus M'$ and the tensor product representation $M \otimes M'$. The dual representation $M^\vee$ is defined by the natural action on $\mathrm{Hom}(M, \mathbb{C})$, and the determinant representation $\det(M)$ is defined by the action on the top exterior power. Since $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ is abelian, the representation $\det(M)$ factors through the quotient $W(k)^{ab} \to k^\times$ of $WD(k)$.

We now define certain selfdual representations of $WD(k)$. We say the representation M is orthogonal if there is a non-degenerate bilinear form

$$B : M \times M \to \mathbb{C}$$

which satisfies

$$\begin{cases} B(\tau m, \tau n) = B(m, n) \\ B(n, m) = B(m, n), \end{cases}$$

for all τ in $WD(k)$.

We say M is symplectic if there is a non-degenerate bilinear form B on M which satisfies

$$\begin{cases} B(\tau m, \tau n) = B(m, n) \\ B(n, m) = -B(m, n), \end{cases}$$

for all τ in $WD(k)$.

In both cases, the form B gives an isomorphism of representations

$$f : M \to M^\vee,$$

whose dual

$$f^\vee : M = M^{\vee\vee} \to M^\vee$$

satisfies

$$f^\vee = b \cdot f, \quad \text{with } b = \text{the sign of } B.$$

We now note:

Lemma 3.1. *Given any two non-degenerate forms B and B' on M preserved by $WD(k)$ with the same sign $b = \pm 1$, there is an automorphism T of M which commutes with $WD(k)$ and such that $B'(m, n) = B(Tm, Tn)$.*

Proof. Since M is semisimple as a representation of $WD(k)$, we may write

$$M = \bigoplus_i V_i \otimes M_i$$

as a direct sum of irreducible representations with multiplicity spaces V_i. Each M_i is either selfdual or else $M_i^\vee \cong M_j$ for some $i \neq j$, in which case $\dim V_i = \dim V_j$. So we may write

$$M = \left(\bigoplus_i V_i \otimes M_i \right) \oplus \left(\bigoplus_j V_j \otimes (P_j + P_j^\vee) \right)$$

with M_i irreducible selfdual and P_j irreducible but $P_j \ncong P_j^\vee$. Since any non-degenerate form B remains non-degenerate on each summand above, we are reduced to the cases:

(a) $M = V \otimes N$ with N irreducible and selfdual, in which case the centralizer of the action of $WD(k)$ is $\mathrm{GL}(V)$;

(b) $M = (V \otimes P) \oplus (V \otimes P^\vee)$, with P irreducible and $P \ncong P^\vee$, in which case the centralizer of the action of $WD(k)$ is $\mathrm{GL}(V) \times \mathrm{GL}(V)$.

In case (a), since N is irreducible and selfdual, there is a unique (up to scaling) $WD(k)$-invariant non-degenerate bilinear form on N; such a form on N has some sign b_N. Thus, giving a $WD(k)$-invariant non-degenerate bilinear form B on M of sign b is equivalent to giving a non-degenerate bilinear form on V of sign $b \cdot b_N$. But it is well-known that any two non-degenerate bilinear forms of a given sign are conjugate under $\mathrm{GL}(V)$. This takes care of (a).

In case (b), the subspaces $V \otimes P$ and $V \otimes P^\vee$ are necessarily totally isotropic. Moreover, there is a unique (up to scaling) $WD(k)$-invariant pairing on $P \times P^\vee$. Thus to give a $WD(k)$-invariant non-degenerate bilinear form B on M of sign b is equivalent to giving a non-degenerate bilinear form on V. But any two such forms are conjugate under the action of $\mathrm{GL}(V) \times \mathrm{GL}(V)$ on $V \times V$. This takes care of (b) and the lemma is proved. □

When M is symplectic, $\dim(M)$ is even and $\det(M) = 1$. When M is orthogonal, $\det(M)$ is an orthogonal representation of dimension 1. These representations correspond to the quadratic characters

$$\chi : k^\times \to \langle \pm 1 \rangle.$$

Since $\mathrm{char}(k) \neq 2$, the Hilbert symbol gives a perfect pairing

$$(-,-) : k^\times / k^{\times 2} \times k^\times / k^{\times 2} \to \langle \pm 1 \rangle.$$

We let $\mathbb{C}(d)$ be the one dimensional orthogonal representation given by the character $\chi_d(c) = (c, d)$.

We also note the following elementary result:

Lemma 3.2. *If M_i is selfdual with sign b_i, for $i = 1$ or 2, then $M_1 \otimes M_2$ is selfdual with sign $b_1 \cdot b_2$.*

Proof. If M_i is selfdual with respect to a form B_i of sign b_i, then $M_1 \otimes M_2$ is selfdual with respect to the tensor product $B_1 \otimes B_2$ which has sign $b_1 \cdot b_2$. $\square$

Next, assume that σ is a nontrivial involution of k, with fixed field k_0. Let s be an element of $W(k_0)$ which generates the quotient group

$$W(k_0)/W(k) = \mathrm{Gal}(k/k_0) = \langle 1, \sigma \rangle.$$

If M is a representation of $WD(k)$, let M^s denote the conjugate representation, with the same action of $\mathrm{SL}_2(\mathbb{C})$ and the action $\tau_s(m) = s\tau s^{-1}(m)$ for τ in $W(k)$.

We say the representation M is conjugate-orthogonal if there is a non-degenerate bilinear form $B : M \times M \to \mathbb{C}$ which satisfies

$$\begin{cases} B(\tau m, s\tau s^{-1} n) = B(m, n) \\ B(n, m) = B(m, s^2 n), \end{cases}$$

for all τ in $WD(k)$. We say M is conjugate-symplectic if there is a non-degenerate bilinear form on M which satisfies

$$\begin{cases} B(\tau m, s\tau s^{-1} n) = B(m, n) \\ B(n, m) = -B(m, s^2 n), \end{cases}$$

for all τ in $WD(k)$. In both cases, the form B gives an isomorphism of representations

$$f : M^s \to M^\vee,$$

whose conjugate-dual

$$(f^\vee)^s : M^s \longrightarrow ((M^s)^\vee)^s \xrightarrow{\varphi(s^2)} M^\vee$$

satisfies

$$(f^\vee)^s = b \cdot f \quad \text{with } b = \text{the sign of } B.$$

We now note:

Lemma 3.3. *Given two such non-degenerate forms B and B' on M with the same sign and preserved by $WD(k)$, there is an automorphism of M which commutes with $WD(k)$ and such that $B'(m, n) = B(Tm, Tn)$.*

Proof. The proof is similar to that of Lemma 3.1. As before, we may reduce to the following two cases:

(a) $M = V \otimes N$ with N irreducible and conjugate-dual, in which case the centralizer of the action of $WD(k)$ is $\mathrm{GL}(V)$;

(b) $M = (V \otimes P) \oplus (V \otimes (P^s)^\vee)$ with P irreducible and $P \ncong (P^s)^\vee$, in which case the centralizer of the action of $WD(k)$ is $\mathrm{GL}(V) \times \mathrm{GL}(V)$.

In case (a), if the conjugate-duality of N has sign b_N, then giving a $WD(k)$-invariant non-degenerate bilinear form on M of sign b is equivalent to giving a non-degenerate bilinear form on V of sign $b \cdot b_N$, and all such are conjugate under $\mathrm{GL}(V)$. Similarly, in case (b), giving a $WD(k)$-invariant nondegenerate bilinear form on M of sign b is equivalent to giving a non-degenerate bilinear form on V, and all such are conjugate under $\mathrm{GL}(V) \times \mathrm{GL}(V)$. □

The isomorphism class of the representation M^s is independent of the choice of s in $W(k_0) - W$. If $s' = ts$ is another choice, then the map

$$f \; : \; \begin{array}{l} M^s \to M^{s'} \\ m \mapsto t(m) \end{array}$$

is an isomorphism of representations of $WD(k)$. We denote the isomorphism class of M^s and $M^{s'}$ simply by M^σ. If M is conjugate-orthogonal or conjugate-symplectic by the pairing B relative to s, then it is conjugate-orthogonal or conjugate-symplectic by the pairing

$$B'(m, f(n)) = B(m, n)$$

relative to s'. In both cases, M^σ is isomorphic to the dual representation $M^\vee$.

If M is conjugate-dual via a pairing B with sign $b = \pm 1$, then $\det(M)$ is conjugate-dual with sign $= (b)^{\dim(M)}$. Any conjugate-dual representation of $WD(k)$ of dimension 1 gives a character $\chi : k^\times \to \mathbb{C}^\times$ which satisfies $\chi^{1+\sigma} = 1$. Hence χ is trivial on the subgroup $\mathbb{N}k^\times$, which has index 2 in $k_0^\times$. We denote this 1-dimensional representation by $\mathbb{C}(\chi)$.

Lemma 3.4. *The representation $\mathbb{C}(\chi)$ is conjugate-orthogonal if and only if χ is trivial on $k_0^\times$, and conjugate-symplectic if and only if χ is nontrivial on $k_0^\times$ but trivial on $\mathbb{N}k^\times$.*

Proof. : Since the action of WD(k) on $\mathbb{C}(\chi)$ factors through the quotient $W(k)^{ab}$, we may compute with the quotient $W(k/k_0)$ of $W(k_0)$. The Weil group $W(k/k_0)$ is isomorphic to the normalizer of $k^\times$ in the multiplicative group of the quaternion division algebra over k_0 [We1, Appendix III, Theorem 2]. It is therefore generated by $k^\times$ and s, with $s\alpha = \alpha^\sigma s$ for $\alpha \in k^\times$, and s^2 in $k_0^\times$ generating the quotient $k_0^\times / \mathbb{N}k^\times$.

If $\chi(s^2) = +1$, then the form $B(z, w) = zw$ on $\mathbb{C}(\chi)$ is conjugate-orthogonal. If $\chi(s^2) = -1$, then this form is conjugate-symplectic. □

We also note:

Lemma 3.5. *(i) The representation M of $WD(k)$ is conjugate-dual with sign b if and only if $N = \mathrm{Ind}_{WD(k)}^{WD(k_0)} M$ is selfdual with sign b and has maximal isotropic subspace M (which is naturally a $WD(k)$-submodule of N).*

(ii) If M_i is conjugate-dual with sign b_i, for $i = 1$ or 2, then $M_1 \otimes M_2$ is conjugate-dual with sign $b_1 \cdot b_2$.

Proof. For (i), suppose that M is conjugate-dual with respect to a form B. As a vector space, $N = M \oplus s^{-1} \cdot M$ for $s \in WD(k_0) \smallsetminus WD(k)$. We define a non-degenerate bilinear form B_N on N by decreeing that M and $s^{-1} \cdot M$ are isotropic spaces and setting

$$\begin{cases} B_N(m, s^{-1} \cdot m') = B(m, m'), \\ B_N(s^{-1} \cdot m', m) = b \cdot B_N(m, s^{-1} \cdot m'). \end{cases}$$

It is easy to check that B_N is preserved by $WD(k_0)$. Conversely, if the induced representation N is selfdual with respect to a form B_N which has M as an isotropic subspace, then the pairing induced by B_N on $M \times s^{-1} \cdot M$ is necessarily nondegenerate. Thus we may define a nondegenerate form on M by

$$B(m, m') = B_N(m, s^{-1} \cdot m').$$

It is easy to check that B gives a conjugate-duality on M with the same sign as B_N; this proves (i). The assertion (ii) is also straightforward: if M_i is conjugate-dual with respect to the form B_i of sign b_i, then $M_1 \otimes M_2$ is conjugate-dual with respect to the tensor product $B_1 \otimes B_2$ which has sign $b_1 \cdot b_2$. $\square$

Remark: M. Weissman has pointed out that the representation M^σ can be more canonically defined (without resorting to the choice of $s \in WD(k) \smallsetminus WD(k_0)$) in the following way. Consider the induced representation $\mathrm{Ind}_{WD(k)}^{WD(k_0)} M$ which can be realized on:

$$\{f : WD(k_0) \to M : f(\tau \cdot s) = \tau(f(s)) \text{ for all } \tau \in WD(k) \text{ and } s \in WD(k_0)\}.$$

Then the representation M^σ of $WD(k)$ can be realized on the subspace of such functions which are supported on $WD(k_0) \smallsetminus WD(k)$. We note:

(i) Any $WD(k)$-equivariant map $M \to N$ induces a natural $WD(k)$-equivariant map $M^\sigma \to N^\sigma$.

(ii) There is a natural isomorphism $(M^\sigma)^\vee \to (M^\vee)^\sigma$, via the perfect duality on $M^\sigma \times (M^\vee)^\sigma$ defined by

$$(f, f^\vee) \mapsto \langle f(s), f'(s) \rangle, \quad \text{for } f \in M^\sigma \text{ and } f^\vee \in (M^\vee)^\sigma,$$

for any $s \in WD(k_0) \smallsetminus WD(k)$ and where $\langle -, - \rangle$ denotes the natural pairing on $M \times M^\vee$. The above pairing is clearly independent of the choice of s.

(iii) On this model of M^σ, there is a canonical isomorphism

$$(M^\sigma)^\sigma \longrightarrow M$$

given by

$$F \mapsto F[s](s^{-1})$$

for any $s \in WD(k_0) \smallsetminus WD(k)$. This isomorphism is independent of the choice of s.

Thus a conjugate-duality with sign b is a $WD(k)$-equivariant isomorphism

$$f : M^\sigma \to M^\vee$$

whose conjugate-dual

$$(f^\vee)^\sigma : M^\sigma \to ((M^\sigma)^\vee)^\sigma \cong ((M^\vee)^\sigma)^\sigma \cong M^\vee$$

satisfies

$$(f^\vee)^\sigma = b \cdot f.$$

This treatment allows one to suppress the somewhat mysterious looking identity $B(n,m) = b \cdot B(m, s^2 n)$.

4. The centralizer and its group of components

The centralizer $C(M)$ of a representation M of $WD(k)$ is the subgroup of $\mathrm{GL}(M)$ which centralizes the image. Write

$$M = \bigoplus m_i M_i$$

as a direct sum of irreducible representations M_i, with multiplicities $m_i \geq 1$. Then by Schur's lemma

$$C(M) \simeq \prod \mathrm{GL}(m_i, \mathbb{C}).$$

In particular, $C(M)$ is a connected reductive group.

The situation is more interesting for representations M which are either selfdual or conjugate-dual, via a pairing B with sign $b = \pm 1$. We define $C = C(M, B)$ as the subgroup of $\mathrm{Aut}(M, B) \subset \mathrm{GL}(M)$ which centralizes the image of $WD(k)$. Up to isomorphism, the reductive group C depends only on the representation M and can be described more explicitly as follows.

If we write M as a direct sum of irreducible representations M_i, with multiplicities m_i, and consider their images in $M^\vee$ under the isomorphism $M^\sigma \to M^\vee$ provided by B, we find that there are three possibilities:

(1) M_i^σ is isomorphic to $M_i^\vee$, via a pairing B_i of the same sign b as B.
(2) M_i^σ is isomorphic to $M_i^\vee$, via pairing B_i of the opposite sign $-b$ as B. In this case the multiplicity m_i is even.
(3) M_i^σ is isomorphic to $M_j^\vee$, with $j \neq i$. In this case $m_i = m_j$.

Hence, we have a decomposition

$$M = \bigoplus V_i \otimes M_i + \bigoplus W_i \otimes N_i + \bigoplus U_i \otimes (P_i + (P_i^\sigma)^\vee),$$

where

(a) the M_i's are selfdual or conjugate-dual of the same sign b,

(b) the N_i's are selfdual or conjugate-dual of the opposite sign $-b$;
(c) P_i^σ is not isomorphic to $P_i^\vee$, so that P_i and $P_j = (P_i^\sigma)^\vee$ are distinct irreducible summands.

Moreover, the restriction of the form B to each summand in the above decomposition of M induces a nondegenerate pairing on the multiplicity space V_i, W_i or U_i. The induced pairing on V_i necessarily has sign $+1$, whereas the pairing on W_i necessarily has sign -1. On the other hand, the induced pairing on U_i need not have a sign.

We can now determine the centralizer C. As in the proofs of Lemmas 3.1 and 3.3, giving an element T of C is equivalent to giving elements T_i in $\mathrm{GL}(V_i)$, $\mathrm{GL}(W_i)$ or $\mathrm{GL}(U_i) \times \mathrm{GL}(U_i)$, such that T_i preserves the induced nondegenerate pairing on V_i, W_i or U_i. Thus, we conclude that (cf. [GP, §6-7], [P1] and [P3]):

$$C \simeq \prod \mathrm{O}(V_i) \times \prod \mathrm{Sp}(W_i) \times \prod \mathrm{GL}(U_i).$$

In particular, the component group of C is

$$A = \pi_0(C) \simeq (\mathbb{Z}/2)^k,$$

where k is the number of irreducible summands M_i of the same type as M, or equivalently the number of V_i's in the above decomposition.

For each such M_i, let a_i be a simple reflection in the orthogonal group $\mathrm{O}(m_i)$. The images of the elements a_i in A give a basis over $\mathbb{Z}/2\mathbb{Z}$. For any semisimple element a in C, we define

$$M^a = \{m \in M : am = -m\}$$

to be the -1 eigenspace for a on M. This is a representation of $WD(k)$, and the restricted pairing $B : M^a \times M^a \to \mathbb{C}$ is non-degenerate, of the same type as M. For the simple reflections a_i in C,

$$M^{a_i} = M_i$$

are the irreducible summands of the same type as M.

We can use these representations to define characters $\chi : A \to \langle \pm 1 \rangle$. The basic idea is to define signed invariants $d(M) = \pm 1$ of representations M of $WD(k)$, which are either selfdual or conjugate-dual.

Proposition 4.1. *Let $d(M)$ be an invariant of selfdual or conjugate-dual representations, taking values in ± 1. Assume that*

(1) $d(M + M') = d(M) \cdot d(M')$
(2) the value $d(M^a)$ depends only on the image of a in the quotient group $A = C_M/C_M^0$.

Then the function

$$\chi(a) = d(M^a)$$

defines a character of A.

Indeed, the different classes in A are all represented by commuting involutions in C, and for two commuting involutions a and b we have the formula:

$$M^{ab} + 2(M^a \cap M^b) = M^a + M^b$$

as representations of $WD(k)$. Hence $\chi(ab) = \chi(a) \cdot \chi(b)$.

The simplest example of such an invariant, which applies in both the conjugate-dual and the selfdual cases, is

$$d(M) = (-1)^{\dim M}.$$

To see that $\dim M^a \pmod 2$ depends only on the coset of $a \pmod{C_M^0}$, we recall that

$$M = \bigoplus V_i \otimes M_i + \bigoplus W_i \otimes N_i + \bigoplus U_i \otimes (P_i + (P_i^\sigma)^\vee)$$

and let $a = \prod_i a_i$ be a semisimple element of the product

$$C_M = \prod_i \mathrm{O}(V_i) \times \prod_i \mathrm{Sp}(W_i) \times \prod_i \mathrm{GL}(U_i).$$

Then -1 occurs with even multiplicity as an eigenvalue of $a_i \in \mathrm{Sp}(W_i)$. On the other hand, for $a_i \in \mathrm{O}(V_i)$, one has

$$\det a_i = (-1)^{\text{multiplicity of } -1 \text{ as an eigenvalue of } a_i}.$$

Hence all the summands of M^a have even dimension, except for the terms $V_i^{a_i} \otimes M_i$ which have odd dimension precisely when $\det(a_i) = -1$ and $\dim M_i \equiv 1 \pmod 2$. Thus it follows that the parity of $\dim M^a$ depends only on the coset of $a \pmod{C_M^0}$. In particular, one obtains a character of A:

$$\eta(a) = (-1)^{\dim M^a}.$$

Now assume M is selfdual. The character η is trivial when M is symplectic, as $\dim M^a$ is even for all a in $\mathrm{Sp}(M) = G(M, B)$. In the orthogonal case, $\dim M^a$ is even precisely for elements a in the centralizer which lie in the subgroup $\mathrm{SO}(M, B)$ of index 2 in $\mathrm{O}(M, B)$. We denote this subgroup by C^+.

An element c of $k^\times/k^{\times 2}$ gives a character

$$\eta_c(a) = (\det M^a)(c)$$

of A. Indeed, the quadratic character $\det M^a$ depends only on the coset of $a \pmod{C^0}$. Since $\eta_{cd} = \eta_c \eta_d$, we get a pairing

$$(c, a) : k^\times/k^{\times 2} \times A \to \langle \pm 1 \rangle$$

which is trivial in the symplectic case.

To construct other characters of A, we need more sophisticated signed invariants $d(M)$ of selfdual or conjugate-dual representations. We will obtain these from local root numbers, after recalling that theory in the next section.

5. Local root numbers

Let M be a representation of the Weil-Deligne group $WD(k)$ of a local field k, and let ψ be a nontrivial additive character of k. In this section, we define the local root number $\epsilon(M,\psi)$, following the articles of Tate [T] and Deligne [De1]. We then study the properties of these constants for selfdual and conjugate-dual representations, and give explicit formulae in the orthogonal and conjugate-orthogonal cases. The local root numbers are more mysterious in the symplectic and conjugate-symplectic cases. Indeed, they form the basis of our conjectures on the restriction of representations of classical groups over local fields.

Let dx be the unique Haar measure on k which is selfdual for Fourier transform with respect to ψ. For a representation M of the Weil group $W(k)$, we define

$$\epsilon(M,\psi) = \epsilon(M,\psi,dx,1/2) \quad \text{in} \quad \mathbb{C}^\times,$$

in the notation of [De1, §4-5]. This is the local constant $\epsilon_L(M,\psi)$ in [T, 3.6.1]. In the non-archimedean case, if M is a representation of $WD(k) = W(k)\times \mathrm{SL}_2(\mathbb{C})$, we may write

$$M = \sum_{n\geq 0} M_n \otimes \mathrm{Sym}^n$$

with each M_n a representation of the Weil group. We define (cf. [GR, §2]):

$$\epsilon(M,\psi) = \prod_{n\geq 0} \epsilon(M_n,\psi)^{n+1}\cdot \det(-F|M_n^I)^n.$$

This constant depends only on the isomorphism class of M.

The following formulae involving $\epsilon(M,\psi)$ are well-known [T, 3.6], for representations M of the Weil group $W(k)$. For a in $k^\times$, let ψ_a be the nontrivial additive character $\psi_a(x)=\psi(ax)$. Then

$$\epsilon(M,\psi_a) = \det M(a)\cdot \epsilon(M,\psi),$$

$$\epsilon(M,\psi)\cdot\epsilon(M^\vee,\psi^{-1}) = 1.$$

Since $\psi^{-1}=\psi_{-1}$, we conclude that

$$\epsilon(M,\psi)\cdot\epsilon(M^\vee,\psi) = \det M(-1).$$

For representations $M=\sum M_n\otimes \mathrm{Sym}^n$ of $WD(k)$ in the non-archimedean case, we have

$$M^\vee = \sum M_n^\vee \otimes \mathrm{Sym}^n,$$

$$\det(M) = \prod_{n\geq 0} \det(M_n)^{n+1}.$$

This allows us to extend the above formulas to the local root numbers $\epsilon(M,\psi)$ of representations of $WD(k)$.

Now let σ be an involution of k, and define $\psi^\sigma(x) = \psi(x^\sigma)$. Then $\epsilon(M^\sigma,\psi^\sigma) = \epsilon(M,\psi)$. (Indeed, this is true for any continuous isomorphism $\sigma : k \to k'$. For M of dimension 1, this follows from Tate's integral formula [T] for $\epsilon(M,\psi)$. It then follows from general M from the inductivity of ϵ-factors.) If we assume further that $\psi^\sigma = \psi^{-1}$, then

$$\begin{aligned}\epsilon(M,\psi)\cdot\epsilon((M^\vee)^\sigma,\psi) &= \epsilon(M,\psi)\cdot\epsilon(M^\vee,\psi^\sigma)\\ &= \epsilon(M,\psi)\cdot\epsilon(M^\vee,\psi^{-1})\\ &= 1.\end{aligned}$$

When we apply these formulas to selfdual and conjugate-dual representations, we obtain the following.

Proposition 5.1. *(1) Assume that M is a selfdual representation of $WD(k)$ with $\det(M) = 1$. Then $\epsilon(M) = \epsilon(M,\psi)$ is independent of the choice of ψ and satisfies*

$$\epsilon(M)^2 = 1.$$

Furthermore, if M is of the form $M = N + N^\vee$, then $\epsilon(M) = \det N(-1)$.

(2) Assume that M is a conjugate-dual representation of $WD(k)$ and that the additive character ψ of k satisfies $\psi^\sigma = \psi^{-1}$. Then

$$\epsilon(M,\psi)^2 = 1.$$

Furthermore, if M is of the form $M = N + {}^\sigma N^\vee$, then $\epsilon(M,\psi) = 1$.

Since we are assuming that the characteristic of k is not equal to 2, the characters ψ of k which satisfy $\psi^\sigma = \psi^{-1}$ are precisely those characters which are trivial on k_0, the fixed field of σ. These characters form a principal homogeneous space for the group $k_0^\times$, and the value $\epsilon(M,\psi)$ depends only on the $\mathbb{N}k^\times$-orbit of ψ. Indeed $\det M$ is conjugate-dual and hence trivial on $\mathbb{N}k^\times \subset k_0^\times$. If $\det M$ is conjugate-orthogonal, the restriction of $\det M$ to $k_0^\times$ is trivial, and hence the value $\epsilon(M) = \epsilon(M,\psi)$ is independent of the choice of ψ.

Following Deligne [De2], we can say more when the selfduality or conjugate-duality of M is given by a pairing B with sign $b = +1$. Recall the spin covering of the special orthogonal group, which gives an exact sequence:

$$1 \to \mathbb{Z}/2 \to \mathrm{Spin}(M) \to \mathrm{SO}(M) \to 1.$$

Proposition 5.2. *(1) Assume that M is an orthogonal representation and that $\det(M) = 1$. Then the root number $\epsilon(M) = \epsilon(M,\psi)$ is independent of the choice of ψ and satisfies $\epsilon(M)^2 = 1$. Furthermore $\epsilon(M) = +1$ if and only if the representation $\varphi : WD(k) \to \mathrm{SO}(M)$ lifts to a homomorphism $\varphi : WD(k) \to \mathrm{Spin}(M)$.*

(2) *Assume that M is a conjugate-orthogonal representation and that $\psi^\sigma = \psi^{-1}$. Then the root number $\epsilon(M) = \epsilon(M, \psi)$ is independent of the choice of ψ and satisfies $\epsilon(M) = +1$.*

Proof. The orthogonal case was proved by Deligne [De2]; we note that in our case the characteristic of k is not equal to 2. We will deduce the second result for conjugate-orthogonal representations of $W(k)$ from Deligne's formula, combined with the work of Frohlich and Queyrut [FQ]. The extension of the second result to conjugate-orthogonal representations of the Weil-Deligne group $WD(k)$ is then an amusing exercise, which we leave to the reader (cf. [De2, §5]).

If M is conjugate-orthogonal and $\dim(M) = 1$, we have seen that M corresponds to a complex character χ of the group $k^\times/k_0^\times$. By [FQ, Thm 3], we have the formula

$$\epsilon(\chi, \psi_0(\mathrm{Tr})) = \chi(e)$$

where ψ_0 is any nontrivial additive character of k_0 and e is any nonzero element of k with $\mathrm{Tr}(e) = 0$ in k_0. The element e is well-defined up to multiplication by $k_0^\times$, and e^2 is an element of $k_0^\times$. If we define the additive character ψ of k by

$$\psi(x) = \psi_0(\mathrm{Tr}(ex))$$

then $\psi^\sigma = \psi^{-1}$, and

$$\epsilon(M) = \epsilon(\chi, \psi) = \chi(e)^2 = +1.$$

This establishes the formula when M has dimension 1.

Since the desired formula is additive in the representation M of $W(k)$, and is true when $\dim(M) = 1$, we are reduced to the case of conjugate-orthogonal representations M of even dimension. Then $N = \mathrm{Ind}(M)$ is an orthogonal representation of determinant 1. Let ψ_0 be a nontrivial additive character of k_0; by the inductivity of local epsilon factors in dimension zero [De1]:

$$\epsilon(N, \psi_0)/\epsilon(P, \psi_0)^{\dim(M)} = \epsilon(M, \psi_0(\mathrm{Tr}))/\epsilon(\mathbb{C}, \psi_0(\mathrm{Tr}))^{\dim(M)}$$

with $\mathbb{C}$ the trivial representation and $P = \mathrm{Ind}(\mathbb{C})$ the corresponding induced representation, which is orthogonal of dimension 2 and determinant ω. Since $\epsilon(\mathbb{C}, \psi_0(\mathrm{Tr})) = 1$ and $\epsilon(P, \psi_0)^2 = \omega(-1)$, we obtain the formula

$$\epsilon(N) = \epsilon(N, \psi_0) = \omega(-1)^{\dim(M)/2} \cdot \epsilon(M, \psi_0(\mathrm{Tr})).$$

On the other hand, we have

$$\epsilon(M) = \epsilon(M, \psi) = \det(M)(e) \cdot \epsilon(M, \psi_0(\mathrm{Tr})),$$

where e is a nonzero element of k with $\mathrm{Tr}(e) = 0$ in k_0. Hence, to show that $\epsilon(M) = +1$, we are reduced to proving the formula

$$\epsilon(N) = \det(M)(e) \cdot \omega(-1)^{\dim(M)/2}$$

for the root number of the orthogonal induced representation N. To do this, we combine Deligne's formula for the orthogonal root number with the following lemma.

Lemma 5.3. *Let M be a conjugate-orthogonal representation of $W(k)$ of even dimension. Then $N = \mathrm{Ind}(M)$ is an orthogonal representation of $W(k_0)$ of determinant 1. The homomorphism $\varphi : W(k_0) \to \mathrm{SO}(N)$ lifts to a homomorphism $\varphi : W(k_0) \to \mathrm{Spin}(N)$ if and only if* $\det M(e) \cdot \omega(-1)^{\dim(M)/2} = +1$.

Proof. Let T be the maximal torus in $\mathrm{SO}(N)$ which consists of the rotations z_i in $n = \dim(M)$ orthogonal planes. The restriction of the spin covering $\mathrm{Spin}(N) \to \mathrm{SO}(N)$ to the torus T is the two-fold covering obtained by pulling back the spin covering $z \to z^2$ of $\mathbb{C}^\times$ under the map $F(z_1, \cdots, z_n) = \prod z_i$.

The image of the map $\varphi : W \to \mathrm{SO}(N)$ lies in the normalizer of the Levi subgroup $\mathrm{GL}(M)$ which fixes the decomposition $N = M + M^\vee$ into maximal isotropic dual subspaces. There is an involution j of N which switches the subspaces M and $M^\vee$. Since $\det(j) = (-1)^n$ and $n = \dim(M)$ is even, this involution lies in $\mathrm{SO}(N)$. The normalizer of the Levi is the semi-direct product $\mathrm{GL}(M) \cdot \langle j \rangle$. We denote the resulting homomorphism of W to $\mathrm{GL}(M) \cdot \langle j \rangle$ also by φ.

Since j has n eigenvalues which are $+1$, and n eigenvalues which are -1, if we view this involution as a product of rotations $(z_1, \cdots, z_n)$ in orthogonal planes, we get $n/2$ values $z_i = -1$ and $n/2$ values $z_i = +1$. Hence the involution j lifts to an element of order 2 in $\mathrm{Spin}(N)$ if and only if $n = \dim(M)$ is divisible by 4. Note that the quadratic extension k of k_0 can be embedded in a cyclic quartic extension of k_0 if and only if the character ω of $k_0^\times$ is a square, or equivalently, if and only if $\omega(-1) = 1$. We therefore conclude that the natural homomorphism $W \to \mathbb{Z}/2\mathbb{Z} = \langle j \rangle$ (given by the quadratic extension k/k_0) lifts to the restriction of the spin cover to $\langle j \rangle = \mathbb{Z}/2\mathbb{Z}$ if and only if

$$\omega(-1)^{\dim(M)/2} = +1. \tag{5.4}$$

We now consider the homomorphism

$$\phi : W \xrightarrow{\varphi} \mathrm{GL}(M) \cdot \langle j \rangle \xrightarrow{\det} \mathbb{C}^\times \cdot \langle j \rangle$$

whose projection to the quotient $\langle j \rangle$ is the quadratic character ω of $\mathrm{Gal}(k/k_0)$. The resulting homomorphism

$$\phi : W \to \mathbb{C}^\times \cdot \langle j \rangle$$

is given by its restriction to $W(k)$, which is nothing but the character $\chi = \det(M)$, a character of $k^\times/k_0^\times$ by the local class field theory. Hence the homomorphism $\phi : W \to \mathbb{C}^\times \cdot \langle j \rangle$ lifts to a homomorphism from W to $\mathbb{C}^\times \cdot \langle j \rangle$:

$$\begin{tikzcd} & \mathbb{C}^\times \cdot \langle j \rangle \arrow[d, "z^2"] \\ \phi : W \arrow[r] \arrow[ur, dotted] & \mathbb{C}^\times \cdot \langle j \rangle \end{tikzcd}$$

if and only if the character χ of $k^\times/k_0^\times$ has a square-root. Clearly, the character χ of $k^\times/k_0^\times$ is a square if and only if

$$\chi(e) = +1, \tag{5.5}$$

where e is a nonzero element of k with trace zero to k_0. Indeed, e generates the 2-torsion subgroup of the one dimensional torus $k^\times/k_0^\times$.

Since the subgroup $\mathrm{SL}(M)$ is simply-connected, it always lifts to $\mathrm{Spin}(N)$. Hence the restriction of the spin covering of $\mathrm{SO}(N)$ to $\mathrm{GL}(M)$ is obtained by taking the square root of the determinant of M, via the formula for the covering of T given above.

It is easy to see that the 2-fold covering of W afforded by $\varphi : W \to \mathrm{GL}(M)\cdot\langle j\rangle \subset \mathrm{SO}(N)$ is the sum of two coverings, one of which is the 2-fold cover of $\langle j\rangle$ pulled back to give a 2-fold cover of $\mathrm{GL}(M)\cdot\langle j\rangle$, and the other of which is the 2-fold cover $\mathbb{C}^\times \xrightarrow{z^2} \mathbb{C}^\times$ pulled back to $\mathrm{GL}(M)\cdot\langle j\rangle$ via the determinant map from $\mathrm{GL}(M)$ to $\mathbb{C}^\times$. From our calculations above, these two 2-fold covers of W are respectively trivial if and only if we have the conditions as in (5.4) and (5.5).

We now observe that $H^2(W,\mathbb{Z}/2\mathbb{Z})$ classifying the 2-fold coverings of W is an abelian group under fiber product which, by local class field theory, is nothing but $\mathbb{Z}/2\mathbb{Z}$. Therefore the sum of two elements in $H^2(W,\mathbb{Z}/2\mathbb{Z})$ is zero if either both of them are trivial, or both of them are nontrivial. Hence by (5.4) and (5.5), the parameter

$$\phi : W \to \mathrm{GL}(M)\cdot\langle j\rangle \to \mathrm{SO}(N)$$

lifts to $\mathrm{Spin}(N)$ if and only if $\chi(e)\cdot\omega(-1)^{\dim(M)/2} = +1$. □

Together with the extension to representations of $WD(k)$, this completes the proof of Proposition 5.2. □

6. Characters of component groups

In this section, we will use the results of the previous section on local root numbers, together with Proposition 4.1, to construct characters of the group A of components of the centralizer C of (M,B).

First assume M and N are conjugate selfdual representations, with signs $b(M)$ and $b(N)$. Fix ψ with $\psi^\sigma = \psi^{-1}$, and for semisimple a in $C_M \subset G(M,B)$, define

$$\chi_N(a) = \epsilon(M^a \otimes N, \psi).$$

Theorem 6.1. *(1) The value $\chi_N(a)$ depends only on the image of a in A_M, and defines a character $\chi_N : A_M \to \langle\pm1\rangle$*

(2) If $b(M)\cdot b(N) = +1$, then $\chi_N = 1$ on A_M.

(3) *If $b(M) \cdot b(N) = -1$, let $\psi'(x) = \psi(tx)$ with t the nontrivial class in $k_0^\times/\mathbb{N}k^\times$, and define*

$$\chi'_N(a) = \epsilon(M^a \otimes N, \psi').$$

Then

$$\chi'_N = \chi_N \cdot \eta^{\dim(N)} \in \mathrm{Hom}(A_M, \pm 1),$$

where the character η of A_M is defined by $\eta(a) = (-1)^{\dim M^a}$.

Proof. (1) Write

$$M = \bigoplus V_i \otimes M_i + \bigoplus W_i \otimes N_i + \bigoplus U_i \otimes (P_i + (P_i^\sigma)^\vee$$

as in §4., so that

$$C_M = \prod_i \mathrm{O}(V_i) \times \prod_i \mathrm{Sp}(W_i) \times \prod_i \mathrm{GL}(U_i).$$

It suffices to check (1) for semisimple elements a which are nontrivial in exactly one of the factors in the above product expression for C_M. Suppose, for example, that $a = a_i \times 1$ with $a_i \in \mathrm{O}(V_i)$. Then

$$M^a \otimes N = \dim V_i^{a_i} \cdot (M_i \otimes N).$$

The parity of $\dim V_i^{a_i}$ depends only on the image of a_i in $\mathrm{O}(V_i)/\mathrm{SO}(V_i)$, or equivalently only on the image of a in the component group A_M. Since

$$\epsilon(M_i \otimes N, \psi) = \pm 1,$$

we see that

$$\chi(a) = \epsilon(M^a \otimes N, \psi) = \epsilon(M_i \otimes N, \psi)^{\dim V_i^{a_i}}$$

depends only on the image of a in A_M. The other cases are similarly treated: when $a = a_i \times 1$ with $a_i \in \mathrm{Sp}(W_i)$ or $a_i \in \mathrm{GL}(U_i)$, one finds that $\epsilon(M^a \otimes N, \psi) = +1$. For the details, see [GP1, §10].

(2) When $b(M) \cdot b(N) = +1$, the representations $M^a \otimes N$ are all conjugate-orthogonal by Lemma 3.5, so $\chi_N = 1$.

(3) The final statement follows from the formula

$$\chi'_N(a) = \chi_N(a) \cdot \det(M^a \otimes N)(t)$$

and the calculation of the sign of the conjugate-dual representation which is the determinant of the tensor product. $\square$

We use this theorem to define the quadratic character

$$\chi_N(a_M) \cdot \chi_M(a_N)$$

on elements (a_M, a_N) in the component group $A_M \times A_N$. Here M and N are two conjugate-dual representations, although by part 2 of Theorem 6.1 the character $\chi_N \times \chi_M$ can only be nontrivial when $b(M) \cdot b(N) = -1$.

The case when M and N are selfdual with signs $b(M)$ and $b(N)$ is more complicated. First, with ψ a nontrivial additive character of k, the function

$$\chi_N(a) = \epsilon(M^a \otimes N, \psi)$$

on C_M need not take values in ± 1. Indeed

$$\chi_N(a)^2 = \det(M^a \otimes N)(-1) = \pm 1.$$

Even when $\det(M^a \otimes N) = 1$ for all a in C_M, the value $\chi_N(a) = \pm 1$ may not be constant on the cosets of C_M^0. For example, when $M = P + P^\vee$ with P irreducible and not isomorphic to $P^\vee$ and N is the trivial representation $\mathbb{C}$ of dimension 1, we have $C_M = \mathrm{GL}(1, \mathbb{C})$. But

$$\chi_N(-1) = \epsilon(M \otimes N, \psi) = \epsilon(M, \psi) = \det P(-1),$$

which need not be equal to $\chi_N(1) = 1$.

We will therefore only consider selfdual representations M and N of *even* dimension, and semisimple elements a in the subgroup C_M^+ of C_M, where $\det(a|M) = +1$. Then $\dim(M^a)$ is also even, and

$$\det(M^a \otimes N) = \det(M^a)^{\dim(N)} \cdot \det(N)^{\dim(M^a)}$$

is clearly trivial. In particular, $\epsilon(M^a \otimes N, \psi) = \epsilon(M^a \otimes N)$ is independent of the choice of additive character ψ and satisfies $\epsilon(M^a \otimes N)^2 = +1$. We correct this sign by another square root of $\det(M^a \otimes N)(-1)$, and define (for $a \in C_M^+$)

$$\chi_N(a) = \epsilon(M^a \otimes N) \cdot \det(M^a)(-1)^{\dim(N)/2} \cdot \det(N)(-1)^{\dim(M^a)/2}.$$

Now let A_M^+ be the image of C_M^+ in $A_M = C_M/C_M^0$. Note that $C_M^0 \subset C_M^+ \subset C_M$, so the component group A_M^+ has index either 1 or 2 in A_M. Then we have:

Theorem 6.2. *Assume that M and N are even dimensional selfdual representations of $WD(k)$.*

1. *The value $\chi_N(a)$ depends only on the image of a in A_M^+, and defines a character $\chi_N : A_M^+ \to \langle \pm 1 \rangle$.*
2. *If $b(M) \cdot b(N) = +1$, then $\chi_N(a) = (\det M^a, \det N)$, where $(-,-)$ is the Hilbert symbol.*

Proof. (1) This follows from the method of [GP1, Prop. 10.5], analogous to the proof of Theorem 6.1(1).

(2) When M and N are both symplectic, the tensor product representation $M^a \otimes N$ of the simply-connected group $\mathrm{Sp}(M^a) \times \mathrm{Sp}(N)$ lifts to $\mathrm{Spin}(M^a \otimes N)$ and $\epsilon(M^a \otimes N) = +1$. This proves (2) since $\det M^a = \det N = 1$.

When M and N are both orthogonal of even dimension, so are M^a and N. Hence $M^a \otimes N$ is orthogonal of dimension divisible by 4 and determinant 1. In this case, we have

$$\epsilon(M^a \otimes N) = w_2(M^a \otimes N)$$

by Deligne [De2], where w_2 refers to the second Stiefel-Whitney class [MiSt], which is valued in $H^2(k, \mathbb{Z}/2\mathbb{Z}) = \{\pm 1\}$. On the other hand, for two representations V and W of $WD(k)$, $w_2(V \otimes W)$ is given by [MiSt, Problem 7-C, Pg. 87-88]

$$\begin{aligned} w_2(V \otimes W) &= w_2(V) \cdot \dim W + \dim(V) \cdot w_2(W) + w_1(V)^2 \cdot \binom{\dim W}{2} \\ &\quad + \binom{\dim V}{2} \cdot w_1(W)^2 + w_1(V) \cdot w_1(W) \cdot (\dim V \cdot \dim W + 1) \end{aligned}$$

as elements of $H^2(k, \mathbb{Z}/2\mathbb{Z}) = \{\pm 1\}$. Here, w_1 refers to the first Stiefel-Whitney class, which is valued in $H^1(k, \mathbb{Z}/2\mathbb{Z}) = k^\times/k^{\times 2}$, and the various operations refer to addition and cup product in the cohomology ring $H^*(k, \mathbb{Z}/2\mathbb{Z})$. In particular, if V and W are both even-dimensional, we have

$$w_2(V \otimes W) = \frac{\dim V}{2} \cdot w_1(W)^2 + \frac{\dim W}{2} \cdot w_1(V)^2 + w_1(V) \cdot w_1(W) \in \{\pm 1\}.$$

For the even dimensional orthogonal representations M^a and N of interest, we have:

$$w_1(N) = \det N \quad \text{and} \quad w_1(M^a) = \det M^a,$$

and the cup product pairing $H^1(k, \mathbb{Z}/2\mathbb{Z}) \times H^1(k, \mathbb{Z}/2\mathbb{Z}) \to H^2(k, \mathbb{Z}/2\mathbb{Z})$ is given by the Hilbert symbol. Hence, we have

$$\begin{aligned} \epsilon(M^a \otimes N) &= (\det N, \det N)^{\frac{1}{2}\dim M^a} \cdot (\det M^a, \det M^a)^{\frac{1}{2}\dim N} \cdot (\det N, \det M^a) \\ &= (\det N)(-1)^{\frac{1}{2}\dim M^a} \cdot (\det M^a)(-1)^{\frac{1}{2}\dim N} \cdot (\det N, \det M^a), \end{aligned}$$

as claimed. □

We use this theorem to define the quadratic character

$$\chi_N(a_M) \cdot \chi_M(a_N)$$

on elements (a_M, a_N) in the component group $A_M^+ \times A_N^+$, for two selfdual representations M and N of even dimension. As in the conjugate-dual case, it follows from Theorem 6.2(2) that the character $\chi_N \times \chi_M$ is only interesting when $b(M) \cdot b(N) = -1$. When the representations M and N are both symplectic, $\chi = 1$. When M and N are both orthogonal, the character χ is given by a product of Hilbert symbols

$$\chi_N(a_M) \cdot \chi_M(a_N) = (\det M^{a_M}, \det N) \cdot (\det M, \det N^{a_N}).$$

7. L-groups of classical groups

Having defined representations, selfdual representations, and conjugate-dual representations of the Weil-Deligne group $WD(k)$ of k, our next goal is to relate these to the Langlands parameters of classical groups. Before doing that, we recall the L-group attached to each of the classical groups, with particular attention to the L-groups of unitary groups.

If G is a connected reductive group over k_0, the L-group of G is a semi-direct product

$$^{L}G = \widehat{G} \rtimes \mathrm{Gal}(K/k_0)$$

where $\widehat{G}$ is the complex dual group and K is a splitting field for the quasi-split inner form of G, with $\mathrm{Gal}(K/k_0)$ acting on $\widehat{G}$ via pinned automorphisms. (Alternatively, one could use $W(k_0)$, acting on $\widehat{G}$ through its quotient $\mathrm{Gal}(K/k_0)$).

Recall that in this paper, our classical group $G = G(V)$ comes equipped with an underlying space V, i.e. with a standard representation. We shall see that this extra data equips the L-group ^{L}G or the dual group $\widehat{G}$ with a standard representation.

For $G = \mathrm{GL}(V)$, we have $\widehat{G} = \mathrm{GL}(M)$ with $\dim M = \dim V$. If $\{e_1, \cdots, e_n\}$ is the basis of the character group of a maximal torus $T \subset G$ given by the weights of V, then the weights of the dual torus $\widehat{T}$ on M are the dual basis $\{e_1^\vee, \cdots, e_n^\vee\}$.

Now assume that $G \subset \mathrm{GL}(V/k)$ is a connected classical group, defined by a σ-sesquilinear form $\langle , \rangle : V \times V \to k$ of sign ϵ. The group G and its dual group $\widehat{G}$, as well as the splitting field K of its quasi-split inner form, are given by the following table:

(k, ϵ)	G	$\widehat{G}$	K	^{L}G
$k = k_0$ $\epsilon = 1$	$\mathrm{SO}(V)$, $\dim V = 2n+1$	$\mathrm{Sp}_{2n}(\mathbb{C})$	k_0	$\mathrm{Sp}_{2n}(\mathbb{C})$
$k = k_0$ $\epsilon = 1$	$\mathrm{SO}(V)$, $\dim V = 2n$	$\mathrm{SO}_{2n}(\mathbb{C})$	$k_0(\sqrt{\mathrm{disc}(V)})$	$\mathrm{O}_{2n}(\mathbb{C})$ $(\mathrm{disc}(V) \notin k^{\times 2})$ $\mathrm{SO}_{2n}(\mathbb{C})$ $(\mathrm{disc}(V) \in k^{\times 2})$
$k = k_0$ $\epsilon = -1$	$\mathrm{Sp}(V)$, $\dim V = 2n$	$\mathrm{SO}_{2n+1}(\mathbb{C})$	k_0	$\mathrm{SO}_{2n+1}(\mathbb{C})$
$k \neq k_0$ $\epsilon = \pm 1$	$\mathrm{U}(V)$, $\dim V = n$	$\mathrm{GL}_n(\mathbb{C})$	k	$\mathrm{GL}_n(\mathbb{C}) \rtimes \mathrm{Gal}(k/k_0)$

We make a few remarks on the table. Firstly, when $k = k_0$, the dual group $\widehat{G}$ is a special orthogonal or symplectic group and thus has a unique standard representation, as indicated in the table. However, when V is an even dimensional quadratic space and $\mathrm{disc}(V) \notin k^{\times 2}$, this standard representation has two extensions to ${}^L G$. In the above table, we have identified ${}^L G$ with $\mathrm{O}_{2n}(\mathbb{C})$ by regarding the nontrivial element in $\mathrm{Gal}(K/k_0)$ as a simple reflection in $\mathrm{O}_{2n}(\mathbb{C})$: this picks out one of these two extensions. Thus, when $k = k_0$, the L-group ${}^L G$ comes equipped with a standard self-dual representation M in each case.

Secondly, when $k \neq k_0$, we fix a standard representation M for the identity component ${}^L G^0 = \widehat{G}$ as follows. Extending scalars from k_0 to k, one has

$$\mathrm{U}(V) \times_{k_0} k \hookrightarrow \mathrm{Res}_{k/k_0}(\mathrm{GL}(V)) \times_{k_0} k \cong \mathrm{GL}(V) \times \mathrm{GL}(V^\sigma).$$

Via the first projection, one has a k-isomorphism

$$\mathrm{U}(V) \times_{k_0} k \cong \mathrm{GL}(V).$$

In the above, we have already fixed the L-group of $\mathrm{GL}(V)$ with a standard representation M. Thus, we see that ${}^L G = \mathrm{GL}(M) \rtimes \mathrm{Gal}(k/k_0)$ and ${}^L G^0 = \mathrm{GL}(M)$ comes equipped with a standard representation.

In addition, we note that the L-group $\mathrm{GL}(M) \rtimes \mathrm{Gal}(k/k_0)$ of $\mathrm{U}(V)$ is isomorphic as a complex Lie group to the L-group of the anisotropic real group $\mathrm{U}(n)$, associated to a definite hermitian space of the same dimension as V over $\mathbb{C}$. We will now use this fact to study the parity of selfdual complex representations of the L-group of $\mathrm{U}(V)$.

Let us make some general observations on the representation theory of the L-groups of anisotropic groups over $\mathbb{R}$. In this case, we have

$${}^L G = \widehat{G} \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R}),$$

where the Galois group acts by a pinned involution (possibly trivial), which maps to the opposition involution in $\mathrm{Out}(\widehat{G})$ and takes any representation V to its dual $V^\vee$. The pinning of $\widehat{G}$ gives a principal SL_2 in $\widehat{G}$, which is fixed by the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, so that we have

$$\delta : \mathrm{SL}_2 \times \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \widehat{G} \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

Let ϵ in $Z(\widehat{G})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$ be the image of $-I$ in SL_2. Then $\epsilon^2 = 1$ and ϵ acts as a scalar on every irreducible representation of ${}^L G$. The following result is due to Deligne.

Proposition 7.1. *Let G be an anisotropic group over $\mathbb{R}$. Then every complex representation N of ${}^L G$ is selfdual. If N is irreducible, the ${}^L G$-invariant pairing $\langle , \rangle_N : N \times N \to \mathbb{C}$ is unique up to scaling, and is $(\epsilon|N)$-symmetric.*

Proof. The proof is similar to Bourbaki [B, Ch. VIII, §7, Prop. 12 and Ex. 6], which treats the irreducible, selfdual representations M of $\widehat{G}$. There one restricts the pairing on M to the highest weight summand for the principal SL_2, which occurs with multiplicity one. In this case, we restrict the pairing on N to the subgroup $\mathrm{SL}_2 \times$

$\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, which again has an irreducible summand which occurs with multiplicity 1. The sign $(\epsilon|N)$ is the sign of $-I$ on this summand, which determines the sign of the pairing. □

If M is an irreducible representation of $\widehat{G}$ which is selfdual, then M extends in two ways to ${}^LG$. If M is *not* isomorphic to $M^\vee$, then the induced representation

$$N = \mathrm{Ind}(M)$$

of ${}^LG$ is irreducible. If we restrict N to $\widehat{G}$, it decomposes as a direct sum

$$M + \alpha \cdot M \simeq M + M^\vee$$

where α generates $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Since M is not isomorphic to $M^\vee$, the subspaces M and $\alpha \cdot M$ of N are isotropic for the pairing $\langle,\rangle_N$, which gives a non-degenerate pairing $\langle,\rangle_M : M \times M \to \mathbb{C}$ defined by

$$\langle m, m' \rangle_M = \langle m, \alpha m' \rangle_N. \tag{7.2}$$

This is a conjugate-duality on M : for g in $\widehat{G}$, we have

$$\begin{cases} \langle gm, \alpha g \alpha^{-1} m' \rangle_M = \langle m, m' \rangle_M \\ \langle m', m \rangle_M = (\epsilon|M) \cdot \langle m, m' \rangle_M \end{cases}$$

We now specialize these arguments to the case of unitary groups. Since the representation of the principal $\mathrm{SL}_2 \to \mathrm{GL}(M)$ on M is irreducible and isomorphic to Sym^{n-1}, we have

$$(\epsilon|M) = (-1)^{n-1}.$$

Hence the self-duality on $N = \mathrm{Ind}M$ and the conjugate-duality on M are $(-1)^{n-1}$-symmetric. In particular, we have

Proposition 7.3. *If $G = \mathrm{U}(V)$, with $\dim_k V = n$, then*

$${}^LG \hookrightarrow \begin{cases} \mathrm{Sp}(N) & = \mathrm{Sp}_{2n}(\mathbb{C}), \ \textit{if } n \textit{ is even;} \\ \mathrm{O}(N) & = \mathrm{O}_{2n}(\mathbb{C}), \ \textit{if } n \textit{ is odd.} \end{cases}$$

In each case, ${}^LG$ is identified with the normalizer of a Levi subgroup of a Siegel parabolic subgroup in $\mathrm{Sp}_{2n}(\mathbb{C})$ or $\mathrm{O}_{2n}(\mathbb{C})$.

Finally, it is instructive to describe the L-groups of the classical groups from the point of view of invariant theory. As we explain above, the L-groups of symplectic and special orthogonal groups $G(V)$ are themselves classical groups over $\mathbb{C}$ and have natural realizations as subgroups of $\mathrm{GL}(M)$ for complex vector spaces M of appropriate dimensions. These subgroups can be described as follows. One has a decomposition

$$M \otimes M \cong \mathrm{Sym}^2 M \bigoplus \bigwedge^2 M$$

of $\mathrm{GL}(M)$-modules. The action of $\mathrm{GL}(M)$ on $\mathrm{Sym}^2 M$ or $\wedge^2 M$ has a unique open orbit consisting of nondegenerate symmetric or skew-symmetric forms on $M^\vee$. Then we note:

(i) The stabilizer of a nondegenerate vector B in $\mathrm{Sym}^2 M$ (resp. $\wedge^2 M$) is the orthogonal group $\mathrm{O}(M, B)$ (resp. the symplectic group $\mathrm{Sp}(M, B)$); these groups exhaust the L-groups of symplectic and orthogonal groups.

(ii) The action of this stabilizer on the other representation $\wedge^2 M$ (resp. $\mathrm{Sym}^2 M$) is its adjoint representation.

(iii) The two representations $\mathrm{Sym}^2 M$ and $\wedge^2 M$ are also useful for characterizing the selfdual representations of $WD(k)$ introduced in §3: a representation M of $WD(k)$ is orthogonal (resp. symplectic) if and only if $WD(k)$ fixes a nondegenerate vector in $\mathrm{Sym}^2 M$ (resp. $\wedge^2 M$).

These rather obvious remarks have analogs for the unitary group $\mathrm{U}(V)$, which we now describe. Suppose that the L-group of $\mathrm{U}(V)$ is $\mathrm{GL}(M) \rtimes \mathrm{Gal}(k/k_0)$. Consider the semi-direct product

$$H = (\mathrm{GL}(M) \times \mathrm{GL}(M)) \rtimes \mathbb{Z}/2\mathbb{Z}$$

where $\mathbb{Z}/2\mathbb{Z}$ acts by permuting the two factors of $\mathrm{GL}(M)$; this is the L-group of $\mathrm{Res}_{k/k_0}(\mathrm{GL}(V/k))$ with $\dim_k V = \dim M$. The irreducible representation $M \boxtimes M$ of $H^0 = \mathrm{GL}(M) \times \mathrm{GL}(M)$ is invariant under $\mathbb{Z}/2\mathbb{Z}$ and thus has two extensions to H. In one such extension, the group $\mathbb{Z}/2\mathbb{Z} = S_2$ simply acts by permuting the two copies of M; the other extension is then given by twisting by the nontrivial character of H/H^0. In honor of Asai, we denote these two extensions by $\mathrm{As}^+(M)$ and $\mathrm{As}^-(M)$ respectively. They can be distinguished by

$$\mathrm{Trace}(c|\mathrm{As}^+(M)) = \dim M \quad \text{and} \quad \mathrm{Trace}(c|\mathrm{As}^-(M)) = -\dim M,$$

where c is the nontrivial element in $\mathbb{Z}/2\mathbb{Z}$. One has

$$\mathrm{Ind}_{H^0}^{H}(M \boxtimes M) = \mathrm{As}^+(M) \bigoplus \mathrm{As}^-(M).$$

The action of H^0 on $\mathrm{As}^\pm(M)$ has an open dense orbit, consisting of isomorphisms $M^\vee \to M$ and whose elements we call nondegenerate. Now we have

Proposition 7.4. *If* $\dim M = n$*, then the stabilizer in* H *of a nondegenerate vector in* $\mathrm{As}^{(-1)^{n-1}}(M)$ *is isomorphic as a complex Lie group to the* L*-group of* $\mathrm{U}(V)$*. Moreover, the action of this stabilizer on the other representation* $\mathrm{As}^{(-1)^n}(M)$ *is the adjoint representation of* ${}^L U(V)$*.*

The representations $\mathrm{As}^\pm(M)$ are also useful for characterizing conjugate-dual representations of $WD(k)$, which were discussed in §3. Indeed, given a representation

$$\varphi : WD(k) \to \mathrm{GL}(M),$$

one obtains a map

$$\tilde{\varphi} : WD(k_0) \to H$$

by setting

$$\tilde{\varphi}(\tau) = (\varphi(\tau), \varphi(s\tau s^{-1})) \in \mathrm{GL}(M) \times \mathrm{GL}(M),$$

for $\tau \in WD(k)$, and

$$\tilde{\varphi}(s) = (1, \varphi(s^2)) \cdot c \in H \smallsetminus H^0.$$

The choice of s is unimportant, since the maps $\tilde{\varphi}$'s thus obtained for different choices of s are naturally conjugate under H^0. Through this map, $WD(k_0)$ acts on $\mathrm{As}^{\pm}(M)$. In fact, the representation $\mathrm{As}^{+}(M)$ of $WD(k_0)$ is obtained from M by the process of *multiplicative induction* [P2] or *twisted tensor product*; it is an extension of the representation $M \otimes M^{\sigma}$ of $WD(k)$ to $WD(k_0)$, and $\mathrm{As}^{-}(M)$ is the twist of $\mathrm{As}^{+}(M)$ by the quadratic character ω_{k/k_0} associated to the quadratic extension k/k_0.

Now we have:

Proposition 7.5. *If M is a representation of $WD(k)$, then*

(i) M is conjugate-orthogonal if and only if $WD(k_0)$ fixes a nondegenerate vector in $\mathrm{As}^{+}(M)$. *When M is irreducible, this is equivalent to* $\mathrm{As}^{+}(M)^{WD(k_0)} \neq 0$.

(ii) M is conjugate-symplectic if and only if $WD(k_0)$ fixes a nondegenerate vector in $\mathrm{As}^{-}(M)$. *When M is irreducible, this is equivalent to* $\mathrm{As}^{-}(M)^{WD(k_0)} \neq 0$.

8. Langlands parameters for classical groups

In this section, we discuss the Langlands parameters of classical groups. In particular, we show that these Langlands parameters can be understood in terms of selfdual or conjugate-dual representations M of $WD(k)$.

If G is a connected, reductive group over k_0, a Langlands parameter is a homomorphism

$$\varphi : WD(k_0) \to {}^{L}G = \widehat{G} \rtimes \mathrm{Gal}(K/k_0).$$

This homomorphism is required to be continuous on $WD(k_0) = W(k_0)$ when $k_0 = \mathbb{R}$ or $\mathbb{C}$. In the non-archimedean case, $WD(k_0) = W(k_0) \times \mathrm{SL}_2(\mathbb{C})$ and φ is required to be trivial on an open subgroup of the inertia group in $W(k_0)$, the image of Frobenius is required to be semi-simple and the restriction of φ to $\mathrm{SL}_2(\mathbb{C})$ is required to be algebraic. In all cases, the projection onto $\mathrm{Gal}(K/k_0)$ is the natural map $W(k_0)/W(K) \to \mathrm{Gal}(K/k_0)$. Finally, two Langlands parameters are considered equivalent if they are conjugate by an element in $\widehat{G}$.

Associated to any Langlands parameter is the reductive group

$$C_{\varphi} \subset \widehat{G}$$

which centralizes the image, and its component group

$$A_\varphi = C_\varphi / C_\varphi^0.$$

The isomorphism class of both C_φ and A_φ are determined by the equivalence class of the parameter φ.

For $G = \mathrm{GL}(V/k_0)$, we have $\widehat{G} = \mathrm{GL}(M)$ with $\dim M = \dim V$. If $\langle e_1, \cdots, e_n \rangle$ is the basis of the character group of a maximal torus $T \subset G$ given by the weights of V, then the weights of the dual torus $\widehat{T}$ on M are the dual basis $\langle e_1^\vee, \cdots, e_n^\vee \rangle$. The Langlands parameters for G are simply equivalence classes of representations of $WD(k_0)$ on M.

Now assume that $G \subset \mathrm{GL}(V/k)$ is a connected classical group, defined by a σ-sesquilinear form $\langle -, - \rangle : V \times V \to k$ of sign ϵ. Recall that the L-group of G or its identity component $\widehat{G}$ comes equipped with a standard representation M. We will see that for each classical group G, a Langlands parameter φ for G corresponds to a natural complex representation

$$WD(k) \to \mathrm{GL}(M)$$

with some additional structure, as given in the following theorem.

Theorem 8.1. *(i) A Langlands parameter φ of the connected classical group $G \subset \mathrm{GL}(V/k)$ determines a selfdual or conjugate-dual representation M of $WD(k)$, with the following structure:*

G	$\dim(V)$	M	$\dim M$	
$\mathrm{Sp}(V)$	$2n$	Orthogonal	$2n+1$	$\det M = 1$
$\mathrm{SO}(V)$	$2n+1$	Symplectic	$2n$	
$\mathrm{SO}(V)$	$2n$	Orthogonal	$2n$	$\det M = \operatorname{disc} V$
$\mathrm{U}(V)$	$2n+1$	Conjugate-orthogonal	$2n+1$	
$\mathrm{U}(V)$	$2n$	Conjugate-symplectic	$2n$	

(ii) The isomorphism class of the representation M determines the equivalence class of the parameter φ, except in the case when M is orthogonal and every irreducible orthogonal summand M_i of M has even dimension. In the exceptional case, M and V have even dimension and there are two equivalence classes $\{\varphi, \varphi'\}$ of parameters for $\mathrm{SO}(V)$ which give rise to the same orthogonal representation M.

(iii) In the unitary cases, the group $C_\varphi \subset \widehat{G}$ which centralizes the image of φ is isomorphic to the group C of elements a in $\mathrm{Aut}(M,B)$ *which centralize the image* $WD(k) \to \mathrm{GL}(M)$*. In the orthogonal and symplectic cases, the group $C_\varphi \subset \widehat{G}$ is isomorphic to the subgroup C^+ of C, consisting of those elements of* $\mathrm{Aut}(M,B)$ *which satisfy* $\det(a|M) = 1$.

Proof. This is well-known if G is an orthogonal or symplectic group. Indeed, the L-group ${}^LG$ is essentially the automorphism group of a nondegenerate symmetric or skew-symmetric bilinear form B on a complex vector space M of appropriate dimension. So the theorem amounts to the assertion that if two homomorphisms $WD(k) \to \mathrm{Aut}(M,B) \subset \mathrm{GL}(M)$ are conjugate in $\mathrm{GL}(M)$, then they are conjugate in $\mathrm{Aut}(M,B)$. This is the content of Lemma 3.1. Moreover, the description of the component group C_φ follows directly from the results of §4.

Henceforth we shall focus on the unitary case. For $G = \mathrm{U}(V)$, a parameter is a homomorphism
$$\varphi : WD(k_0) \to {}^LG = \mathrm{GL}(M) \rtimes \mathrm{Gal}(k/k_0)$$
with $\dim M = \dim V = n$. The restriction of φ to $WD(k)$ gives the desired representation M. We must next show that M is conjugate-dual with sign $(-1)^{n-1}$.

If $s \in W(k)$ generates the quotient $WD(k_0)/WD(k)$, then
$$\varphi(s) = (A, \alpha) = (A, 1) \cdot (1, \alpha) \text{ in } {}^LG$$
with A in $\mathrm{GL}(M)$. In the previous section (cf. equation (7.2)), we have seen that the standard representation M of $\mathrm{GL}(M)$ has a conjugate-duality $\langle -, - \rangle_M$ of sign $(-1)^{n-1}$ with respect to the nontrivial element $\alpha \in \mathrm{Gal}(k/k_0) \subset {}^LG$. We define the bilinear form
$$B(m, m') = B_s(m, m') = \langle m, A^{-1}m' \rangle_M,$$
Then the form B is non-degenerate on M and satisfies
$$B(\tau m, s\tau s^{-1} m') = B(m, m')$$
for all τ in $WD(k)$, and
$$B(m', m) = (-1)^{n-1} \cdot B(m, s^2 m').$$
Hence, as a representation of $WD(k)$, M is conjugate-dual with sign $= (-1)^{n-1}$.

It is clear that the conjugation of a parameter φ by an element of $\mathrm{GL}(M)$ gives an isomorphism of the associated conjugate-dual representations. Hence we are reduced to showing that every conjugate-dual representation M of sign $= (-1)^{n-1}$ extends to a Langlands parameter φ of $WD(k_0)$, and that the isomorphism class of M determines the equivalence class of φ.

Suppose then that M is a conjugate-dual representation of $WD(k)$ of sign $= (-1)^{n-1}$ with $n = \dim M$. To obtain an extension, consider the induced representation $N =$

$\mathrm{Ind}(M)$ of $WD(k_0)$. By Lemma 3.5(i), N is selfdual of sign $= (-1)^{n-1}$. Moreover, the proof of Lemma 3.5 shows that the image of $WD(k_0)$ in $\mathrm{Sp}(4d)$ or $\mathrm{O}(4d+2)$ (depending on whether $n = 2d$ or $n = 2d+1$) is contained in the normalizer of a Levi subgroup in a Siegel parabolic subgroup. By Proposition 7.3, this normalizer is isomorphic to the L-group of $\mathrm{U}(V)$: it splits as a semi-direct product $\mathrm{GL}(M) \rtimes \langle\alpha\rangle$, with $\det(\alpha|N) = (-1)^n$. Thus, we have produced an L-parameter for $\mathrm{U}(V)$ whose restriction to $WD(k)$ is the given M.

Finally we need to show that the extension obtained above is unique, up to conjugacy by $\widehat{G}$. If φ and φ' are two parameters extending $\rho : WD(k) \to \mathrm{GL}(M)$, we must show that the elements

$$\begin{aligned} \varphi(s) &= (A, \alpha) \\ \varphi'(s) &= (A', \alpha) \text{ of } {}^L G \end{aligned}$$

are conjugate by an element of $\mathrm{GL}(M)$ centralizing the image of ρ. The bilinear forms

$$\begin{aligned} B(m, m') &= \langle m, A^{-1} m' \rangle_M \\ B'(m, m') &= \langle m, (A')^{-1} m' \rangle_M \end{aligned}$$

give two conjugate-dualities of M which are preserved by $WD(k)$ and non-degenerate with sign $(-1)^{n-1}$. By Lemma 3.3, there is an element T in $\mathrm{GL}(M)$ centralizing the image of ρ with

$$B'(m, m') = B(Tm, Tm').$$

This gives the identity

$$\langle m, (A')^{-1} m' \rangle = \langle m, (T^{-1})^{\alpha} A^{-1} T m' \rangle$$

for all m and m'. Hence

$$A' = T^{-1} A T^{\alpha}$$

and the elements are conjugate by the element $T \in \mathrm{GL}(M) = \widehat{G}$.

The argument identifying the group C_φ with either the group C or C^+ associated to (M, B) is contained in §4. This completes the proof of the theorem. □

Corollary 8.2. *A representation M of $WD(k)$ gives rise to a Langlands parameter for a quasi-split unitary group $\mathrm{U}(V)$ if and only if $WD(k_0)$ fixes a non-degenerate vector in $\mathrm{As}^{(-1)^{n-1}}(M)$, with $n = \dim M$.*

Proof. This is a consequence of the theorem and Proposition 7.5. □

Remark: When M is orthogonal of even dimension, it is often convenient to view it as defining a unique Langlands parameter for the full orthogonal group $\mathrm{O}(V)$ (which is not connected), with the equivalence being defined by $\mathrm{O}(M)$-conjugacy; see [P1].

We conclude this section by recalling certain simple invariants of the representation M of $WD(k)$. For $G = \mathrm{GL}(V)$, we have the character

$$\det M : k^\times \to \mathbb{C}^\times.$$

For $G = \mathrm{U}(V)$, we obtain the character

$$\det M : k^\times / k_0^\times \to \mathbb{C}^\times$$

as the sign of $\det M$ is $(-1)^{n(n-1)} = +1$. Finally, for $G = \mathrm{Sp}(V)$ or $G = \mathrm{SO}(V)$ with $\dim(V)$ even and $\mathrm{disc}(V) = 1$, the representation M is orthogonal with $\det(M) = 1$. Hence we have the root number $\epsilon(M) = \epsilon(M, \psi)$ independent of the additive character ψ of k, and

$$\epsilon(M) = \pm 1.$$

We will relate these invariants to the central characters of certain representations of G after introducing Vogan L-packets in the next section.

9. Vogan L-packets - Desiderata

Let G be a quasi-split, connected, reductive group over a local field k_0. In this section, we will discuss Langlands parameters φ as the conjectural parameters for the isomorphism classes of irreducible smooth admissible complex representations of the locally compact group $G(k_0)$. Before coming to that, we briefly recall the notions of smooth and admissible representations of $G(k_0)$ when k_0 is a local field.

When k_0 is local and discretely valued, a smooth representation π is simply a homomorphism

$$\pi : G(k_0) \longrightarrow \mathrm{GL}(E)$$

for a complex vector space E (possibly infinite-dimensional) such that

$$E = \cup_K E^K,$$

where the union is over all open compact subgroups K of $G(k_0)$. Such a smooth representation is admissible if E^K is finite dimensional for any open compact subgroup K. A homomorphism from (π, E) to (π', E') is simply a linear map $E \longrightarrow E'$ which commutes with the action of $G(k_0)$.

When $k_0 = \mathbb{R}$ or $\mathbb{C}$, we will consider the category of smooth Frechet representations (π, E) of moderate growth, as introduced by Casselman [C] and Wallach [W1]. An admissible representation is such a representation whose subspace of K-finite vectors (where K is a maximal compact subgroup of $G(k_0)$) is the direct sum of irreducible representations of K with finite multiplicities. A homomorphism $(\pi, E) \longrightarrow (\pi', E')$ is a continuous linear map $E \longrightarrow E'$ which commutes with the action of $G(k_0)$.

We come now to the local Langlands conjecture. We shall present this conjecture in a form proposed by Vogan [Vo], which treats representations π of all pure inner forms G' of G simultaneously. Recall from Section 2 that the pure inner forms of G are the groups G' over k_0 which are obtained from G via inner twisting by elements in the finite pointed set $H^1(k_0, G)$.

All of the pure inner forms G' of G have the same center Z over k_0, and each irreducible representation π of $G'(k_0)$ has a central character

$$\omega_\pi : Z(k_0) \to \mathbb{C}^\times.$$

The adjoint group $G'_{ad}(k_0)$ acts on $G'(k_0)$ by conjugation, and hence acts on the set of its irreducible complex representations. The quotient group $G'_{ad}(k_0)/\mathrm{Im}\ G'(k_0)$ acts on the set of isomorphism classes of representations of $G'(k_0)$. This quotient is abelian, and canonically isomorphic to the cohomology group

$$E = G'_{ad}(k_0)/\mathrm{Im}\ G'(k_0) = \ker(H^1(k_0, Z) \to H^1(k_0, G')).$$

It can be seen that the group E is independent of the choice of the inner form G' of G.

Let B be a Borel subgroup of G over k_0, with unipotent radical N. The quotient torus $T = B/N$ acts on the group $\mathrm{Hom}(N, \mathbb{C}^\times)$. We call a character $\theta : N(k_0) \to \mathbb{C}^\times$ *generic* if its stabilizer in $T(k_0)$ is equal to the center $Z(k_0)$. If π is an irreducible representation of $G(k_0)$ and θ is a generic character, then the complex vector space $\mathrm{Hom}_{N(k_0)}(\pi, \theta)$ has dimension ≤ 1. When the dimension is 1, we say π is θ-generic. This depends only on the $T(k_0)$-orbit of θ.

When $Z = 1$, the group $T(k_0)$ acts simply-transitively on the set of generic characters. In general, the set D of $T(k_0)$-orbits on the set of all generic characters θ of $N(k_0)$ forms a principal homogeneous space for the abelian group E':

$$E' = T_{ad}(k_0)/\mathrm{Im}\ T(k_0) = \ker(H^1(k_0, Z) \to H^1(k_0, T)).$$

By Lemma 16.3.6(iii) of [Sp],

$$T_{ad}(k_0)/\mathrm{Im}\ T(k_0) = G_{ad}(k_0)/\mathrm{Im}\ G(k_0),$$

hence we have the equality $E' = E$

We are now ready to describe the desiderata for Vogan L-packets, which will be assumed in the rest of this paper. These properties are known to hold for the groups $\mathrm{GL}(V)$ and $\mathrm{SL}(V)$, as well as for some classical groups of small rank.

(1) Every irreducible representation π of $G'(k_0)$ (up to isomorphism) determines a Langlands parameter

$$\varphi : WD(k_0) \to \widehat{G} \rtimes \mathrm{Gal}(K/k_0)$$

(up to equivalence).

Each Langlands parameter φ for G corresponds to a finite set Π_φ of irreducible representations of $G(k_0)$ and its pure inner forms $G'(k_0)$. Moreover, the cardinality of the finite set Π_φ is equal to the number of irreducible representations χ of the finite group $A_\varphi = \pi_0(C_\varphi)$.

(2) Each choice of a $T(k_0)$-orbit θ of generic characters for $G(k_0)$ gives a bijection of finite sets

$$J(\theta) : \Pi_\varphi \to \mathrm{Irr}(A_\varphi).$$

For archimedean v, this bijection was established by Vogan in [Vo, Thm. 6.3].

The L-packet Π_φ contains at most one θ'-generic representation, for each $T(k_0)$-orbit of generic characters θ' of $G(k_0)$. We conjectured in [GP1, Conjecture 2.6] that the L-packet Π_φ contains a generic representation if and only if the adjoint L-function $L(\varphi, Ad, s)$ is regular at the point $s = 1$. In this case, we say the L-packet Π_φ is generic.

Assume that the L-packet Π_φ is generic. In the bijection $J(\theta)$, the unique θ-generic representation π in Π_φ corresponds to the trivial representation of A_φ. The θ'-generic representations correspond to the one dimensional representations η_g described below.

(3) The finite set Π_φ of irreducible representations of $G(k_0)$ is stable under the adjoint action of $G^{ad}(k_0)$, which permutes the different generic representations for $G(k_0)$ in an L-packet transitively.

In any of the bijections $J(\theta)$, the action of $g \in G'_{ad}(k_0)$ on $\mathrm{Irr}(A_\varphi)$ is given by tensor product with the one-dimensional representation η_g of A_φ alluded to above.

More precisely, Tate local duality gives a perfect pairing

$$H^1(k_0, Z) \times H^1(K/k_0, \pi_1(\widehat{G})) \to \mathbb{C}^\times.$$

The coboundary map $C_\varphi \to H^1(K/k_0, \pi_1(\widehat{G}))$ factors through the quotient A_φ, and gives a pairing

$$H^1(k_0, Z) \times A_\varphi \to \mathbb{C}^\times.$$

The adjoint action by the element g in $G'_{ad}(k_0) \to H^1(k_0, Z)$, viewed as a one dimensional representation η_g of A_φ, will take $\pi(\varphi, \chi)$ to the representation $\pi(\varphi, \chi \otimes \eta_g)$.

(4) In any of the bijections $J(\theta)$, the pure inner form which acts on the representation with parameter (φ, χ) is constrained by the restriction of the irreducible representation χ to the image of the group $\pi_0(Z(\widehat{G}))^{\mathrm{Gal}(K/k_0)}$ in A_φ.

More precisely, when $k_0 \neq \mathbb{R}$, Kottwitz has identified the pointed set $H^1(k_0, G)$ with the group of characters of the component group of $Z(\widehat{G})^{\mathrm{Gal}(K/k_0)}$. The inclusion

$$Z(\widehat{G})^{\mathrm{Gal}(K/k_0)} \to C_\varphi$$

induces a map on component groups, whose image is central in A_φ. Hence an irreducible representation χ of A_φ has a central character on $\pi_0(Z(\widehat{G}))^{\mathrm{Gal}(K/k_0)}$, and determines a class in $H^1(k_0, G)$. This is the pure inner form G' that acts on the representation $\pi(\varphi, \chi)$.

(5) All of the irreducible representations π in Π_φ have the same central character ω_π. This character is determined by φ, using the recipe in [GR, §8].

10. Vogan L-packets for the classical groups

We now make the desiderata of Vogan L-packets completely explicit for the classical groups $G \subset \mathrm{GL}(V/k)$. We have already described the Langlands parameters φ for G explicitly, as certain representations M of $WD(k)$, in Section 6. In all cases, the component group A_φ is an elementary abelian 2-group, so $\mathrm{Irr}(\mathrm{A}_\varphi) = \mathrm{Hom}(\mathrm{A}_\varphi, \pm 1)$. We treat each family of groups in turn.

The General Linear Group $G = \mathrm{GL}(V)$

(1) A Langlands parameter is a representation M of $WD(k)$, with $\dim(M) = \dim(V)$. The group $C_\varphi = C(M)$ is connected, so $A_\varphi = 1$. Hence Π_φ consists of a single element. In this case, the full Langlands conjecture is known (by [HT] and [He]).

(2) There is a unique T-orbit on the generic characters, and the regularity of the adjoint L-function of φ at $s = 1$ detects generic L-packets Π_φ.

(3) The adjoint action is trivial, as the center Z of G has trivial first cohomology.

(4) The only pure inner form is $G = \mathrm{GL}(V)$.

(5) The center $Z(k) = k^\times$, and the central character of $\pi(\varphi)$ has parameter $\det(M)$.

The Symplectic Group $G = \mathrm{Sp}(V)$

(1) A Langlands parameter is an orthogonal representation M of $WD(k)$, with

$$\dim(M) = \dim(V) + 1 \quad \text{and} \quad \det(M) = 1.$$

The group $A_\varphi = A_M^+$ has order 2^{m-1}, where m is the number of distinct irreducible orthogonal summands M_i in M. The full Langlands conjecture is known when $\dim(V) = 2$ [LL] or 4 [GT2].

(2) The set D of T-orbits on generic characters is a principal homogeneous space for the group $E = H^1(k, Z) = k^\times / k^{\times 2}$. We will see in §12 that the choice of the symplectic space V identifies the set D with the set of $k^{\times 2}$-orbits on the nontrivial additive characters ψ of k.

(3) The adjoint action is via elements c in the group $E = k^\times/k^{\times 2}$. This acts on the irreducible representations of A_φ via tensor product with the character $\eta_c(a) = \det(M^a)(c)$, and on the set D of orbits of generic characters by mapping $\psi(x)$ to $\psi(cx)$.

(4) The only pure inner form is $G = \mathrm{Sp}(V)$.

(5) The center $Z(k) = \langle \pm 1 \rangle$, and the central character of $\pi(\varphi)$ maps the element -1 in $Z(k)$ to the local root number $\epsilon(M)$.

The Odd Special Orthogonal Group $G = \mathrm{SO}(V)$**,** $\dim(V) = 2n+1$

(1) A Langlands parameter is a symplectic representation M of $WD(k)$, with $\dim(M) = \dim(V)-1$. The group $A_\varphi = A_M$ has order 2^m, where m is the number of distinct irreducible symplectic summands M_i in M. The full Langlands conjecture is known when $\dim(V) = 3$ [Ku] or 5 [GT1].

(2) Since G is an adjoint group, there is a unique T-orbit on the set of generic characters, and hence a single natural bijection $J : \Pi_\varphi \to \mathrm{Hom}(A_\varphi, \pm 1)$.

(3) The adjoint action on the L-packet is trivial.

(4) The pure inner forms of G are the groups $G' = \mathrm{SO}(V')$, where V' is an orthogonal space over k with $\dim(V') = \dim(V)$ and $\mathrm{disc}(V') = \mathrm{disc}(V)$ [KMRT, (29.29)].

If k is non-archimedean and $n \geq 1$, there is a unique non-split pure inner form G', which has k-rank $(n-1)$. The representation $\pi(\varphi, \chi)$ is a representation of G if $\chi(-1) = +1$ and a representation of G' if $\chi(-1) = -1$. If $k = \mathbb{R}$ and $G = \mathrm{SO}(p, q)$, then the pure inner forms are the groups $G' = \mathrm{SO}(p', q')$ with $q' \equiv q \mod 2$, and $\pi(\varphi, \chi)$ is a representation of one of the groups G' with $(-1)^{(q-q')/2} = \chi(-1)$.

(5) The center Z of G is trivial.

The Even Special Orthogonal Group $G = \mathrm{SO}(V)$**,** $\dim(V) = 2n$**,** $\mathrm{disc}(V) = d$

(1) A Langlands parameter determines an orthogonal representation M of $WD(k)$, with

$$\dim(M) = \dim(V) \quad \text{and} \quad \det(M) = \mathbb{C}(d).$$

The group $A_\varphi = A_M^+$ has order 2^m, where m is either the number of distinct irreducible orthogonal summands M_i in M, or the number of distinct orthogonal summands minus 1.

The latter case occurs if some irreducible orthogonal summand M_i has odd dimension (in which case the orthogonal representation M determines the parameter φ). If every irreducible orthogonal summand M_i of M has even dimension, then $A_M^+ = A_M$ and there are two parameters $\{\phi, \phi^*\}$ which determine the same orthogonal representation M. The representations $\pi(\phi, \chi)$ and $\pi(\phi^*, \chi)$ are conjugate under the outer action of $\mathrm{O}(V)$ on $\mathrm{SO}(V)$.

The full Langlands conjecture is known when $\dim(V) = 2$ or 4 or 6.

(2) The set D of T-orbits on generic characters is a principal homogeneous space for the group $E = \mathbb{N}K^\times/k^{\times 2}$, where K is the splitting field of G. We will see in §12 that the choice of the orthogonal space V identifies the set D with the set of G-orbits on the set of non-isotropic lines $L \subset V$, such that the space $L^\perp$ is split.

(3) The adjoint action is via elements c in the group $E = \mathbb{N}K^\times/k^{\times 2}$. This acts on the irreducible representations of A_φ via tensor product with the character $\eta_c(a) = \det(M^a)(c)$, and on the set D of orbits of generic characters by mapping a line $L = kv$ with $\langle v, v\rangle = \alpha$ in $k^\times$ to a line $L' = kv'$ with $\langle v', v'\rangle = c \cdot \alpha$.

(4) The pure inner forms of G are the groups $G' = \mathrm{SO}(V')$, where V' is an orthogonal space over k with $\dim(V') = \dim(V)$ and $\mathrm{disc}(V') = \mathrm{disc}(V)$ [KMRT, (29.29)].

If k is non-archimedean and V is not the split orthogonal space of dimension 2, there is a unique pure inner form G', such that the Hasse-Witt invariant of V' is distinct from the Hasse-Witt invariant of V. The representation $\pi(\varphi, \chi)$ is a representation of G if $\chi(-1) = +1$ and a representation of G' if $\chi(-1) = -1$. If $k = \mathbb{R}$ and $G = \mathrm{SO}(p, q)$, then the pure inner forms are the groups $G' = \mathrm{SO}(p', q')$ with $q' \equiv q \mod 2$, and $\pi(\varphi, \chi)$ is a representation of one of the groups G' with $(-1)^{(q-q')/2} = \chi(-1)$.

(5) If $\dim(V) = 2$, then $Z = G$. If $\dim(V) \geq 4$, then $Z(k) = \langle \pm 1\rangle$ and the central character of $\pi(\varphi)$ maps the element -1 in $Z(k)$ to $\epsilon(M, \psi)/\epsilon(\det M, \psi)$.

The Odd Unitary Group $G = \mathrm{U}(V)$, $\dim V = 2n+1$

(1) A Langlands parameter is a conjugate-orthogonal representation M of $WD(k)$, with $\dim(M) = \dim(V)$. The group $A_\varphi = A_M$ has order 2^m, where m is the number of distinct irreducible conjugate-orthogonal summands M_i in M. The full Langlands conjecture is known when $\dim(V) = 1$ or 3 [Ro].

(2) There is a unique T-orbit on the set of generic characters, and hence a single natural isomorphism $J : \Pi_\varphi \to \mathrm{Hom}(\mathrm{A}_\varphi, \pm 1)$.

(3) The adjoint action on the L-packet is trivial.

(4) The pure inner forms of G are the groups $G' = \mathrm{U}(V')$, where V' is a hermitian (or skew-hermitian) space over k with $\dim(V') = \dim(V)$ [KMRT, (29.19)].

If k_0 is non-archimedean, there is a unique pure inner form G' such that the discriminant of V' is distinct from the discriminant of V. The representation $\pi(\varphi, \chi)$ is a representation of G if $\chi(-1) = +1$ and a representation of G' if $\chi(-1) = -1$. If $k_0 = \mathbb{R}$ and $G = \mathrm{U}(p, q)$, then the pure inner forms are the groups $G' = \mathrm{U}(p', q')$, and $\pi(\varphi, \chi)$ is a representation of one of the groups G' with $(-1)^{q-q'} = \chi(-1)$.

(5) The center $Z(k_0) = k^\times / k_0^\times = \mathrm{U}(1)$, and the central character of $\pi(\varphi, \chi)$ has parameter $\det(M)$.

The Even Unitary Group $G = \mathrm{U}(V)$, $\dim V = 2n$

(1) A Langlands parameter is a conjugate-symplectic representation M of $WD(k)$, with $\dim(M) = \dim(V)$. The group $A_\varphi = A_M$ has order 2^m, where m is the number of distinct irreducible conjugate-symplectic summands M_i in M. The full Langlands conjecture is known when $\dim(V) = 2$ [Ro].

(2) The set D of T-orbits on generic characters is a principal homogeneous space for the group $E = H^1(k, Z) = k_0^\times / \mathbb{N}k^\times$ of order 2. We will see in §12 that the choice of a hermitian space V identifies the set D with the set of $\mathbb{N}k^\times$-orbits on the nontrivial additive characters ψ of k/k_0. Similarly, the choice of a skew-hermitian space V identifies the set D with the set of $\mathbb{N}k^\times$-orbits on the nontrivial additive characters ψ_0 of k_0.

(3) The adjoint action is via elements c in the group $E = k_0^\times / \mathbb{N}k^\times$. The nontrivial class c acts on the irreducible representations of A_φ via tensor product with the character $\eta(a) = (-1)^{\dim(M^a)}$, and on the set D of orbits of generic characters by mapping $\psi(x)$ to $\psi(cx)$, or $\psi_0(x)$ to $\psi_0(cx)$.

(4) The pure inner forms of G are the groups $G' = \mathrm{U}(V')$, where V' is a hermitian (or skew-hermitian) space over k with $\dim(V') = \dim(V)$ [KMRT, (29.19)].

If k_0 is non-archimedean, there is a unique pure inner form G' such that the discriminant of V' is distinct from the discriminant of V. The representation $\pi(\varphi, \chi)$ is a representation of G if $\chi(-1) = +1$ and a representation of G' if $\chi(-1) = -1$. If $k_0 = \mathbb{R}$ and $G = \mathrm{U}(p, q)$, then the pure inner forms are the groups $G' = \mathrm{U}(p', q')$, and $\pi(\varphi, \chi)$ is a representation of one of the groups G' with $(-1)^{q-q'} = \chi(-1)$.

(5) The center $Z(k_0) = k^\times/k_0^\times = \mathrm{U}(1)$, and the central character of $\pi(\varphi, \chi)$ has parameter $\det(M)$.

The forthcoming book of Arthur [A3] and the papers [Mo1, Mo2] of Moeglin should establish most of the above expectations.

11. Vogan L-packets for the metaplectic group

Let $(W, \langle -, - \rangle_W)$ be a symplectic space of dimension $2n \geq 0$ over the local field k. We assume, as usual, that $\mathrm{char}(k) \neq 2$. In this section, we also assume that $k \neq \mathbb{C}$.

Let $\widetilde{\mathrm{Sp}}(W)$ denote the nontrivial double cover of the symplectic group $\mathrm{Sp}(W)(k)$. We will use the Howe duality correspondence (also known as the theta correspondence) to describe the (genuine) representation theory of $\widetilde{\mathrm{Sp}}(W)$ in terms of the representation theory of the groups $\mathrm{SO}(V)$ over k, with $\dim V = 2n+1$. Assuming the Langlands-Vogan parametrization of irreducible representations of $\mathrm{SO}(V)$ over k, with $\dim V = 2n+1$, we then obtain a notion of Vogan L-packets for the genuine irreducible representations $\widetilde{\pi}$ of $\widetilde{\mathrm{Sp}}(W)$. More precisely, the Langlands parameter of a genuine representation of $\widetilde{\mathrm{Sp}}(W)$ will be a symplectic representation

$$\varphi : WD(k) \to \mathrm{Sp}(M) \quad \text{with} \quad \dim M = 2n,$$

and the individual representations $\widetilde{\pi}(\varphi, \chi)$ in the Vogan packet Π_φ will be indexed by quadratic characters

$$\chi : A_\varphi = A_M \to \langle \pm 1 \rangle.$$

This parametrization of the irreducible genuine representations of $\widetilde{\mathrm{Sp}}(W)$ will depend on the choice of a nontrivial additive character ψ of k, up to multiplication by $k^{\times 2}$. As we shall see in §12, such an orbit of additive characters ψ determines an orbit of generic characters $\theta : N \to \mathbb{C}^\times$ for $\mathrm{Sp}(W)$. The character θ also determines a character $\widetilde{\theta}$ of the unipotent radical $\widetilde{N} \simeq N$ of $\widetilde{\mathrm{Sp}}(W)$. Our parametrization is normalized so that for generic parameters φ, the unique representation $\widetilde{\pi} \in \Pi_\varphi$ which is $\widetilde{\theta}$-generic corresponds to the trivial character $\chi = 1$ of A_φ. Such a dependence of the Langlands parametrization on the choice of an additive character ψ is already present in the case of the linear classical groups discussed in the previous section (through fixing of a generic character on a quasi-split form of the group). For the metaplectic groups, the dependence is more serious: even the Langlands parameter φ associated to $\widetilde{\pi}$ depends on the choice of ψ.

To define the parameters (φ, χ) of $\widetilde{\pi}$, we let (V, q) be a quadratic space over k with

$$\dim V = 2n + 1 \quad \text{and} \quad \mathrm{disc}(V) = (-1)^n \det(V) \equiv 1 \in k^\times / k^{\times 2}.$$

Note that the discriminant above refers to the discriminant of the *quadratic* space (V, q). The quadratic form q on V gives rise to a symmetric bilinear form

$$\langle x, y \rangle_V = q(x + y) - q(x) - q(y)$$

so that

$$\langle x, x \rangle_V = 2 \cdot q(x),$$

and

$$\mathrm{disc}(V, \langle -, - \rangle_V) = 2 \cdot \mathrm{disc}(V, q) = 2 \in k^\times / k^{\times 2}.$$

The space $W \otimes V$ is a symplectic space over k with the skew-symmetric form

$$\langle -, - \rangle_W \otimes \langle -, - \rangle_V,$$

and one has the associated Heisenberg group

$$H(W \otimes V) = k \oplus (W \otimes V),$$

which has a one dimensional center k. Associated to ψ, $H(W \otimes V)$ has a unique irreducible representation ω_ψ with central character ψ (by the Stone-von-Neumann theorem). Now $\mathrm{Sp}(W \otimes V)$ acts as automorphisms of $H(W \otimes V)$ via its natural action on $W \otimes V$ and the trivial action on k. Thus ω_ψ gives rise to a projective representation of $\mathrm{Sp}(W \otimes V)$ and it was shown by Weil that this projective representation is a linear representation of $\widetilde{\mathrm{Sp}}(W \otimes V)$. We thus have a representation ω_ψ of the semi-direct product

$$\widetilde{\mathrm{Sp}}(W \otimes V) \ltimes H(W \otimes V).$$

This is the so-called Weil representation (associated to ψ). As a representation of $\widetilde{\mathrm{Sp}}(W \otimes V)$, it is the direct sum of two irreducible representations, and its isomorphism class depends only on the $k^{\times 2}$-orbit of ψ.

Via a natural homomorphism

$$\widetilde{\mathrm{Sp}}(W) \times \mathrm{O}(V) \longrightarrow \widetilde{\mathrm{Sp}}(W \otimes V),$$

we regard the Weil representation ω_ψ as a representation $\omega_{W,V,\psi}$ of $\widetilde{\mathrm{Sp}}(W) \times \mathrm{O}(V)$. The theory of Howe duality gives a correspondence between irreducible genuine representations $\widetilde{\pi}$ of $\widetilde{\mathrm{Sp}}(W)$ and certain irreducible representations σ of $\mathrm{O}(V)$.

More precisely, given an irreducible representation σ of $\mathrm{O}(V)$, the maximal σ-isotypic quotient of $\omega_{W,V,\psi}$ has the form

$$\sigma \boxtimes \Theta_{W,V,\psi}(\sigma)$$

for some smooth representation $\Theta_{W,V,\psi}(\sigma)$ (the big theta lift of σ) of $\widetilde{\mathrm{Sp}}(W)$. It is known ([K] and [MVW]) that $\Theta_{W,V,\psi}(\sigma)$ is either zero or has finite length. Let $\theta_{W,V,\psi}(\sigma)$ (the small theta lift of σ) denote the maximal semisimple quotient of $\Theta_{W,V,\psi}(\sigma)$. It is known by Howe [Ho] and Waldspurger [Wa3] that when the residue characteristic of k is different from 2, then $\theta_{W,V,\psi}(\sigma)$ is irreducible or zero; this is the so-called Howe's conjecture. In the following, we will assume that the same holds when the residue characteristic of k is 2.

Analogously, if $\widetilde{\pi}$ is an irreducible representation of $\widetilde{\mathrm{Sp}}(W)$, we have the representations $\Theta_{W,V,\psi}(\widetilde{\pi})$ and $\theta_{W,V,\psi}(\widetilde{\pi})$ of $\mathrm{O}(V)$.

Now we have the following theorem, which is due to Adams-Barbasch [AB] when $k = \mathbb{R}$ and follows from fundamental results of Kudla-Rallis [KR] when k is non-archimedean.

Theorem 11.1. *Assume that the local field k is either real or non-archimedean with odd residual characteristic. Then corresponding to the choice of an additive character ψ of k, there is a natural bijection given by the theta correspondence:*

$$\left\{\textit{irreducible genuine representations } \widetilde{\pi} \textit{ of } \widetilde{\mathrm{Sp}}(W)\right\}$$
$$\updownarrow$$
$$\coprod \left\{\textit{irreducible representations } \sigma' \textit{ of } \mathrm{SO}(V')\right\}$$

where the union is disjoint, and taken over all the isomorphism classes of orthogonal spaces V' over k with $\dim V' = 2n+1$ and $\mathrm{disc}(V') = 1$.

More precisely, given an irreducible representation $\widetilde{\pi}$ of $\widetilde{\mathrm{Sp}}(W)$, there is a unique V' as above such that $\theta_{W,V',\psi}(\widetilde{\pi})$ is nonzero, in which case the image of $\widetilde{\pi}$ under the above bijection is the restriction of $\theta_{W,V',\psi}(\widetilde{\pi})$ to $\mathrm{SO}(V')$ (note that this restriction is irreducible since $\mathrm{O}(V') = \mathrm{SO}(V') \times \langle \pm 1 \rangle$).

Proof. We give a sketch of the proof of Theorem 11.1 when k is non-archimedean; a detailed proof can be found in [GS]. Let's begin by noting that there are 2 isomorphism classes of quadratic space of dimension $2n+1$ and trivial discriminant; we denote these by V and V', and assume that V is split. To simplify notation, we shall write Θ in place of $\Theta_{W,V,\psi}$ and Θ' in place of $\Theta_{W,V',\psi}$.

We now divide the proof into two steps:

(i) Given an irreducible representation $\widetilde{\pi}$ of $\widetilde{\mathrm{Sp}}(W)$, exactly one of $\Theta(\widetilde{\pi})$ or $\Theta'(\widetilde{\pi})$ is nonzero.

This dichotomy was also shown in the recent paper of C. Zorn [Z]. In any case, [KR, Thm. 3.8] shows that any irreducible representation $\widetilde{\pi}$ of $\widetilde{\mathrm{Sp}}(W)$ participates in theta correspondence with at most one of $\mathrm{O}(V)$ or $\mathrm{O}(V')$. We claim however that $\widetilde{\pi}$ does have nonzero theta lift to $\mathrm{O}(V)$ or $\mathrm{O}(V')$. To see this, note that [KR, Prop. 4.1] shows that $\widetilde{\pi}$ has nonzero theta lift to $\mathrm{O}(V)$ if and only if

$$\mathrm{Hom}_{\widetilde{\mathrm{Sp}}(W)\times\widetilde{\mathrm{Sp}}(W)}(R(V), \widetilde{\pi}\boxtimes\widetilde{\pi}^{\vee}) \neq 0,$$

where $R(V)$ is the big theta lift of the trivial representation of $\mathrm{O}(V)$ to $\widetilde{\mathrm{Sp}}(W+W^{-})$ (where W^{-} is the symplectic space obtained from W by scaling its form by -1). Similarly, one has the analogous statement for V'. On the other hand, if $I_P(s)$ denotes the degenerate principal series representation of $\widetilde{\mathrm{Sp}}(W+W^{-})$ unitarily induced from the character $\chi_\psi\cdot|\det|^s$ of the Siegel parabolic subgroup stabilizing the maximal isotropic subspace ΔW, the diagonal W in $W\oplus W^{-}$, and χ_ψ a genuine character of $\widetilde{GL}(\Delta W)$ defined in §16, then it was shown by Sweet [Sw] that

$$I_P(0) = R(V)\oplus R(V').$$

It follows that $R(V)$ and $R(V')$ are unitarizable and thus irreducible (since they have a unique irreducible quotient). In particular, we conclude that $\widetilde{\pi}$ has nonzero theta lift to one of $\mathrm{O}(V)$ or $\mathrm{O}(V')$ if and only if

$$\mathrm{Hom}_{\widetilde{\mathrm{Sp}}(W)\times\widetilde{\mathrm{Sp}}(W)}(I_P(0), \widetilde{\pi}\boxtimes\widetilde{\pi}^{\vee}) \neq 0.$$

We thus need to show that this Hom space is nonzero. This can be achieved by the doubling method of Piatetski-Shapiro and Rallis [GPSR] (cf. also [GS] and [Z]), which provides a zeta integral

$$Z(s): I_P(s)\otimes\widetilde{\pi}^{\vee}\otimes\widetilde{\pi}\longrightarrow\mathbb{C}.$$

The precise definition of $Z(s)$ need not concern us here; it suffices to note that for a flat section $\Phi(s)\in I_P(s)$ and $f\otimes f^{\vee}\in\widetilde{\pi}\otimes\widetilde{\pi}^{\vee}$, $Z(s,\Phi(s),f\otimes f^{\vee})$ is a meromorphic function in s. Moreover, at any $s=s_0$, the leading term of the Laurent expansion of $Z(s)$ gives a nonzero element

$$Z^*(s_0)\in\mathrm{Hom}_{\widetilde{\mathrm{Sp}}(W)\times\widetilde{\mathrm{Sp}}(W)}(I_P(s_0), \widetilde{\pi}\boxtimes\widetilde{\pi}^{\vee}).$$

For these basic properties of zeta integrals, see [GS] or [Z]. This proves our contention that $\widetilde{\pi}$ participates in the theta correspondence with exactly one of $\mathrm{O}(V)$ or $\mathrm{O}(V')$.

By (i), one obtains a map

$$\begin{array}{c}\left\{\text{irreducible genuine representations } \widetilde{\pi} \text{ of } \widetilde{\mathrm{Sp}}(W)\right\}\\ \Big\downarrow\\ \coprod\{\text{irreducible representations } \sigma' \text{ of } \mathrm{O}(V')\}\end{array}$$

Moreover, this map is injective by the theorem of Waldspurger [Wa3] proving Howe's conjecture.

(ii) An irreducible representation π_0 of $\mathrm{SO}(V)$ has two extensions to $\mathrm{O}(V) = \mathrm{SO}(V) \times \langle \pm 1 \rangle$, and exactly one of these extensions participates in the theta correspondence with $\widetilde{\mathrm{Sp}}(W)$. The same assertion holds for representations of $\mathrm{SO}(V')$.

Suppose on the contrary that π is an irreducible representation of $\mathrm{O}(V)$ such that both π and $\pi \otimes \det$ participate in theta correspondence with $\widetilde{\mathrm{Sp}}(W)$, say

$$\widetilde{\pi} = \theta_{W,V,\psi}(\pi) \quad \text{and} \quad \widetilde{\pi}' = \theta_{W,V,\psi}(\pi \otimes \det).$$

Now consider the seesaw diagram:

$$\begin{tikzcd}
\widetilde{\mathrm{Sp}}(W + W^-) \arrow[d, dash] \arrow[drr, dash] & & \mathrm{O}(V) \times \mathrm{O}(V) \arrow[d, dash] \arrow[dll, dash] \\
\widetilde{\mathrm{Sp}}(W) \times \widetilde{\mathrm{Sp}}(W) & & \mathrm{O}(V).
\end{tikzcd}$$

The seesaw identity implies that

$$\mathrm{Hom}_{\widetilde{\mathrm{Sp}}(W) \times \widetilde{\mathrm{Sp}}(W)}(\Theta_{W+W^-,V,\psi}(\det), \widetilde{\pi}' \boxtimes \widetilde{\pi}^\vee) \supset \mathrm{Hom}_{\mathrm{O}(V)}((\pi \otimes \det) \otimes \pi^\vee, \det) \neq 0.$$

This implies that

$$\Theta_{W+W^-,V,\psi}(\det) \neq 0.$$

However, a classical result of Rallis [R, Appendix] says that the determinant character of $\mathrm{O}(V)$ does not participate in the theta correspondence with $\widetilde{\mathrm{Sp}}(4r)$ for $r \leq n$. This gives the desired contradiction.

We have thus shown that at most one of π or $\pi \otimes \det$ could have nonzero theta lift to $\widetilde{\mathrm{Sp}}(W)$. On the other hand, the analog of the zeta integral argument in (i) shows that one of π or $\pi \otimes \det$ does lift to $\widetilde{\mathrm{Sp}}(2n)$; for this, one needs the structure of the degenerate principal series representation $I_{P(\Delta V)}(0)$ of $\mathrm{O}(V + V^-)$ which is determined in [BJ] (see also [Y, Prop. 3.3]). This proves (ii).

Putting (i) and (ii) together, we have established the theorem. $\square$

The only reason for the assumption of odd residue characteristic in the theorem is that Howe's conjecture for local theta correspondence is only known under this assumption.

Since V is an odd dimensional quadratic space, $\mathrm{SO}(V)$ is an adjoint group, there is a unique orbit of generic characters on it, and the Vogan parametrization of irreducible representations σ' of the groups $\mathrm{SO}(V')$ requires no further choices. So we label $\widetilde{\pi} = \widetilde{\pi}(M, \chi)$ using the Vogan parameters (M, χ) of the representation $\sigma' = \Theta_{W,V',\psi}(\widetilde{\pi})$. The theorem thus gives the following corollary.

Corollary 11.2. *Assume that the residue characteristic of k is odd. Suppose that the local Langlands-Vogan parametrization holds for* $\mathrm{SO}(V')$. *Then one has a parametrization (depending on ψ) of*

$$\{\textit{irreducible genuine representations } \widetilde{\pi} \textit{ of } \widetilde{\mathrm{Sp}}(W)\}$$

by the set of isomorphism classes of pairs (φ, χ) such that

$$\varphi : WD(k) \longrightarrow \mathrm{Sp}(M)$$

is a symplectic representation of $WD(k)$ and χ is an irreducible character of the component group A_φ.

It follows that the various desiderata for the Vogan packets of $\widetilde{\mathrm{Sp}}(W)$ can be obtained from those of $\mathrm{SO}(V')$ if one understands the properties of the theta correspondence. For example, in the theta correspondence, generic representations of the split group $\mathrm{SO}(V)$ lift to $\widetilde{\theta}$-generic representations of $\widetilde{\mathrm{Sp}}(W)$. Hence the $\widetilde{\theta}$-generic element in the L-packet of M corresponds to the trivial character of the component group A_M. Also, when k is non-archimedean, $\widetilde{\pi}(M, \chi)$ is lifted from the split group $\mathrm{SO}(V)$ precisely when $\chi(-1) = 1$. For these and other similar issues, see [GS].

One difference between metaplectic and linear groups is in the description of the action of the adjoint group by outer automorphisms on the set of irreducible representations. The adjoint action of the symplectic similitude group $\mathrm{GSp}(W)$ on the set of genuine irreducible representations of $\widetilde{\mathrm{Sp}}(W)$ factors through the quotient

$$k^\times/k^{\times 2} = \mathrm{PGSp}(W)(k)/\mathrm{Image}\ \mathrm{Sp}(W)(k).$$

In the metaplectic case, this outer action does not permute the representations $\widetilde{\pi}$ in an individual Vogan L-packet, and we predict a more complicated recipe, as follows.

Conjecture 11.3. *If $\widetilde{\pi}$ has ψ-parameter (M, χ) and c is a class in $k^\times/k^{\times 2}$, the conjugated representation $\widetilde{\pi}^c$ has ψ-parameter $(M(c), \chi \cdot \eta[c])$. Here $M(c)$ is the twist of M by the one-dimensional orthogonal representation $\mathbb{C}(c)$ so that its component group $A_{M(C)}$ is canonically isomorphic to A_M. The character $\eta[c]$ is defined by*

$$\eta[c] = \chi_N : A_M \to \langle \pm 1 \rangle,$$

where N is the two dimensional orthogonal representation $N = \mathbb{C} + \mathbb{C}(c)$, so that

$$\eta[c](a) = \epsilon(M^a) \cdot \epsilon(M(c)^a) \cdot (c, -1)^{\frac{1}{2}\dim M^a}.$$

This conjecture is known when $\dim W = 2$, where it is a result of Waldspurger ([W1] and [W2]); our recipe above is suggested by his results.

The above conjecture has the following consequence. If one replaces the character ψ by the character

$$\psi_c : x \mapsto \psi(cx)$$

of k, then the new Vogan parameter (relative to ψ_c) of $\widetilde{\pi}$ will be $(M(c), \chi \cdot \eta[c])$.

A consequence of this is the following. Suppose that $\widetilde{\pi}$ is such that

$$\theta_{W,V,\psi}(\widetilde{\pi}) \neq 0 \quad \text{and} \quad \theta_{W,V,\psi_c}(\widetilde{\pi}) \neq 0$$

as representations of $\mathrm{SO}(V)$. Then, when $\dim W = 2$, a basic result of Waldspurger says that

$$\theta_{W,V,\psi}(\widetilde{\pi}) \cong \theta_{W,V,\psi_c}(\widetilde{\pi}) \otimes \chi_c,$$

where χ_c is the character

$$\mathrm{SO}(V)(k) \xrightarrow{\text{spinor norm}} k^\times/k^{\times 2} \xrightarrow{(c,-)} \langle \pm 1 \rangle.$$

However, according to the conjecture above, if the Vogan parameter of $\theta_{W,V\psi}(\widetilde{\pi})$ is (M, χ), then that of $\theta_{W,V,\psi_c}(\widetilde{\pi}) \otimes \chi_c$ is $(M, \chi \cdot \eta[c])$. So the two representations are equal if and only if the character $\eta[c]$ is trivial. The assumption that $\widetilde{\pi}$ has nonzero theta lift to $\mathrm{SO}(V)$ with respect to both ψ and ψ_c implies that

$$\eta[c](-1) = 1.$$

When $\dim W = 2$, this is equivalent to saying that $\eta[c]$ is trivial. But when $\dim W > 2$, this is no longer the case and one can construct such counterexamples already when $\dim W = 4$.

12. The representation ν of H and generic data

In this section, we shall describe the remaining ingredient in the restriction problem to be studied. Suppose as before that k is a local field with an involution σ (possibly trivial) and k_0 is the fixed field of σ. Let V be a k-vector space endowed with a non-degenerate sesquilinear form $\langle -, - \rangle$ with sign ϵ. Moreover, suppose that $W \subset V$ is a non-degenerate subspace satisfying:

(1) $\epsilon \cdot (-1)^{\dim W^\perp} = -1$

(2) $W^\perp$ is a split space.

So we have

$$\dim W^\perp = \begin{cases} \text{odd, if } \epsilon = 1, \text{ i.e. } V \text{ is orthogonal or hermitian;} \\ \text{even, if } \epsilon = -1, \text{ i.e. } V \text{ is symplectic or skew-hermitian.} \end{cases}$$

Let $G(V)$ be the identity component of the automorphism group of V and $G(W) \subset G(V)$ the subgroup which acts as identity on $W^\perp$. Set

$$G = G(V) \times G(W).$$

As explained in Section 2, G contains a subgroup H defined as follows. Since $W^\perp$ is split, we may write

$$W^\perp = X + X^\vee \quad \text{or} \quad W^\perp = X + X^\vee + E$$

depending on whether $\dim W^\perp$ is even or odd, where in the latter case, E is a non-isotropic line. Let P be a parabolic subgroup which stabilizes a complete flag of (isotropic) subspaces in X. Then $G(W)$ is a subgroup of a Levi subgroup of P and thus acts by conjugation on the unipotent radical N of P. We set

$$H = N \rtimes G(W).$$

Note that there is a natural embedding $H \hookrightarrow G$ which is the natural inclusion

$$H \subset P \subset G(V)$$

in the first factor and is given by the projection

$$H \longrightarrow H/N = G(W)$$

in the second factor. When $G' = G(V') \times G(W')$ is a relevant pure inner form of G, a similar construction gives a distinguished subgroup H'.

The goal of this section is to describe a distinguished representation ν of H (and similarly H'). It will turn out that $\dim \nu = 1$ if $\dim W^\perp$ is odd (orthogonal and hermitian cases), whereas ν has Gelfand-Kirillov dimension $1/2 \cdot \dim(W/k_0)$ when $\dim W^\perp$ is even (symplectic and skew-hermitian cases). Because of this, we will treat the cases when $\dim W^\perp$ is even or odd separately.

Orthogonal and Hermitian Cases (Bessel Models)

Assume that $\dim W^\perp = 2n + 1$ and write

$$W^\perp = X + X^\vee + E \quad \text{with} \quad E = \langle e \rangle,$$

where X and $X^\vee$ are maximal isotropic subspaces which are in duality using the form $\langle -, - \rangle$ of V, and e is a non-isotropic vector. Let $P(X)$ be the parabolic subgroup in $G(V)$ stabilizing the subspace X, and let $M(X)$ be the Levi subgroup of $P(X)$ which stabilizes both X and $X^\vee$, so that

$$M(X) \cong \mathrm{GL}(X) \times G(W \oplus E).$$

We have

$$P(X) = M(X) \ltimes N(X)$$

where $N(X)$ is the unipotent radical of $P(X)$. The group $N(X)$ sits in an exact sequence of $M(X)$-modules,

$$0 \longrightarrow Z(X) \longrightarrow N(X) \longrightarrow N(X)/Z(X) \longrightarrow 0,$$

and using the form on V, one has natural isomorphisms

$$Z(X) \cong \{\text{skew-hermitian forms on } X^\vee\},$$

and

$$N(X)/Z(X) \cong \mathrm{Hom}(W+E, X) \cong (W+E) \otimes X.$$

Here, when $k = k_0$, skew-hermitian forms on $X^\vee$ simply mean symplectic forms. In particular, $Z(X)$ is the center of $N(X)$ unless $k = k_0$ and $\dim X = 1$, in which case $Z(X)$ is trivial and $N(X)$ is abelian.

Now let

$$\ell_X : X \to k$$

be a nonzero k-linear homomorphism, and let

$$\ell_W : W \oplus E \longrightarrow k$$

be a nonzero k-linear homomorphism which is zero on the hyperplane W. Together, these give a map

$$\ell_X \otimes \ell_W : X \otimes (W+E) \longrightarrow k,$$

and one can consider the composite map

$$\ell_{N(X)} : N(X) \longrightarrow N(X)/Z(X) \cong X \otimes (W+E) \xrightarrow{\ell_X \otimes \ell_W} k.$$

Let U_X be any maximal unipotent subgroup of $\mathrm{GL}(X)$ which stabilizes ℓ_X. Then the subgroup

$$U_X \times G(W) \subset M(X)$$

fixes the homomorphism $\ell_{N(X)}$.

Now the subgroup

$$H \subset G = G(V) \times G(W)$$

is given by

$$H = (N(X) \rtimes (U_X \cdot G(W)) = N \rtimes G(W).$$

We may extend the map $\ell_{N(X)}$ of $N(X)$ to H, by making it trivial on $U_X \times G(W)$. If ψ is a nontrivial additive character of k, and

$$\lambda_X : U_X \longrightarrow \mathbb{S}^1$$

is a generic character of U_X, then the representation ν of H is defined by

$$\nu = (\psi \circ \ell_{N(X)}) \boxtimes \lambda_X.$$

The pair (H, ν) is uniquely determined up to conjugacy in the group $G = G(V) \times G(W)$ by the pair $W \subset V$.

One can give a more explicit description of (H, ν), by explicating the choices of ℓ_X, U_X and λ_X above. To do this, choose a basis $\{v_1, \cdots, v_n\}$ of X, with dual basis $\{v_i^\vee\}$ of $X^\vee$. Let $P \subset G(V)$ be the parabolic subgroup which stabilizes the flag

$$0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \cdots, v_n \rangle = X,$$

and let

$$L = (k^\times)^n \times G(W+E)$$

be the Levi subgroup of P which stabilizes the lines $\langle v_i \rangle$ as well as the subspace $W+E$. The torus $T = (k^\times)^n$ scales these lines:

$$t(v_i) = t_i v_i,$$

and $G(W+E)$ acts trivially on $X + X^\vee$.

Let N be the unipotent radical of P, so that

$$N = U_X \ltimes N(X)$$

where U_X is the unipotent radical of the Borel subgroup in $\mathrm{GL}(X)$ stabilizing the chosen flag above. Now define a homomorphism $f : N \to k^n$ by

$$\begin{array}{rcl} f(u) & = & (x_1, \cdots, x_{n-1}, z), \\ x_i & = & \langle u v_{i+1}, v_i^\vee \rangle, \qquad i = 1, 2, \cdots, n-1 \\ z & = & \langle ue, v_n^\vee \rangle. \end{array}$$

The subgroup of L which fixes f is $G(W)$, the subgroup of $G(W+E)$ fixing the vector e. The torus acts on f by

$$f(tut^{-1}) = ((t_1/t_2)x_1, (t_2/t_3)x_2, \cdots, t_n z).$$

Consider the subgroup $H = N \cdot G(W)$ of $G = G(V) \times G(W)$. Then, for a nontrivial additive character ψ of k, the representation ν is given by:

$$\begin{array}{rcl} \nu & : & H \to \mathbb{C}^\times \\ & & (u, g) \mapsto \psi(\sum x_i + z). \end{array}$$

It is as regular as possible on N, among the characters fixed by $G(W)$. As noted above, up to G-conjugacy, the pair (H, ν) depends only on the initial data $W \subset V$, and not on the choices of ψ, $\{v_i\}$, or e used to define it.

A special case of (H, ν) is worth noting. If V is orthogonal of even dimension and W has dimension 1, then $\mathrm{SO}(W) = 1$ and $H = N$ is the unipotent radical of a Borel subgroup $P \subset G = \mathrm{SO}(V)$. In this case, ν is simply a generic character θ_W of N. By choosing different non-isotropic lines L in the 2-dimensional orthogonal space $W + E$, so that $L^\perp = X + X^\vee + L'$, and using L in place of W in the above construction, the map $L \mapsto \theta_L$ gives a bijection

$$\begin{array}{c} \{T\text{-orbits of generic characters on } N\} \\ \updownarrow \\ \{\mathrm{SO}(V)\text{-orbits of non-isotropic lines } L \text{ with } L^\perp \text{ split}\} \end{array}$$

in the even orthogonal case. This bijection was described in [GP1, Prop. 7.8].

Symplectic and Skew-Hermitian Cases (Fourier-Jacobi Models)

We now treat the symplectic and skew-hermitian cases, so that $W^\perp$ is split of even dimension $2n$ and we may write

$$W^\perp = X + X^\vee,$$

where X and $X^\vee$ are maximal isotropic subspaces which are in duality using the form on V. In this case, $G(W)$ is a subgroup of $\mathrm{Sp}(W/k_0)$, preserving the form $\mathrm{Tr}_{k/k_0} \circ \langle -, - \rangle$. It will turn out that the representation ν of H depends on some other auxiliary data besides the spaces $W \subset V$. As in the case of Bessel models, we include the case $k = k_0 \times k_0$ in our discussion below.

We assume first that $\dim X > 0$. Let $P(X) = M(X) \cdot N(X)$ be the parabolic subgroup in $G(V)$ stabilizing the subspace X, with Levi subgroup

$$M(X) \cong \mathrm{GL}(X) \times G(W)$$

stabilizing both X and $X^\vee$. Let $Z(X)$ be the center of the unipotent radical $N(X)$ of $P(X)$, so that one has the exact sequence of $M(X)$-modules:

$$0 \longrightarrow Z(X) \longrightarrow N(X) \longrightarrow N(X)/Z(X) \longrightarrow 0,$$

Using the form on V, one has natural isomorphisms

$$Z(X) \cong \{\text{hermitian forms on } X^\vee\}$$

and

$$N(X)/Z(X) \cong \mathrm{Hom}(W, X) \cong W \otimes X.$$

Here, if $k = k_0$, then hermitian forms on $X^\vee$ simply mean symmetric bilinear forms.

The commutator map

$$[-,-] : N(X) \times N(X) \to N(X)$$

factors through $N(X)/Z(X)$ and takes value in $Z(X)$. It thus gives rise to a skew-symmetric k_0-bilinear map

$$\Lambda^2_{k_0}(X \otimes W) \longrightarrow Z(X) = \{\text{hermitian forms on } X^\vee\},$$

or equivalently by duality, a map

$$\{\text{hermitian forms on } X\} \longrightarrow \Lambda^2_{k_0}(X^\vee \otimes W) = \{\text{symplectic forms on } \mathrm{Res}_{k/k_0}(X \otimes W)\}.$$

Indeed, this last map is a reflection of the fact that, using the skew-hermitian structure on W, the space of hermitian forms on X can be naturally embedded in the space of skew-hermitian forms on $X \otimes W$, and then by composition with the trace map if necessary, in the space of symplectic forms on $\mathrm{Res}_{k/k_0}(X \otimes W)$.

Now let

$$\ell_X : X \to k$$

be a nonzero homomorphism, and let $U_X \subset \mathrm{GL}(X)$ be a maximal unipotent subgroup which fixes ℓ_X. Then the group H is defined by

$$H = N(X) \rtimes (U_X \times G(W)) = N \rtimes G(W)$$

with $N = N(X) \rtimes U_X$. If

$$\lambda_X : U_X \longrightarrow \mathbb{S}^1$$

is a generic character of U_X, then by composing with the projection from H to U_X, we may regard λ_X as a character of H.

On the other hand, by pulling back, the homomorphism ℓ_X gives rise to a linear map

$$k_0 = \{\text{hermitian forms on } k\} \longrightarrow \{\text{hermitian forms on } X\},$$

and hence by duality

$$\ell_{Z(X)} : Z(X) = \{\text{hermitian forms on } X^\vee\} \longrightarrow k_0.$$

Moreover, ℓ_X gives a k-linear map

$$\ell_W : X \otimes W \to W$$

making the following diagram commute:

$$\begin{tikzcd}
\Lambda^2_{k_0}(X \otimes W) \arrow[r, "{[-,-]}"] \arrow[d, "\ell_W"'] & Z(X) \arrow[d, "\ell_{Z(X)}"] \\
\Lambda^2_{k_0}(W) \arrow[r, "\frac{2}{[k:k_0]} \cdot Tr_{k/k_0}(\langle -,- \rangle)"'] & k_0.
\end{tikzcd}$$

For example, when $k = k_0$ and $\dim X = 1$, then the commutator map $[-,-]$ is given by the skew-symmetric form $2 \cdot \langle -,- \rangle_W$ on $N(X)/Z(X) = W$. On the other hand, when $k \neq k_0$ and $\dim X = 1$, it is given by the skew-symmetric form $Tr_{k/k_0}(\langle -,- \rangle_W)$ on W/k_0. In any case, let us set

$$V_1 = \begin{cases} \text{the rank 1 quadratic space with discriminant 1 if } k = k_0; \\ \text{the rank 1 hermitian space with discriminant 1 if } k \neq k_0, \end{cases}$$

and let $H(V_1 \otimes W)$ be the Heisenberg group associated to the symplectic vector space $V_1 \otimes W$ over k_0 with form

$$Tr_{k/k_0}(\langle -,- \rangle_{V_1} \otimes \langle -,- \rangle_W).$$

Here, given a quadratic space (V, q) over k_0, the associated symmetric bilinear form is

$$\langle v_1, v_2 \rangle = q(v_1 + v_2) - q(v_1) - q(v_2).$$

Thus, when $k = k_0$, the form on V_1 is such that $\langle v, v \rangle_V = 2$, so that $H(V_1 \otimes W)$ is the Heisenberg group associated to the symplectic vector space $(W, 2 \cdot \langle -,- \rangle_W)$.

Now one has the following commutative diagram of algebraic groups over k_0:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z(X) & \longrightarrow & N(X) & \longrightarrow & X \otimes W & \longrightarrow & 0 \\ & & {\scriptstyle \ell_{0,X}} \big\downarrow & & \big\downarrow & & {\scriptstyle \ell_W} \big\downarrow & & \\ 0 & \longrightarrow & k_0 & \longrightarrow & H(V_1 \otimes W) & \longrightarrow & W & \longrightarrow & 0. \end{array}$$

Given a nontrivial character $\psi_0 : k_0 \to \mathbb{S}^1$, one may consider the unique irreducible unitarizable representation ω_{W,ψ_0} of $H(V_1 \otimes W)$ of Gelfand-Kirillov dimension $\frac{1}{2} \cdot \dim_{k_0} W$, on which the center of $H(V_1 \otimes W)$ acts by ψ_0. Pulling back by the above diagram, one obtains an irreducible representation ω_{ψ_0} of $N(X)$ with central character $\psi_0 \circ \ell_{Z(X)}$. Up to conjugation by $M(X)$, the representation ω_{ψ_0} depends only on ψ_0 up to multiplication by $(k^\times)^{1+\sigma}$. The representation ω_{ψ_0} can be extended trivially to U_X. Moreover, the group $G(W)$ acts as outer automorphisms of $H(V_1 \otimes W)$, so the theory of Weil representations furnishes us with a projective representation of $G(W)$ on ω_{ψ_0}. Thus, one has a projective representation ω_{ψ_0} of H.

As in the orthogonal and hermitian cases, we can make the above discussion completely explicit by making specific choices of ℓ_X, U_X and λ_X. Assuming that $\dim X = n > 0$, choose a basis $\{v_i\}$ for X and let $\{v_i^\vee\}$ be the dual basis of $X^\vee$. Let $P \subset G(V)$ be the subgroup stabilizing the flag

$$0 \subset \langle v_1 \rangle \subset \cdots \subset \langle v_1, \cdots, v_n \rangle = X,$$

and let

$$L = G(W) \times (k^\times)^n$$

be the Levi subgroup of P stabilizing the lines $\langle v_i \rangle$ as well as the subspace W.

Let N be the unipotent radical of P and define a homomorphism to a vector group

$$f : N \to k^{n-1} \oplus W$$

given by:

$$\begin{aligned} f(u) &= (x_1, \cdots, x_{n-1}, y) \\ x_i &= \langle u v_{i+1}, v_i^\vee \rangle \\ y &= \textstyle\sum_j \langle u w_j, v_n^\vee \rangle . w_j^\vee . \end{aligned}$$

Here $\{w_1, \cdots, w_n\}$ is a basis for W over k and $\langle w_i, w_j^\vee \rangle = \delta_{ij}$. Thus y is the unique vector in W with

$$\langle w, y \rangle = \langle uw, v_n^\vee \rangle$$

for all w in W.

The torus $T = (k^\times)^n$ acts on f by

$$f(tut^{-1}) = ((t_1/t_2)x_1, \cdots, (t_{n-1}/t_n)x_n, t_n \cdot y)$$

and an element $g \in G(W)$ acts by

$$f(gug^{-1}) = (x_1, \cdots, x_n, g(y)).$$

Now the maps $(x_1, \cdots, x_{n-1})$ give a functional

$$\begin{aligned} \ell \;\; : \;\; & N \to k_0 \\ & n \mapsto \mathrm{Tr}(\textstyle\sum x_i) \end{aligned}$$

which is fixed by $G(W)$, and is as regular as possible subject to this condition. Choose a nontrivial additive character ψ_0 of k_0. Then the character

$$\begin{aligned} \lambda \;\; : \;\; & N \to \mathbb{S}^1 \\ & u \mapsto \psi_0(\ell(u)) = \psi_0(\mathrm{Tr}(\textstyle\sum x_i)) \end{aligned}$$

is regular, and up to conjugacy by the torus, independent of the choice of ψ_0. Since it is fixed by $G(W)$, we may extend it trivially to $G(W)$ and obtain a character λ of H.

On the other hand, one may define a homomorphism of N to a Heisenberg group as follows. Let $N_0 \lhd N$ be the kernel of the map

$$\begin{aligned} N &\to W \\ u &\mapsto y \end{aligned}$$

and define a homomorphism

$$\begin{aligned} f_0 \;\; : \;\; & N_0 \to k \\ & u_0 \mapsto z = \langle u_0 v_n^\vee, v_n^\vee \rangle. \end{aligned}$$

Note that the element z lies in the subfield k_0 of k, since

$$u_0 v_n^\vee \text{ is isotropic} \Longrightarrow z - z^\sigma = 0.$$

Hence, we have

$$f_0 : N_0 \longrightarrow k_0.$$

The torus act by

$$f_0(tut^{-1}) = t_n^{1+\sigma} \cdot z$$

and $G(W)$ acts trivially. The above two maps combine to give a homomorphism from N to the Heisenberg group $H(W/k_0)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_0 & \longrightarrow & N & \longrightarrow & W & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle f_0} & & \Big\downarrow & & \Big\downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & k_0 & \longrightarrow & H(V_1 \otimes W) & \longrightarrow & W & \longrightarrow & 0 \end{array}$$

which is equivariant for the action of $G(W)$ on N and $G(W) \subset \mathrm{Sp}(V_1 \otimes W)$ on $H(V_1 \otimes W)$. The nontrivial additive character ψ_0 then gives rise to the projective representation ω_ψ of H as above.

It is now more convenient to consider the symplectic and skew-hermitian cases separately.

(i) (symplectic case) When $k = k_0$, we have $G(W) = \mathrm{Sp}(W)$. In this case, for each character ψ of $k = k_0$, it is known that the projective representation ω_ψ of $G(W)$ lifts to a linear representation of the double cover $\widetilde{G}(W) = \widetilde{\mathrm{Sp}}(W)$, the metaplectic group. Recalling that

$$H = N \rtimes G(W),$$

we thus obtain a unitary representation

$$\nu_\psi = \omega_\psi \otimes \lambda$$

of

$$\widetilde{H} = N \rtimes \widetilde{G}(W)$$

in the case when $\dim W^\perp > 0$.

When $W^\perp = 0$, so that $W = V$, we have

$$N = \{1\} \quad \text{and} \quad H = G(W) = G(V).$$

In this case, we simply set

$$\nu_\psi = \omega_\psi,$$

which is a representation of $\widetilde{H}$.

In each case, the representation ν_ψ has Gelfand-Kirillov dimension $1/2 \cdot \dim_k(W)$. Up to conjugation by the normalizer of H in G, ν_ψ depends only on ψ up to the action of $(k^\times)^2$.

A particular case of this is worth noting. When $W = 0$, so that $G(W)$ is trivial, the group H is simply the unipotent radical N of the Borel subgroup P and ν_ψ is simply a generic character of N. This gives a bijection

$$\begin{array}{c}\{T\text{-orbits of generic characters of } N\} \\ \updownarrow \\ \{k^{\times 2}\text{-orbits of nontrivial characters } \psi \text{ of } k\}\end{array}$$

in the symplectic case.

(ii) (skew-hermitian case) When $k \neq k_0$, $G(W) = \mathrm{U}(W)$. In this case, the projective representation ω_{ψ_0} of $G(W) = \mathrm{U}(W)$ lifts to a linear representation of $G(W)$, but when $\dim W > 0$, the lifting is not unique: it requires the choice of a character

$$\mu : k^\times \to \mathbb{C}^\times$$

whose restriction to $k_0^\times$ is the quadratic character ω_{k/k_0} associated to k/k_0 [HKS]. Equivalently, when k is a field, it requires the choice of a 1-dimensional, conjugate-dual representation of $WD(k)$ with sign $c = -1$. Given such a μ, we let $\omega_{\psi_0,\mu}$ be the corresponding representation of $G(W)$ and set

$$\nu_{\psi_0,\mu} = \omega_{\psi_0,\mu} \otimes \lambda.$$

Hence, we have defined an irreducible unitary representation $\nu_{\psi_0,\mu}$ of $H = N.\mathrm{U}(W)$ when $\dim W^\perp > 0$.

When $W^\perp = 0$, so that $W = V$, we have

$$N = \{1\} \quad \text{and} \quad H = G(W) = G(V).$$

In this case, we simply set

$$\nu_{\psi_0,\mu} = \omega_{\psi_0,\mu}.$$

In each case, the representation $\nu_{\psi_0,\mu}$ has Gelfand-Kirillov dimension $1/2 \cdot \dim_{k_0}(W)$. It depends, up to conjugation by the normalizer of H in $G = \mathrm{U}(V) \times \mathrm{U}(W)$, on ψ_0 up to the action of $\mathbb{N}k^\times$ (as well as the choice of μ).

A particular case of this is noteworthy. When $W = 0$ (and V is even dimensional), so that $G(W)$ is trivial, the group H is simply the unipotent radical N of the Borel subgroup P and there is no need to choose μ. Hence, $\nu_{\psi_0,\mu} = \nu_{\psi_0}$ is simply a generic character of N. This gives a bijection

$$\begin{array}{c} \{T\text{-orbits of generic characters of } N\} \\ \updownarrow \\ \{\mathbb{N}k^\times\text{-orbits of nontrivial characters } \psi_0 \text{ of } k_0\} \end{array}$$

in the even skew-hermitian case.

Remarks: Note that when $W = 0$, there is no need to invoke the Weil representation at all. Hence, the above description of generic characters could be carried out for hermitian spaces (of even dimension $2n$) as well. One would consider the situation

$$W = 0 \subset V \quad \text{of hermitian spaces.}$$

and note that the homomorphisms

$$f : N \longrightarrow k^{n-1}$$

and

$$f_0 : N = N_0 \longrightarrow k$$

can still be defined by the same formulas. But now the image of f_0 lies in the subspace of trace zero elements of k (as opposed to the subfield k_0 when

V is skew-hermitian). The torus actions on f and f_0 are again given by the same formulas. Thus, giving a T-orbit of generic characters of N in the even hermitian case amounts to giving a nontrivial character of k trivial on k_0, up to the action of $\mathbb{N}k^\times$.

This completes our definition of the representation ν of H.

As we noted in the course of the discussion above, special cases of the pair (H, ν) give the determination of T-orbits of generic characters. Recall that if $G = G(V)$ is quasi-split, with Borel subgroup $B = T \cdot N$, then the set D of $T(k_0)$-orbits of generic characters of N is a principal homogeneous space for the abelian group

$$E = T^{ad}(k_0)/\mathrm{Im}\ T(k_0) = \ker(H^1(k_0, Z) \to H^1(k_0, T)).$$

When $G = \mathrm{GL}(V)$ or $\mathrm{U}(V)$ with $\dim V$ odd or $\mathrm{SO}(V)$ with $\dim V$ odd or $\dim V = 2$, the group E is trivial. In the remaining cases, E is a finite elementary abelian 2-group. In Section 10, we have described the E-torsor D explicitly for the various classical groups $G(V)$, but did not say how this was done. Our discussion of (H, ν) above has thus filled this gap, and we record the result in the following proposition for ease of reference.

Proposition 12.1. *(1) If V is symplectic, $E = k^\times/k^{\times 2}$, and we have constructed an explicit bijection of E-spaces*

$$D \longleftrightarrow k^{\times 2}\text{-orbits on nontrivial } \psi : k \to \mathbb{C}^\times.$$

(2) If V is hermitian of even dimension, $E = k_0^\times/\mathbb{N}k^\times$, and we have constructed an explicit bijection of E-spaces

$$D \longleftrightarrow \mathbb{N}k^\times\text{-orbits on nontrivial } \psi : k/k_0 \to \mathbb{C}$$

(3) If V is skew-hermitian of even dimension, $E = k_0^\times/\mathbb{N}k^\times$, and we have constructed an explicit bijection of E-spaces

$$D \longleftrightarrow \mathbb{N}k^\times\text{-orbits on nontrivial } \psi_0 : k_0 \to \mathbb{C}^\times$$

(4) If V is orthogonal of even dimension and split by the quadratic algebra K, then $E = \mathbb{N}K^\times/k^{\times 2}$, and we have constructed an explicit bijection of E-spaces

$$D \longleftrightarrow \mathrm{SO}(V)\text{-orbits on non-isotropic lines } L \subset V, \text{ with } L^\perp \text{ split.}$$

We stress that the bijections constructed in Proposition 12.1 depend crucially on the form $\langle -, - \rangle$ on V.

13. Bessel and Fourier-Jacobi models for GL(n)

The construction of the pair (H, ν) given in the previous section includes the case when $k = k_0 \times k_0$ is the split quadratic algebra. In this case, the groups $G(V)$ and $G(W)$ are general linear groups, and it is useful to give a direct construction of (H, ν) in the context of general linear groups, rather than regarding them as unitary groups of hermitian or skew-hermitian spaces over k. We describe this direct construction in this section.

We first give a brief explanation of how one translates from the context of unitary groups to that of general linear groups. The hermitian or skew-hermitian space V has the form

$$V = V_0 \times V_0^\vee$$

for a vector space V_0 over k_0. Moreover, up to isomorphism, the hermitian form on V can be taken to be

$$\langle (x, x^\vee), (y, y^\vee) \rangle = (\langle x, y^\vee \rangle, \langle y, x^\vee \rangle) \in k,$$

whereas the skew-hermitian form on V can be taken to be

$$\langle (x, x^\vee), (y, y^\vee) \rangle = (\langle x, y^\vee \rangle, -\langle y, x^\vee \rangle) \in k.$$

Then, by restriction to V_0, one has an isomorphism

$$G(V) \cong \mathrm{GL}(V_0)$$

of linear algebraic groups over k_0.

If $W \subset V$ is a nondegenerate subspace, then $W = W_0 \times W_0^\vee$ gives rise to $W_0 \subset V_0$. If, further, $W^\perp$ is split, and $X \subset W^\perp$ is a maximal isotropic subspace, then X has the form

$$X = X_0 \times Y_0^\vee \subset V_0 \times V_0^\vee$$

with the natural pairing of X_0 and $Y_0^\vee$ equal to zero, so that X_0 is contained in the kernel of $Y_0^\vee$. Writing the kernel of $Y_0^\vee$ as $X_0 + W_0$, we see that the isotropic space X determines a decomposition

$$V_0 = X_0 + W_0 + Y_0,$$

with a natural perfect pairing between Y_0 and $Y_0^\vee$. Then the parabolic subgroup $P(X)$ stabilizing X in $G(V)$ is isomorphic to the parabolic subgroup of $\mathrm{GL}(V_0)$ stabilizing the flag

$$X_0 \subset X_0 + W_0 \subset V_0.$$

It is now easy to translate the construction of (H, ν) given in the previous section to the setting of $W_0 \subset V_0$, and we simply describe the answer below.

Bessel Models for $\mathrm{GL}(n)$

In this case, we start with a vector space V_0 over k_0 with a decomposition

$$V_0 = X_0 + W_0 + E_0 + X_0^\vee,$$

where $E_0 = \langle e \rangle$ is a line. Consider the (non-maximal) parabolic subgroup Q stabilizing the flag

$$X_0 \subset X_0 + W_0 + E_0 \subset V_0.$$

It has Levi subgroup

$$L = \mathrm{GL}(X_0) \times \mathrm{GL}(W_0 + E_0) \times \mathrm{GL}(X_0^\vee)$$

and unipotent radical U sitting in the exact sequence:

$$0 \longrightarrow \mathrm{Hom}(X_0^\vee, X_0) \longrightarrow U \longrightarrow \mathrm{Hom}(X_0^\vee, W_0 + E_0) + \mathrm{Hom}(W_0 + E_0, X_0) \longrightarrow 0.$$

We may write the above exact sequence as:

$$0 \longrightarrow X_0 \otimes X_0 \longrightarrow U \longrightarrow X_0 \otimes (W_0 + E_0) + (W_0^\vee + E_0^\vee) \otimes X_0 \longrightarrow 0,$$

where $E_0^\vee = \langle f \rangle$ is the dual of E_0.

Let

$$\ell_{X_0} : X_0 \longrightarrow k_0$$

be any nontrivial homomorphism, and let $U_{X_0} \times U_{X_0^\vee}$ be a maximal unipotent subgroup of $\mathrm{GL}(X_0) \times \mathrm{GL}(X_0^\vee)$ which fixes ℓ_{X_0}. On the other hand, let

$$\ell_{W_0} : (W_0 + E_0) + (W_0^\vee + E_0^\vee) \longrightarrow k_0$$

be a linear form which is trivial on $W_0 + W_0^\vee$ but nontrivial on E_0 and $E_0^\vee$. Together, the homomorphisms ℓ_{X_0} and ℓ_{W_0} give a map

$$\ell = \ell_{X_0} \otimes \ell_{W_0} : U \longrightarrow X_0 \otimes (W_0 + E_0) + (W_0^\vee + E_0^\vee) \otimes X_0 \longrightarrow k_0.$$

Since ℓ is fixed by $U_{X_0} \times U_{X_0^\vee} \times \mathrm{GL}(W_0)$, we may extend ℓ trivially to this group. Thus, we may regard ℓ as a map on

$$H = U \rtimes ((U_{X_0} \times U_{X_0^\vee}) \times \mathrm{GL}(W_0)).$$

Choose any nontrivial additive character ψ_0 of k_0 and any generic character

$$\lambda : U_{X_0} \times U_{X_0^\vee} \longrightarrow \mathbb{S}^1,$$

which we may regard as a character of H. Then the representation ν of H is defined by

$$\nu = (\psi_0 \circ \ell) \otimes \lambda.$$

The pair (H, ν) depends only on the spaces $W_0 \subset V_0$, up to conjugacy by $\mathrm{GL}(V_0)$. This completes the construction of (H, ν) in the case when codimension of W_0 in V_0 is odd.

Fourier-Jacobi models for $\mathrm{GL}(n)$

In this case, we consider a vector space V_0 over k_0, together with a decomposition

$$V_0 = X_0 + W_0 + X_0^\vee.$$

As before, let Q be the parabolic subgroup stabilizing the flag

$$X_0 \subset X_0 + W_0 \subset V_0.$$

Thus Q has Levi subgroup

$$L = \mathrm{GL}(X_0) \times \mathrm{GL}(W_0) \times \mathrm{GL}(X_0^\vee),$$

and its unipotent radical U sits in the exact sequence,

$$0 \to \mathrm{Hom}(X_0^\vee, X_0) \to U \to \mathrm{Hom}(X_0^\vee, W_0) + \mathrm{Hom}(W_0, X_0) \to 0,$$

in which $\mathrm{Hom}(X_0^\vee, X_0)$ is central. The group U is completely described by the natural bilinear map

$$\mathrm{Hom}(X_0^\vee, W_0) \times \mathrm{Hom}(W_0, X_0) \to \mathrm{Hom}(X_0^\vee, X_0).$$

Indeed, given a bilinear map

$$\langle -, - \rangle : B \times C \to A,$$

of vector groups, there is a natural central extension of $B \times C$ by A defined by a group structure on $A \times B \times C$ given by

$$(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1 + a_2 + \langle b_1, c_2 \rangle, b_1 + b_2, c_1 + c_2).$$

Given a linear map

$$\ell_{X_0} : X_0 \to k_0,$$

let

$$U_{X_0} \times U_{X_0^\vee} \subset \mathrm{GL}(X_0) \times \mathrm{GL}(X_0^\vee),$$

be a maximal unipotent subgroup fixing ℓ_{X_0}. Let

$$\lambda : U_{X_0} \times U_{X_0^\vee} \longrightarrow \mathbb{S}^1$$

be a generic character, which we may regard as a character of

$$H = U \rtimes (U_{X_0} \times U_{X_0^\vee} \times \mathrm{GL}(W_0))$$

via projection onto $U_{X_0} \times U_{X_0^\vee}$.

On the other hand, the homomorphism ℓ_{X_0} allows one to define a homomorphism from U to the Heisenberg group $H(W_0 + W_0^\vee)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_0 \otimes X_0 & \longrightarrow & U & \longrightarrow & W_0^\vee \otimes X_0 + X_0 \otimes W_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & k_0 & \longrightarrow & H(W_0^\vee + W_0) & \longrightarrow & W_0^\vee + W_0 & \longrightarrow & 0,
\end{array}$$

which is clearly equivariant under the action of $U_{X_0} \times U_{X_0^\vee} \times \mathrm{GL}(W_0)$.

Thus, given any nontrivial additive character ψ_0 of k_0, we may consider the unique irreducible representation of $H(W_0^\vee + W_0)$ with central character ψ_0, and regard it as a representation of U using the above diagram. This representation can be extended trivially to $U_{X_0} \times U_{X_0^\vee}$, and is realized naturally on the space $\mathcal{S}(W_0)$ of Schwarz-Bruhat functions on W_0. For any character $\mu : k_0^\times \longrightarrow \mathbb{C}^\times$, one then obtains a Weil representation $\omega_{\psi_0,\mu}$ of

$$H = (\mathrm{GL}(W_0) \times U_{X_0} \times U_{X_0^\vee}) \ltimes U.$$

on $\mathcal{S}(W_0)$, where the action of $\mathrm{GL}(W_0)$ is given by

$$(g \cdot f)(w) = \mu(\det(g)) \cdot f(g^{-1} \cdot w).$$

When $\dim X_0 > 0$, the representation $\nu_{\psi,\mu}$ of H is then given by

$$\nu_{\psi_0,\mu} = \omega_{\psi_0,\mu} \otimes \lambda.$$

When $\dim X_0 = 0$, we have $W_0 = V_0$ and we take the representation $\nu_{\psi_0,\mu}$ of $H = \mathrm{GL}(W_0)$ to be the representation $\omega_{\psi_0,\mu}$ of $\mathrm{GL}(W_0)$ on $\mathcal{S}(W_0)$ defined above. In either case, the isomorphism class of $\omega_{\psi_0,\mu}$ is independent of ψ_0 and the pair $(H, \nu_{\psi_0,\mu})$ is independent of ψ_0, up to conjugacy in $\mathrm{GL}(V_0)$. This completes the definition of (H, ν) when the codimension of W_0 in V_0 is even.

This concludes our direct construction of the pair (H, ν) for general linear groups.

14. Restriction Problems and Multiplicity One Theorems

We are now ready to formulate the local restriction problems studied in this paper.

Let $W \subset V$ be as in §12, so that $G = G(V) \times G(W)$ contains the subgroup $H = N \rtimes G(W)$. We have defined a representation ν of H (or its double cover), which may depend on some auxiliary data such as ψ or μ. Let $\pi = \pi_V \boxtimes \pi_W$ be an irreducible representation of G (or an appropriate double cover). Then the restriction problem of interest is to determine

$$\dim_{\mathbb{C}} \mathrm{Hom}_H(\pi \otimes \overline{\nu}, \mathbb{C}).$$

More precisely, we have:

(1) In the orthogonal or hermitian cases, the representation ν of H depends only on $W \subset V$ and so we set

$$d(\pi) = \dim_{\mathbb{C}} \mathrm{Hom}_H(\pi \otimes \overline{\nu}, \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Hom}_H(\pi, \nu).$$

In the literature, this restriction problem is usually referred to as a problem about the existence of Bessel models, and for an irreducible representation $\pi = \pi_V \boxtimes \pi_W$ of G, the space $\mathrm{Hom}_H(\pi, \nu)$ usually called the space of $\pi_W^\vee$-Bessel models of π_V.

(2) In the symplectic case, the representation ν_ψ is a representation of the double cover $\widetilde{H} = N \rtimes \widetilde{\mathrm{Sp}}(W)$ and depends on a nontrivial additive character ψ of $k = k_0$ up to the action of $k^{\times 2}$. In this case, for the above Hom space to be nonzero, the representation $\pi = \pi_V \boxtimes \pi_W$ must be a genuine representation when restricted to $\widetilde{H}$. Hence, we have to take an irreducible representation

$$\widetilde{\pi} = \pi_V \boxtimes \widetilde{\pi}_W \quad \text{of} \quad \mathrm{Sp}(V) \times \widetilde{\mathrm{Sp}}(W)$$

or

$$\widetilde{\pi} = \widetilde{\pi}_V \boxtimes \pi_W \quad \text{of} \quad \widetilde{\mathrm{Sp}}(V) \times \mathrm{Sp}(W).$$

In this case, we set

$$d(\tilde{\pi}, \psi) = \dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{H}}(\widetilde{\pi} \otimes \overline{\nu}_\psi, \mathbb{C}).$$

In the literature, this restriction problem is usually referred to as one about Fourier-Jacobi models, and the space $\mathrm{Hom}_{\widetilde{H}}(\widetilde{\pi} \otimes \overline{\nu}_\psi, \mathbb{C})$ usually called the $(\pi_W^\vee, \psi)$-Fourier-Jacobi models of π_V.

(3) In the skew-hermitian case, the representation $\nu_{\psi_0,\mu}$ of H depends on a nontrivial additive character ψ_0 of k_0, up to the action of $\mathbb{N}k^\times$, and also on the choice of a character μ of $k^\times$ whose restriction to $k_0^\times$ is the quadratic character associated to k/k_0. In this case, we set

$$d(\pi, \mu, \psi_0) = \dim_{\mathbb{C}} \mathrm{Hom}_H(\pi \otimes \overline{\nu}_{\psi_0,\mu}, \mathbb{C}).$$

In the literature, this restriction problem is usually referred to as a problem about the existence of Fourier-Jacobi models in the context of unitary groups, and the space $\mathrm{Hom}_H(\widetilde{\pi} \otimes \overline{\nu}_{\psi_0,\mu}, \mathbb{C})$ usually called the $(\pi_W^\vee, \psi_0, \mu)$-Fourier-Jacobi models of π_V.

We remark that in the orthogonal and hermitian cases, since ν is 1-dimensional and unitary, one has:

$$\mathrm{Hom}_H(\pi \otimes \overline{\nu}, \mathbb{C}) \cong \mathrm{Hom}_H(\pi \otimes \nu^\vee, \mathbb{C}) \cong \mathrm{Hom}_H(\pi, \nu).$$

In the symplectic and skew-hermitian cases over non-archimedean fields, the same assertion holds, even though ν is infinite-dimensional. However, over archimedean fields, it is only clear to us that:

$$\mathrm{Hom}_H(\pi, \nu) \subseteq \mathrm{Hom}_H(\pi \otimes \nu^\vee, \mathbb{C}) \cong \mathrm{Hom}_H(\pi \otimes \overline{\nu}, \mathbb{C}).$$

The difficulty arises in the subtlety of duality in the theory of topological vector spaces. In any case, we work with $\mathrm{Hom}_H(\pi \otimes \overline{\nu}, \mathbb{C})$ since this is the space which naturally arises in the global setting. We should also mention that, over archimedean fields, the tensor product $\pi \otimes \overline{\nu}$ refers to the natural completed tensor product of the two spaces (which are nuclear Fréchet spaces) and $\mathrm{Hom}(-, \mathbb{C})$ refers to continuous linear functionals. For a discussion of these archimedean issues, see [AG, Appendix A].

A basic conjecture in the subject is the assertion that

$$d(\pi) \leq 1$$

in the various cases. Recently, there has been much progress in the most basic cases where $\dim W^{\perp} = 0$ or 1. We describe these in the following theorem.

Theorem 14.1. *Assume that k has characteristic zero and $\dim W^{\perp} = 0$ or 1.*
(i) In the orthogonal case, with $G = \mathrm{O}(V) \times \mathrm{O}(W)$ or $\mathrm{SO}(V) \times \mathrm{SO}(W)$, we have

$$d(\pi) \leq 1.$$

(ii) In the hermitian case (including the case when $k = k_0 \times k_0$), we have

$$d(\pi) \leq 1.$$

(iii) In the symplectic case, suppose that k is non-archimedean. Then we have

$$d(\pi, \psi) \leq 1.$$

(iv) In the skew-hermitian case (including the case $k = k_0 \times k_0$), suppose that k is non-archimedean and that $\pi = \pi_1 \boxtimes \pi_2$ with π_2 supercuspidal. Then we have

$$d(\pi, \mu, \psi_0) \leq 1.$$

Proof. For the groups $\mathrm{O}(V) \times \mathrm{O}(W)$ in (i) and $\mathrm{U}(V) \times \mathrm{U}(W)$ in (ii), the results are due to Aizenbud-Gourevitch-Rallis-Schiffmann [AGRS] in the p-adic case and to Sun-Zhu [SZ] and Aizenbud-Gourevitch [AG] in the archimedean case. This is extended to $G = \mathrm{SO}(V) \times \mathrm{SO}(W)$ by Waldspurger [Wa8] in the non-archimedean case and by Sun-Zhu [SZ2] in the archimedean case.

The case (iii) is a result of Sun [S] in the non-archimedean case (the archimedean case seems to be still open). The methods of [S] could likely be adapted to give (iv) completely in the non-archimedean case. However, we shall show how (iv) can also be deduced from (ii) using theta correspondence.

Thus, suppose that W is a skew-hermitian space and π_1 and π_2 are irreducible representations of $\mathrm{U}(W)$ with π_2 supercuspidal. For a hermitian space V of the same dimension as W, one may consider the theta lifting between $\mathrm{U}(W)$ and $\mathrm{U}(V)$, with respect to the fixed additive character ψ_0 and the fixed character μ. By [HKS], there is a (unique) V such that the theta lift of π_1 to $\mathrm{U}(V)$ is nonzero; in other words, there is an irreducible representation τ of $\mathrm{U}(V)$ such that $\Theta_{\psi_0,\mu}(\tau)$ has π_1 as a quotient. Now consider the seesaw diagram

$$\begin{tikzcd} \mathrm{U}(W)\times\mathrm{U}(W) \arrow[d, dash] \arrow[dr, dash] & \mathrm{U}(V\oplus\langle -1\rangle) \arrow[d, dash] \arrow[dl, dash] \\ \mathrm{U}(W) & \mathrm{U}(V)\times\mathrm{U}(\langle -1\rangle). \end{tikzcd}$$

The resulting seesaw identity gives

$$\begin{aligned} d(\pi_1\boxtimes\pi_2,\mu,\psi_0) &= \dim\mathrm{Hom}_{\mathrm{U}(W)}(\pi_1\otimes\overline{\omega_{\psi_0,\mu}},\pi_2^\vee) \\ &\leq \dim\mathrm{Hom}_{\mathrm{U}(W)}(\Theta_{\psi_0,\mu}(\tau)\otimes\overline{\omega_{\psi_0,\mu}},\pi_2^\vee) \\ &= \dim\mathrm{Hom}_{\mathrm{U}(V)}(\Theta_{\psi_0,\mu}(\pi_2^\vee),\tau) \\ &= \dim\mathrm{Hom}_{\mathrm{U}(V)}(\theta_{\psi_0,\mu}(\pi_2^\vee),\tau) \\ &\leq 1, \end{aligned}$$

as desired. Here, we have used the assumption that π_2 is supercuspidal to deduce that $\Theta_{\psi_0,\mu}(\pi_2^\vee)=\theta_{\psi_0,\mu}(\pi_2^\vee)$ is irreducible.

The above seesaw argument works in the case when $k=k_0\times k_0$ as well, thus completing the proof of (iv). Indeed, this argument shows more generally that the multiplicity one results in (ii) and (iv) are equivalent, modulo the issue of whether $\Theta(\tau)=\theta(\tau)$. Similarly, the analogous argument for symplectic-orthogonal dual pairs shows that the results of (i) and (iii) are equivalent, with the same caveat on Θ versus θ. $\square$

Remarks: For the group $\mathrm{GL}_n\times\mathrm{GL}_{n-1}$, the above multiplicity one result has been extended to the case when k has characteristic p by Aizenbud-Avni-Gourevitch [AAG].

15. Uniqueness of Bessel Models

In this section, we show that if k is non-archimedean, the multiplicity one theorem for the general Bessel models can be deduced from Theorem 14.1(i) and (ii) in the orthogonal and hermitian cases. We remind the reader that the case $k=k_0\times k_0$ is included in our discussion. In particular, the results we state below are valid in this case as well, though we frequently write our proofs only for k a field, and leave the adaptation to the case $k=k_0\times k_0$ to the reader.

Thus, we consider the case when $W\subset V$ are orthogonal or hermitian spaces of odd codimension. Then we have

$$W^\perp = X + X^\vee + E$$

where $E = k \cdot e$ is a non-isotropic line and

$$X = \langle v_1, v_2, \cdots, v_n \rangle$$

is an isotropic subspace with $\dim X = n > 0$ and dual basis $\{v_i^\vee\}$ of $X^\vee$. With $G = G(V) \times G(W)$ and $H = N \cdot G(W)$, we would like to show that

$$\dim \operatorname{Hom}_H(\pi_V \otimes \pi_W, \nu) \leq 1$$

for any irreducible representation $\pi_V \boxtimes \pi_W$ of G.

Let

$$E^- = k \cdot f$$

denote the rank 1 space equipped with a form which is the negative of that on E, so that $E + E^-$ is a split rank 2 space. The two isotropic lines in $E + E^-$ are spanned by

$$v_{n+1} = e + f \quad \text{and} \quad v_{n+1}^\vee = \frac{1}{2 \cdot \langle e, e \rangle} \cdot (e - f).$$

Now consider the space

$$W' = V \oplus E^-$$

which contains V with codimension 1 and isotropic subspaces

$$Y = X + k \cdot v_{n+1} = \langle v_1, \cdots, v_{n+1} \rangle$$

and

$$Y^\vee = X^\vee + k \cdot v_{n+1}^\vee = \langle v_1^\vee, \cdots, v_{n+1}^\vee \rangle.$$

Hence we have

$$W' = Y + Y^\vee + W.$$

Let $P = P(Y)$ be the parabolic subgroup of $G(W')$ stabilizing Y and let M be its Levi subgroup stabilizing Y and $Y^\vee$, so that

$$M \cong \mathrm{GL}(Y) \times G(W).$$

Let τ be an irreducible supercuspidal representation of $\mathrm{GL}(Y)$ and π_W an irreducible smooth representation of $G(W)$ and let

$$I(\tau, \pi_W) = \operatorname{Ind}_P^{G(W')}(\tau \boxtimes \pi_W)$$

be the (unnormalized) induced representation of $G(W')$ from the representation $\tau \boxtimes \pi_W$ of P.

Our goal is to prove the following theorem:

Theorem 15.1. *Assume that k is non-archimedean. With the notations as above, we have*

$$\operatorname{Hom}_{G(V)}(I(\tau, \pi_W) \otimes \pi_V, \mathbb{C}) = \operatorname{Hom}_H(\pi_V \otimes \pi_W, \nu)$$

as long as $\pi_V^\vee$ does not belong to the Bernstein component of $G(V)$ associated to $(\mathrm{GL}(Y') \times M', \tau \otimes \mu')$, where $Y' \subset V$ is isotropic of dimension equal to $\dim Y$ *with*

$V = Y' + Y'^{\vee} + V'$, M' is a Levi subgroup of $G(V')$ and μ' is any irreducible supercuspidal representation of M'.

Proof. We assume that k is a field in the proof and calculate the restriction of $\Pi := I(\tau, \pi_W)$ to $G(V)$ by Mackey's orbit method. For this, we begin by observing that $G(V)$ has at most two orbits on the flag variety $G(W')/P(Y)$ consisting of:

(1) $(n+1)$-dimensional isotropic subspaces of W' which are contained in V; these exist if and only if W is isotropic, in which case if Y' is a representative of this closed orbit, then its stabilizer in $G(V)$ is the parabolic subgroup $P_V(Y') = P(Y') \cap G(V)$;

(2) $(n+1)$-dimensional isotropic subspaces of W' which are not contained in V; a representative of this open orbit is the space Y and its stabilizer in $G(V)$ is the subgroup $Q = P(Y) \cap G(V)$.

By Mackey theory, this gives a filtration on the restriction of Π to $G(V)$ as follows:

$$0 \longrightarrow \mathrm{ind}_Q^{G(V)}(\tau \otimes \pi_W)|_Q \longrightarrow \Pi|_{G(V)} \longrightarrow \mathrm{Ind}_{P_V(Y')}^{G(V)} \tau \otimes \pi_W|_{G(V')} \longrightarrow 0,$$

where the induction functors here are unnormalized and where ind denotes the induction with compact support.

Denoting the above short exact sequence by

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

for simplicity, we have an exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{G(V)}(C, \pi_V^{\vee}) \longrightarrow \mathrm{Hom}_{G(V)}(B, \pi_V^{\vee}) \longrightarrow \mathrm{Hom}_{G(V)}(A, \pi_V^{\vee}) \\ \longrightarrow \mathrm{Ext}^1_{G(V)}(C, \pi_V^{\vee}). \end{aligned}$$

By our assumption on $\pi_V^{\vee}$, we have:

$$\mathrm{Hom}_{G(V)}(\mathrm{Ind}_{P_V(Y')}^{G(V)}(\tau \otimes \mu), \pi_V^{\vee}) = 0$$

and

$$\mathrm{Ext}^1_{G(V)}(\mathrm{Ind}_{P_V(Y')}^{G(V)}(\tau \otimes \mu), \pi_V^{\vee}) = 0$$

for any smooth (not necessarily of finite length) representation μ of $G(V')$. Thus, $\mathrm{Hom}_{G(V)}(C, \pi_V^{\vee}) = 0 = \mathrm{Ext}^1(C, \pi_V^{\vee})$ and we have

$$\mathrm{Hom}_{G(V)}(\mathrm{ind}_Q^{G(V)}(\tau \otimes \pi_W)|_Q, \pi_V^{\vee}) = \mathrm{Hom}_{G(V)}(\Pi, \pi_V^{\vee}).$$

It thus suffices to analyze the representations of $G(V)$ which appear on the open orbit. For this, we need to determine the group $Q = P(Y) \cap G(V)$ as a subgroup of $G(V)$ and $P(Y)$.

Recall that $W' = Y \oplus W \oplus Y^\vee$, and V is the codimension 1 subspace $X \oplus W \oplus X^\vee \oplus E$ which is the orthogonal complement of f. It is not difficult to see that as a subgroup of $G(V)$,

$$Q = G(V) \cap P(Y) \subset P_V(X).$$

Indeed, if $g \in Q$, then g fixes f and stabilizes Y, and we need to show that it stabilizes X. If $x \in X$, it suffices to show that $\langle g \cdot x \,, e - f \rangle = 0$. But

$$\langle g \cdot x \,, e - f \rangle = \langle x \,, g^{-1} \cdot (e - f) \rangle = \langle x \,, g^{-1} \cdot v_{n+1} - 2f \rangle = 0,$$

as desired.

Now we claim that as a subgroup of $P_V(X)$,

$$Q = (\mathrm{GL}(X) \times G(W)) \ltimes N_V(X),$$

where $N_V(X)$ is the unipotent radical of $P_V(X)$. To see this, given an element $h \in P_V(X)$, note that $h \in Q$ if and only if

$$h \cdot v_{n+1} \in Y, \quad \text{or equivalently} \quad h \cdot e - e \in Y.$$

We may write

$$h \cdot e = \lambda \cdot e + w + x, \quad \text{with } w \in W \text{ and } x \in X.$$

Then we see that $h \cdot e - e \in Y$ if and only if $\lambda = 1$ and $w = 0$, so that h fixes e modulo X and hence stabilizes W modulo X, in which case $h \in (\mathrm{GL}(X) \times G(W)) \ltimes N_V(X)$, as desired.

Since we are restricting the representation $\tau \boxtimes \pi_W$ of $P(Y)$ to the subgroup Q, we also need to know how Q sits in $P(Y)$. For this, note the following lemma.

Lemma 15.2. *The natural projection $pr : P(Y) \twoheadrightarrow \mathrm{GL}(Y) \times G(W)$ induces the following commutative diagram with exact rows, where the vertical arrows are inclusions:*

$$\begin{tikzcd}
0 \arrow[r] & N(Y) \arrow[r] & P(Y) \arrow[r, "pr"] & \mathrm{GL}(Y) \times G(W) \arrow[r] & 0 \\
0 \arrow[r] & N(Y) \cap Q \arrow[r] \arrow[u] & Q \arrow[r] \arrow[u] & R \times G(W) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & N(Y) \cap Q \arrow[r] \arrow[u, equal] & N_V(X) \arrow[r] \arrow[u] & \mathrm{Hom}(k \cdot v_{n+1}, X) \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

Here

$$R \subset \mathrm{GL}(Y)$$

is the mirabolic subgroup which stabilizes the codimension one subspace $X \subset Y$ and fixes v_{n+1} modulo X and $\mathrm{Hom}(k \cdot v_{n+1}, X)$ *is the unipotent radical of R.*

Proof. The projection *pr* is given by the action of $P(Y)$ on $Y \times (Y+W)/Y$. Consider the restriction of *pr* to the subgroup

$$Q = G(V) \cap P(Y) = (\mathrm{GL}(X) \times G(W)) \ltimes N_V(X)$$

of $P(Y)$. We note:

(i) the subgroup $\mathrm{GL}(X) \times G(W)$ maps isomorphically to its image in $\mathrm{GL}(Y) \times G(W)$, and its image is precisely the Levi subgroup $\mathrm{GL}(X) \times G(W)$ of $R \times G(W)$. This is clear.

(ii) the kernel $N(Y) \cap Q$ of $pr|_Q$ is contained in $N_V(X)$. To see this, suppose that $n \in N(Y) \cap Q$. Then $n \in G(V)$ (since $Q \subset G(V)$), so that $n \cdot f = f$. To show that $n \in N_V(X)$, we need to show that n acts trivially on X and acts trivially on $W \oplus E$ modulo X.

Now, as an element of $N(Y)$, n acts trivially on Y and acts trivially on W modulo Y. Thus, n certainly acts trivially on $X \subset Y$, and for $w \in W$, $n \cdot w - w \in Y \cap V = X$. It remains to show that n acts trivially on E modulo X, i.e. that $n \cdot e - e \in X = Y \cap V$. Since $n \cdot e - e$ lies in V, it suffices to show that $n \cdot e - e$ lies in Y. But we have:

$$n \cdot e - e = n \cdot (e+f) - (e_f) = n \cdot v_{n+1} - v_{n+1} \in Y.$$

This proves the $N(Y) \cap Q \subset N_V(X)$.

(iii) the projection *pr* induces an isomorphism:

$$N_V(X)/(N(Y) \cap Q) \cong \mathrm{Hom}(k \cdot v_{n+1}, X).$$

Indeed, if $n \in N_V(X)$, then n fixes X and fixes W modulo X. Moreover, since n fixes f (as it is in $G(V)$) and fixes e modulo X, we have

$$n \cdot v_{n+1} - v_{n+1} = n \cdot (e+f) - (e+f) \in X.$$

This shows that $pr(n)$ lies in the unipotent radical $\mathrm{Hom}(k \cdot v_{n+1}, X)$ of R. Indeed,

$$pr(n)(v_{n+1}) = n \cdot e - e.$$

It remains to show that $pr|_{N_V(X)}$ is surjective onto $\mathrm{Hom}(kv_{n+1}, X)$. For any $x \in X$, let $n_x \in N_V(X)$ be the element which fixes X and W and such that $n_x(e) = e + x$. Then $pr(n_x)(v_{n+1}) = x$, as desired.

In view of the above, we see that the image of Q under *pr* is precisely $R \times G(W)$ and the image of $N_V(X)$ is the unipotent radical of R. The lemma is proved. $\square$

By the lemma, one has:

$$(\tau \boxtimes \pi_W)|_Q = \tau|_R \boxtimes \pi_W.$$

By a well-known result of Gelfand-Kazhdan, since τ is supercuspidal, one knows that

$$\tau|_R \cong \mathrm{ind}_U^R \chi$$

where U is the unipotent radical of the Borel subgroup of $\mathrm{GL}(Y)$ stabilizing the flag

$$\langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \cdots, v_{n+1} \rangle = Y,$$

and χ is any generic character of U.

Now it is clear from the lemma that the pre-image of $U \times G(W)$ in Q is precisely the subgroup

$$H = (U_X \times G(W)) \ltimes N_V(X) \subset G(V),$$

where U_X is the unipotent subgroup of $\mathrm{GL}(X)$ stabilizing the flag

$$\langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \cdots, v_n \rangle = X.$$

Further, the representation $\chi \boxtimes \pi_W$ of $U \times G(W)$ pulls back to the representation $\nu^\vee \otimes \pi_W$ of H. Indeed, the pre-image of U in Q is the subgroup $U_X \ltimes N_V(X)$, and the pullback of χ is in general position when restricted to U_X. Moreover, when restricted to $N_V(X)$, the pullback of χ is nontrivial and fixed by $U_X \times G(W)$.

Hence, by induction in stages, we conclude that

$$\mathrm{ind}_Q^{G(V)}(\tau \otimes \pi_W)|_Q \cong \mathrm{ind}_H^{G(V)} \pi_W \otimes \nu^\vee.$$

Thus, by dualizing and Frobenius reciprocity, one has

$$\mathrm{Hom}_{G(V)}(I(\tau, \pi_W), \pi_V^\vee) \cong \mathrm{Hom}_H(\pi_V \otimes \pi_W, \nu).$$

This completes the proof of the theorem. □

Corollary 15.3. *In the orthogonal or hermitian cases over a non-archimedean k, with $W \subset V$ of odd codimension, we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_H(\pi, \nu) \leq 1$$

for any irreducible representation π of $G = G(V) \times G(W)$.

Proof. To apply Theorem 15.1, choose a supercuspidal representation τ which does not belong to the Bernstein components in Theorem 15.1. Then, replacing τ by its twist by an unramified character, we may assume that the associated induced representation $I(\tau, \pi_W)$ is irreducible; this is possible by a result of Waldspurger [Sa]. Then, by Theorem 15.1, the corollary is reduced to Theorem 14.1. □

Remarks: In the archimedean case, a recent paper of Jiang-Sun-Zhu [JSZ] adapted the proof of Theorem 15.1 to show the containment

$$\mathrm{Hom}_H(\pi_V \otimes \pi_W, \nu) \subset \mathrm{Hom}_{G(V)}(I(\tau, \pi_W) \otimes \pi_V, \mathbb{C}).$$

Namely, to each element on the left hand side, [JSZ] constructs an associated element on the right hand side, using an explicit integral. This is enough to deduce the multiplicity one result of Corollary 15.3 from the results of [SZ], [SZ2] and [AG].

16. Uniqueness of Fourier-Jacobi Models

In this section, we continue with the assumption that k is non-archimedean and our goal is to establish the analog of Theorem 15.1 in the symplectic and skew-hermitian cases, which will imply that

$$d(\pi,\psi) \leq 1.$$

Before coming to the analogous result, which is given in Theorem 16.1, we need to recall certain structural results about parabolic induction for the metaplectic groups.

Recall that if W is a symplectic space, then a parabolic subgroup $\widetilde{P}$ in $\widetilde{\mathrm{Sp}}(W)$ is nothing but the inverse image of a parabolic P in $\mathrm{Sp}(W)$. It is known that the metaplectic covering splits (uniquely) over unipotent subgroups, so for a Levi decomposition $P = M \cdot N$, it makes sense to speak of the corresponding Levi decomposition

$$\widetilde{P} = \widetilde{M} \cdot N \quad \text{in} \quad \widetilde{\mathrm{Sp}}(W).$$

Furthermore, we note that for a maximal parabolic subgroup $P(X)$ of $\mathrm{Sp}(W)$ with Levi subgroup of the form $M = \mathrm{GL}(X) \times \mathrm{Sp}(W_0)$ in $\mathrm{Sp}(W)$,

$$\widetilde{M} = \left(\widetilde{\mathrm{GL}}(X) \times \widetilde{\mathrm{Sp}}(W_0)\right)/\Delta\mu_2$$

where $\widetilde{\mathrm{GL}}(X)$ is a certain two-fold cover of $\mathrm{GL}(X)$ defined as follows. As a set, we write

$$\widetilde{\mathrm{GL}}(X) = \mathrm{GL}(X) \times \{\pm 1\},$$

and the multiplication is given by

$$(g_1,\epsilon_1)\cdot(g_2,\epsilon_2) = (g_1g_2, \epsilon_1\epsilon_2 \cdot (\det g_1, \det g_2)),$$

where $(-,-)$ denotes the Hilbert symbol on $k^\times$ with values in $\{\pm 1\}$.

The two-fold cover $\widetilde{\mathrm{GL}}(X)$ has a natural genuine 1-dimensional character

$$\chi_\psi : \widetilde{\mathrm{GL}}(X) \longrightarrow \mathbb{C}^\times$$

defined as follows. The determinant map gives rise to a natural group homomorphism

$$\det : \widetilde{\mathrm{GL}}(X) \longrightarrow \widetilde{\mathrm{GL}}(\wedge^{top} X) = \widetilde{\mathrm{GL}}(1).$$

On the other hand, one has a genuine character on $\widetilde{\mathrm{GL}}(1)$ defined by

$$(a,\epsilon) \mapsto \epsilon \cdot \gamma(a,\psi)^{-1},$$

where

$$\gamma(a,\psi) = \gamma(\psi_a)/\gamma(\psi)$$

and $\gamma(\psi)$ is an 8-th root of unity associated to ψ by Weil. Composing this character with det gives the desired genuine character χ_ψ on $\widetilde{\mathrm{GL}}(X)$, which satisfies:

$$\chi_\psi^2(g,\epsilon) = (\det(g), -1).$$

Thus, there is a bijection between the set of irreducible representations of $\mathrm{GL}(X)$ and the set of genuine representations of $\widetilde{\mathrm{GL}}(X)$, given simply by

$$\tau \mapsto \tilde{\tau}_\psi = \tau \otimes \chi_\psi.$$

Note that this bijection depends on the additive character ψ of k. Now associated to a representation τ of $\mathrm{GL}(X)$ and π_0 of $\widetilde{\mathrm{Sp}}(W_0)$, one has the representation

$$\tilde{\tau}_\psi \boxtimes \pi_0 \quad \text{of} \quad \widetilde{M}.$$

Then one can consider the (unnormalized) induced representation

$$I_\psi(\tau, \pi_0) = \mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{Sp}}(W)}(\tilde{\tau}_\psi \boxtimes \pi_0).$$

Here is the analog of Theorem 15.1 in the symplectic case.

Theorem 16.1. *Consider* $W = X \oplus W_0 \oplus X^\vee$ *with* $X \neq 0$ *and fix the additive character* ψ *of the non-archimedean local field* k*. Let*

- τ *be a supercuspidal representation of* $\mathrm{GL}(X)$*;*
- π_0 *be a genuine representation of* $\widetilde{\mathrm{Sp}}(W_0)$*;*
- π *be an irreducible representation of* $\mathrm{Sp}(W)$*,*

and consider the (unnormalized) induced representation $I_\psi(\tau, \pi_0)$ *of* $\widetilde{\mathrm{Sp}}(W)$*. Assume that* $\pi^\vee$ *does not belong to the Bernstein component associated to* $(\mathrm{GL}(X) \times M, \tau \boxtimes \mu)$*, where* M *is any Levi subgroup of* $\mathrm{Sp}(W_0)$ *and* μ *is any supercuspidal representation of* M*. Then*

$$\mathrm{Hom}_{\widetilde{\mathrm{Sp}}(W)}(I_\psi(\tau, \pi_0) \otimes \pi, \omega_{W,\psi}) \cong \mathrm{Hom}_H(\pi \otimes \pi_0, \nu_{W,W_0,\psi}).$$

Proof. We shall compute

$$\mathrm{Hom}_{\widetilde{\mathrm{Sp}}(W)}(I_\psi(\tau, \pi_0) \otimes \omega_{W,\psi}^\vee, \pi^\vee).$$

Let $P(X) = M(X) \cdot N(X)$ be the parabolic subgroup in $\mathrm{Sp}(W)$ stabilizing the subspace X, so that

$$\widetilde{M}(X) \cong \left(\widetilde{\mathrm{GL}}(X) \times \widetilde{\mathrm{Sp}}(W_0) \right) / \Delta\mu_2.$$

The Weil representation $\omega_{W,\overline{\psi}} = \omega_{W,\psi}^\vee$ has a convenient description as a $\widetilde{P}(X)$-module; this is the so-called mixed model of the Weil representation. This model of $\omega_{W,\psi}^\vee$ is realized on the space

$$\mathbb{S}(X^\vee) \otimes \omega_{W_0,\psi}^\vee$$

of Schwartz-Bruhat functions on $X^\vee$ valued in $\omega_{W_0,\psi}^\vee$. In particular, evaluation at 0 gives a $\widetilde{P}(X)$-equivariant map

$$ev : \omega_{W,\psi}^\vee \longrightarrow \chi_{\overline{\psi}} |\det{}_X|^{1/2} \boxtimes \omega_{W_0,\psi}^\vee,$$

where $N(X)$ acts trivially on the target space. In fact, this map is the projection of $\omega_{W,\psi}^\vee$ onto its space of $N(X)$-coinvariants.

On the other hand, to determine the kernel of the map ev, note that $\mathrm{GL}(X)$ acts transitively on the nonzero elements of $X^\vee$. Recall that we have fixed a basis $\{v_1, \cdots, v_n\}$ of X in the definition of the data (H, ν_ψ), with dual basis $\{v_1^\vee, \cdots, v_n^\vee\}$ of $X^\vee$. Let R be the stabilizer of $v_n^\vee$ in $\mathrm{GL}(X)$, so that R is a mirabolic subgroup of $\mathrm{GL}(X)$. Let

$$Q = (R \times \mathrm{Sp}(W_0)) \cdot N(X) \subset P(X)$$

so that its inverse image in $\widetilde{P}(X)$ is

$$\widetilde{Q} = \left((\widetilde{R} \times \widetilde{\mathrm{Sp}}(W_0))/\Delta\mu_2\right) \cdot N(X) \subset \widetilde{P}(X).$$

Then one deduces the following short exact sequence of $\widetilde{P}(X)$-modules:

$$0 \longrightarrow \mathrm{ind}_{\widetilde{Q}}^{\widetilde{P}(X)} \chi_{\overline{\psi}} |\det_X|^{1/2} \boxtimes \omega_{W_0,\psi}^\vee \longrightarrow \omega_{W,\psi}^\vee \xrightarrow{ev} \chi_{\overline{\psi}} |\det_X|^{1/2} \boxtimes \omega_{W_0,\psi}^\vee \longrightarrow 0,$$

where the compact induction functor ind is unnormalized and the action of $\widetilde{Q}$ on $\omega_{W_0,\psi}^\vee$ is via the Weil representation of $\widetilde{\mathrm{Sp}}(W_0) \cdot N(X)$ with respect to the character $\overline{\psi}$.

Tensoring the above short exact sequence by $\tilde{\tau}_\psi \boxtimes \pi_0$ and then inducing to $\widetilde{\mathrm{Sp}}(W)$, one gets a short exact sequence of $\mathrm{Sp}(W)$-modules:

$$\begin{array}{c}
0 \\
\downarrow \\
\mathrm{ind}_Q^{\mathrm{Sp}(W)}(|\det_X|^{1/2} \cdot \tau|_R \otimes (\pi_0 \otimes \omega_{W_0,\psi}^\vee)) = A \\
\downarrow \\
I_\psi(\tau, \pi_0) \otimes \omega_{W,\psi}^\vee = B \\
\downarrow \\
\mathrm{Ind}_{P(X)}^{\mathrm{Sp}(W)}(\tau \cdot |\det_X|^{1/2} \otimes (\pi_0 \otimes \omega_{W_0,\psi}^\vee)) = C \\
\downarrow \\
0.
\end{array}$$

By our assumption on π,

$$\mathrm{Hom}_{\mathrm{Sp}(W)}(C, \pi^\vee) = 0 \quad \text{and} \quad \mathrm{Hom}_{\mathrm{Sp}(W)}(B, \pi^\vee) = \mathrm{Hom}_{\mathrm{Sp}(W)}(A, \pi^\vee).$$

Moreover since τ is supercuspidal, by a well-known result of Gelfand-Kazhdan, one has

$$\tau|_R \cong \mathrm{ind}_U^R \chi,$$

where U is the unipotent radical of the Borel subgroup of $\mathrm{GL}(X)$ stabilizing the flag

$$\langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \cdots, v_n \rangle = X$$

and χ is any generic character of U. Observing that

$$H = (U \times \mathrm{Sp}(W_0)) \cdot N(X),$$

we conclude that

$$A = \mathrm{ind}_H^{\mathrm{Sp}(W)}(\pi_0 \otimes \nu_\psi^\vee).$$

Therefore, the desired result follows by Frobenius reciprocity. $\square$

Corollary 16.2. *In the symplectic case over a non-archimedean local field, we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_H(\pi \otimes \overline{\nu}_\psi, \mathbb{C}) \leq 1$$

for any irreducible representation π of $G = G(V) \times G(W)$.

One can prove an analog of Theorem 16.1 in the skew-hermitian case, including the case when $k = k_0 \times k_0$, and deduce the following corollary; we omit the details.

Corollary 16.3. *In the skew-hermitian case over a non-archimedean k, with $W \subset V$ of even codimension, we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_H(\pi \otimes \overline{\nu}_{\psi_0,\mu}, \mathbb{C}) \leq 1$$

for any irreducible representation $\pi = \pi_V \boxtimes \pi_W$ of $G = G(V) \times G(W)$ with π_V supercuspidal.

17. Local Conjectures

In this section, we propose a conjecture for the restriction problem formulated in Section 14. Recall that we have a pair of spaces $W \subset V$ and we are considering the restriction of irreducible representations $\pi = \pi_V \boxtimes \pi_W$ of $G = G(V) \times G(W)$ to the subgroup $H = N \cdot G(W) \subset G$. Recall also that, with auxiliary data if necessary, we have defined a unitary representation ν of H (or sometimes its double cover $\widetilde{H}$), which has dimension 1 when $W \subset V$ are orthogonal or hermitian, and has Gelfand-Kirillov dimensional $1/2 \cdot \dim(W/k_0)$ when $W \subset V$ are symplectic or skew-hermitian. Then we are interested in

$$d(\pi) = \dim_{\mathbb{C}} \mathrm{Hom}_H(\pi \otimes \overline{\nu}, \mathbb{C}),$$

which is known to be ≤ 1 in almost all cases. In this section, we shall give precise criterion for this Hom space to be nonzero, in terms of the Langlands-Vogan parametrization of irreducible representations of G. We first note the following conjecture, which has been called *multiplicity one in L-packets*.

Recall that a pure inner form $G' = G(V') \times G(W')$ of the group G is *relevant* if the space W' embeds as a non-degenerate subspace of V', with orthogonal complement isomorphic to $W^\perp$. In this case, one can define a subgroup $H' = G(W') \ltimes N' \subset G'$.

Conjecture 17.1. *There is a unique representation π of a relevant pure inner form $G' = G(V') \times G(W')$ in each generic Vogan L-packet Π_φ of G which satisfies*

$$\mathrm{Hom}_{H'}(\pi \otimes \overline{\nu}, \mathbb{C}) \neq 0.$$

In the papers [Wa4-5] and [MW], Waldspurger and Moeglin-Waldspurger have made substantial progress towards this conjecture. Namely, assuming certain natural and expected properties of the characters of representations in a Vogan L-packet, they have shown that the above conjecture holds in the special orthogonal case. There is no doubt that these methods will give the same result in the hermitian case.

We also note that when $k = k_0 \times k_0$, we have $G \cong \mathrm{GL}(V_0) \times \mathrm{GL}(W_0)$ so that the Vogan packets are all singletons. In this case, the above conjecture simply asserts that $\mathrm{Hom}_H(\pi \otimes \overline{\nu}, \mathbb{C}) \neq 0$ for any irreducible generic representation π of G. In this case, we have:

Theorem 17.2. *(i) If $k = k_0 \times k_0$, then Conjecture 17.1 holds when* $\dim W^\perp = 0$ *or* 1.

(ii) If $k = k_0 \times k_0$ is non-archimedean, then Conjecture 17.1 holds in general.

Proof. When $\dim W^\perp = 0$ or 1, Conjecture 17.1 is an immediate consequence of the local Rankin-Selberg theory of Jacquet, Piatetski-Shapiro and Shalika ([JPSS] and [JS]). Indeed, the local Rankin-Selberg integral gives a nonzero element of $\mathrm{Hom}_H(\pi, \nu)$. When k is non-archimedean, the general case then follows from Theorems 15.1 and Theorem 16.1. □

In each of the remaining cases, we will make the above conjecture more precise by specifying a canonical character χ of the component group A_φ. The character χ depends on the choice of a generic character θ of G (used to normalize the Langlands-Vogan parametrization) and on the additional data needed to define the representation ν when $\epsilon = -1$. We then conjecture that the representation π in Conjecture 17.1 has parameter $\pi = \pi(\varphi, \chi)$ in the Vogan correspondence $J(\theta)$.

We treat the various cases separately.

$G = \mathrm{SO}(V) \times \mathrm{SO}(W)$, $\dim W^\perp$ **odd**

Here the character θ is determined by the pair of orthogonal spaces $W \subset V$. In view of Proposition 12.1, specifying θ amounts to giving a non-isotropic line L in the even orthogonal space (with $L^\perp$ split), and we simply take the line L to have discriminant equal to the discriminant of the odd space. The representation ν is also canonical.

The L-packet Π_φ is determined by a parameter

$$\varphi : WD(k) \longrightarrow \mathrm{Sp}(M) \times \mathrm{O}(N)$$

with $\dim N$ even. We define

$$\chi = \chi_N \times \chi_M : A_M \times A_N^+ \longrightarrow \langle \pm 1 \rangle,$$

where the characters χ_N and χ_M were defined in §6.

$G = \mathrm{U}(V) \times \mathrm{U}(W)$**,** $\dim W^\perp$ **odd**

Here, in view of Proposition 12.1, we need to choose a nontrivial character

$$\psi : k/k_0 \longrightarrow \mathbb{S}^1$$

up to the action of $\mathbb{N}k^\times$ in order to define a generic character θ_0 of the even unitary group. If δ is the discriminant of the odd hermitian space, then we define

$$\theta(x) = \theta_0(-2 \cdot \delta \cdot x)$$

and use θ to fix the Vogan parametrization for the even unitary group. Note that θ is simply the generic character of the even unitary group determined by the additive character

$$\psi_{-2\cdot\delta}(x) = \psi(-2 \cdot \delta \cdot x).$$

The representation ν is canonical.

The L-packet is determined by a parameter

$$\varphi : WD(k) \longrightarrow \mathrm{GL}(M) \times \mathrm{GL}(N)$$

with M conjugate-symplectic of even dimension and N conjugate-orthogonal of odd dimension. We define:

$$\chi = \chi_N \times \chi_M : A_M \times A_N \longrightarrow \langle \pm 1 \rangle,$$

using the character ψ to calculate the local epsilon factors which intervene in the definition of χ.

$G = \widetilde{\mathrm{Sp}}(V) \times \mathrm{Sp}(W)$ **or** $\mathrm{Sp}(V) \times \widetilde{\mathrm{Sp}}(W)$**,** $\dim W^\perp$ **even**

Here we need to choose a nontrivial additive character $\psi : k \to \mathbb{S}^1$ to define a generic character θ of the symplectic group, the notion of Vogan parameters for the metaplectic group and the representation ν_ψ of H.

The L-packet is determined by a parameter

$$\varphi : WD(k) \longrightarrow \mathrm{Sp}(M) \times \mathrm{SO}(N)$$

with $\dim N$ odd. Let

$$N_1 = N \oplus \mathbb{C}$$

be the corresponding orthogonal representation of even dimension and define

$$\chi = \chi_{N_1} \times \chi_M : A_M \times A^+_{N_1} \longrightarrow \langle \pm 1 \rangle.$$

The group A_φ is a subgroup of index 1 or 2 in $A_M \times A^+_{N_1}$ and we take the restriction of χ to this subgroup.

$G = \mathrm{U}(V) \times \mathrm{U}(W)$**,** $W \subset V$ **skew-hermitian and** $\dim W \equiv \dim V \equiv 1 \mod 2$

Here there is a unique orbit of generic character θ on the quasi-split group $\mathrm{U}(V) \times \mathrm{U}(W)$. On the other hand, we need to choose

$$\psi_0 : k_0 \to \mathbb{S}^1$$

up to $\mathbb{N}k^\times$ and

$$\mu : k^\times/\mathbb{N}k^\times \longrightarrow \mathbb{C}^\times,$$

nontrivial on $k_0^\times$, to define the representation $\nu_{\psi_0,\mu}$ of H.

Let e be the discriminant of V and W which is a nonzero element of trace 0 in k, well-defined up to $\mathbb{N}k^\times$. Let

$$\psi(x) = \psi_0(\mathrm{Tr}(ex))$$

which is a nontrivial character of k/k_0, well-defined up to $\mathbb{N}k^\times$.

The L-packet has parameter

$$\varphi : WD(k) \longrightarrow \mathrm{GL}(M) \times \mathrm{GL}(N)$$

with M and N conjugate-orthogonal representations of odd dimension. We define:

$$\chi = \chi_N \times \chi_{M(\mu^{-1})} = \chi_{N(\mu^{-1})} \times \chi_M : A_M \times A_N \longrightarrow \langle \pm 1 \rangle,$$

using ψ to calculate the local epsilon actors which intervene in the definition of χ. Here, $M(\mu^{-1})$ and $N(\mu^{-1})$ are the twist of M and N by the character μ^{-1}. Note that the representations $M(\mu^{-1})$ and $N(\mu^{-1})$ are conjugate-symplectic.

$G = \mathrm{U}(V) \times \mathrm{U}(W)$**,** $W \subset V$ **skew-hermitian,** $\dim W \equiv \dim V \equiv 0 \mod 2$

In this case, we must choose $\psi_0 : k_0 \to \mathbb{S}^1$ to define θ for both groups, and $\mu : k^\times/\mathbb{N}k^\times \to \mathbb{C}^\times$, nontrivial on $k_0^\times$, to define $\nu = \nu_{\psi_0,\mu}$.

The parameter of an L-packet is

$$\varphi : WD(k) \longrightarrow \mathrm{GL}(M) \times \mathrm{GL}(N)$$

with M and N conjugate-symplectic representations of even dimension. We define

$$\chi = \chi_N \times \chi_{M(\mu^{-1})} = \chi_{N(\mu^{-1})} \times \chi_M : A_M \times A_N \longrightarrow \langle \pm 1 \rangle.$$

Here, the twisted representations $M(\mu^{-1})$ and $N(\mu^{-1})$ are conjugate-orthogonal. Since the representations both have even dimension, the values of χ are independent of the choice of ψ used to define the epsilon factors.

Now we have:

Conjecture 17.3. *Having fixed the Langlands-Vogan parametrization for the group G and its pure inner forms in the various cases above, the unique representation π in a generic Vogan packet Π_φ which satisfies $\mathrm{Hom}_H(\pi \otimes \overline{\nu}, \mathbb{C}) \neq 0$ has parameters*

$$\pi = \pi(\varphi, \chi)$$

where χ is as defined above.

Note that the character χ defined above satisfies:

$$\chi(-1, -1) = 1,$$

so that $\pi(\varphi, \chi)$ is a representation of a relevant pure inner form G' of G. In the non-archimedean case, the sign

$$\chi(-1, 1) = \chi(1, -1)$$

determines which relevant pure inner form acts on $\pi(\varphi, \chi)$.

18. Compatibilities of local conjectures

In this section, we verify that the precise conjecture 17.3 is independent of:

(1) the bijection
$$J(\theta) : \Pi_\varphi \leftrightarrow \mathrm{Hom}(A_\varphi, \pm 1)$$
given by the choice of a generic character θ of G;

(2) the scaling of the form $\langle -, - \rangle$ on V and hence W, which does not change the groups G and H;

(3) the data needed to define the representation ν of H.

This serves as a check on the internal consistency of the conjecture. Again, we consider the various cases separately.

In the orthogonal case, the generic character θ and the representation ν of H are determined by the pair of spaces $W \subset V$ and are unchanged if the bilinear form on V is scaled by $k^\times$. The character $\chi = \chi_N \times \chi_M$ depends only on the Langlands parameter $\varphi : WD(k) \longrightarrow \mathrm{Sp}(M) \times \mathrm{O}(N)$. So our conjecture is internally consistent in this case, as there is nothing to check.

In the hermitian case, both the generic character θ and the character $\chi = \chi_N \times \chi_M$ depend on a choice of nontrivial $\psi : k/k_0 \to \mathbb{S}^1$, up to multiplication by $\mathbb{N}k^\times$, while the representation ν is determined by the spaces $W \subset V$. If we scale the hermitian form on $W \subset V$ by an element of $k_0^\times$, the generic character θ, the representation ν of H and the character χ of the component group are unchanged.

To see the dependence of our conjecture on the choice of ψ, suppose that t represents the nontrivial coset of $k_0/\mathbb{N}k^\times$ and let θ^t be the generic character associated to $\psi_t(x) = \psi(tx)$. For $(a,b) \in A_M \times A_N$, we have

$$\begin{aligned}\chi^t(a,b) &= \epsilon(M^a \otimes N, \psi_0^t) \cdot \epsilon(M \otimes N^b, \psi_t) \\ &= \det M^a(t) \cdot \epsilon(M^a \otimes N, \psi_0) \cdot \epsilon(M \otimes N^b, \psi) \\ &= (-1)^{\dim M^a} \cdot \chi(a,b) \\ &= \eta(a) \cdot \chi(a,b)\end{aligned}$$

Here we have used the facts that M is conjugate-symplectic of even dimension and N is conjugate-orthogonal of odd dimension.

Now if the parameter of π under $J(\theta)$ is (φ, χ), then its parameter under $J(\theta^t)$ is $(\varphi, \chi \cdot \eta) = (\varphi, \chi^t)$, according to the desiderata in §10. Hence our conjecture is independent of the choice of ψ in the hermitian case.

In the symplectic case, we will discuss representations of $G = \widetilde{\mathrm{Sp}}(W) \times \mathrm{Sp}(V)$; the case of representations of $\mathrm{Sp}(W) \times \widetilde{\mathrm{Sp}}(V)$ is similar. In this case, we used the choice of an additive character $\psi : k \to \mathbb{S}^1$, up to multiplication by $k^{\times 2}$, to

(i) define the notion of L-parameters M for representations of $\widetilde{\mathrm{Sp}}(W)$;

(ii) define a generic character θ for $\mathrm{Sp}(V)$ and a generic character $\widetilde{\theta}$ for $\widetilde{\mathrm{Sp}}(W)$;

(iii) define the representation $\nu = \nu_\psi$ for H.

Note, however, that the character χ of the component group A_M is independent of the choice of ψ.

Suppose that under the ψ-parametrization, the parameter (M, N, χ) corresponds to the representation $\widetilde{\pi}$ of G, so that our conjecture predicts that

$$\mathrm{Hom}_H(\widetilde{\pi}, \nu_\psi) \neq 0.$$

Now replace the character ψ by ψ_c for $c \in k^\times/k^{\times 2}$ and let $\widetilde{\pi}'$ be the representation of G corresponding to (M, N, χ) under the ψ_c-parametrization. By our construction of the Vogan parametrization for metaplectic groups, it is easy to see that $\widetilde{\pi}'$ is isomorphic to the conjugated representation $\widetilde{\pi}^c$. Thus, our conjecture for the character ψ_c predicts that

$$\mathrm{Hom}_H(\widetilde{\pi}^c, \nu_{\psi_c}) \neq 0 \quad \text{and hence} \quad \mathrm{Hom}_H(\widetilde{\pi}, \nu_{\psi_c}^c) \neq 0.$$

Since $\nu_\psi^c = \nu_{\psi_c}$, our conjecture is internally consistent with respect to changing ψ.

Note that the use of ψ in (i) and (ii) above concerns the Vogan parametrization, whereas its use in (iii) concerns the restriction problem in representation theory. Hence there is no reason why one needs to use the same character ψ for these two different purposes.

Suppose that one continues to use ψ for the Vogan parametrization in (i) and (ii), but uses the character $\psi_c(x) = \psi(cx)$ to define the representation ν_{ψ_c} of H. Then for a given Vogan packet of G with ψ-parameter φ, one can ask which representation $\pi \in \Pi_\varphi$ satisfies $\mathrm{Hom}_H(\pi, \nu_{\psi_c}) \neq 0$. This can be answered using Conjecture 17.3, together with Conjecture 11.3 from §11. We have:

Proposition 18.1. *Assume the conjectures 11.3 and 17.3. Let*

$$\varphi : WD(k) \longrightarrow \mathrm{Sp}(M) \times \mathrm{SO}(N)$$

be a generic Langlands parameter for $\widetilde{\mathrm{Sp}}(W) \times \mathrm{Sp}(V)$ *relative to the nontrivial additive character* ψ *of* k*. Let*

$$N_c = N(c) \oplus \mathbb{C} \quad \textit{for } c \in k^\times / k^{\times 2}$$

Then the unique representation π *in* Π_φ *with* $\mathrm{Hom}_H(\pi, \nu_{\psi_c}) \neq 0$ *corresponds under the bijection* $J(\widetilde{\theta} \times \theta)$ *to the restriction of the character*

$$\chi_{N_c} \times \chi_M : A_M \times A^+_{N_c} \longrightarrow \langle \pm 1 \rangle$$

to the subgroup $A_\varphi = A_M \times A_N^+$*, multiplied by the character*

$$\eta_c(a) = \det N^a(c) \qquad \textit{of } A_N^+.$$

Proof. Let π be the representation whose ψ-parameter is (M, N, χ), where χ is as given in the proposition:

$$\chi = \chi_{N_c} \times \chi_M \cdot \eta_c : A_M \times A_N^+ \longrightarrow \langle \pm 1 \rangle.$$

We want to show that

$$\mathrm{Hom}_H(\pi, \nu_{\psi_c}) \neq 0.$$

By Conjecture 11.3, the ψ_c-parameters of π are $(M(c), N, \chi')$ with

$$\chi' = \chi_{N_c} \cdot \eta[c] \times \chi_M \cdot \eta_c^2.$$

Hence, it suffices to show that χ' is equal to the character predicted by Conjecture 17.3 relative to ψ_c. More precisely, we need to show that

$$\chi' = \chi_{N_1} \times \chi_{M(c)}.$$

We now calculate this character on an element

$$(a', a) \in A_M \times A_N^+ = A_{M(c)} \times A_N^+,$$

using the fact that for a in $C_N^+ \to C_{N_c}^+$,

$$N_c^a = N(c)^a.$$

Since this space has even dimension,

$$\det N_c^a = \det N^a = \det N_1^a.$$

Hence we have

$$\begin{aligned}
\chi'(a', a) &= (\chi_{N_c} \cdot \eta[c])(a') \cdot \chi_M(a) \\
&= \epsilon(M^{a'} \otimes N_c) \cdot \epsilon(M^{a'} \otimes (\mathbb{C} \oplus \mathbb{C}(c))) \cdot \epsilon(M \otimes N_c^a) \cdot \det(N_c^a)(-1)^{\frac{1}{2} \cdot \dim M} \\
&= \epsilon(M^{a'} \otimes (N(c) + \mathbb{C}(c)) \cdot \epsilon(M \otimes N^a(c)) \cdot \det(N^a(c))(-1)^{\frac{1}{2} \cdot \dim M} \\
&= \epsilon(M(c)^{a'} \otimes N_1) \cdot \epsilon(M(c) \otimes N_1^a) \cdot \det(N_1^a)(-1)^{\frac{1}{2} \cdot \dim M} \\
&= (\chi_{N_1} \times \chi_{M(c)})(a', a).
\end{aligned}$$

This proves the proposition. $\square$

Finally, we consider the skew-hermitian case with $W \subset V$ of even codimension. We consider the two cases:

(i) $\dim W \equiv \dim V \equiv 1 \mod 2$;

(ii) $\dim W \equiv \dim V \equiv 0 \mod 2$

in turn.

Assume first that $\dim W \equiv \dim V \equiv 1 \mod 2$ and the discriminant of W and V is a trace zero element $e \in k$. In this case, the Vogan parametrization is completely canonical, given the spaces $W \subset V$, with the trivial character of A_φ corresponding to a generic representation of $G(V) \times G(W)$. However, the representation ν of H depends not only on the spaces $W \subset V$ but also on the choice of an additive character $\psi_0 : k_0 \to \mathbb{S}^1$ and on the choice of a multiplicative character $\mu : k^\times \to \mathbb{C}^\times$ which is trivial on $\mathbb{N}(k^\times)$ but nontrivial on $k_0^\times$. Thus, to be completely precise, we denote the group H by $H_{W,V}$ and the representation ν by $\nu_{W,V,\psi_0,\mu}$. Finally, the character χ of the component group is defined using μ and the additive character

$$\psi(x) = \psi_0(\mathrm{Tr}(ex))$$

of k and depends on ψ_0 up to multiplication by $\mathbb{N}k^\times$.

Suppose without loss of generality that the representation π with parameter (φ, χ) is one for the group $G(W) \times G(V)$, so that our conjecture (for the character ψ_0) predicts that

$$\mathrm{Hom}_{H_{W,V}}(\pi, \nu_{W,V,\psi_0,\mu}) \neq 0.$$

If t represents the nontrivial coset of $k_0^\times/\mathbb{N}k^\times$, let χ_t be the character of A_φ defined using the character $\psi_0^t(x) = \psi_0(tx)$. Then we have

$$\begin{aligned}\chi_t(a,b) &= \epsilon(M^a \otimes N(\mu^{-1}), \psi_t) \cdot \epsilon(M \otimes N(\mu^{-1})^b, \psi_t)\\ &= (-1)^{\dim M^a} \cdot (-1)^{\dim N^b} \cdot \chi(a,b)\\ &= \eta(a,b) \cdot \chi(a,b).\end{aligned}$$

Thus, the representation π_t indexed by the character χ_t is one for the pure inner form $G(W')\times G(V')$. Moreover, the spaces $W' \subset V'$ are simply the spaces $tW \subset tV$ obtained from $W \subset V$ by scaling the skew-hermitian forms by t. Thus, our conjecture (for the character ψ_0^t) predicts that

$$\mathrm{Hom}_{H_{tW,tV}}(\pi_t, \nu_{tW,tV,\psi_0^t,\nu}) \neq 0.$$

To see that this is equivalent to the prediction of our conjecture for the character ψ_0, note that $G(W') \times G(V')$ is canonically identified with $G(W) \times G(V)$ as a subgroup of $\mathrm{GL}(W) \times \mathrm{GL}(V)$ and under this identification, one has $\pi_t = \pi$. Moreover, we also have $H_{W',V'} = H_{W,V}$ as subgroups of $G(W) \times G(V)$ and

$$\nu_{tW,tV,\psi_0^t,\mu} = \nu_{W,V,\psi_0,\mu}.$$

This proves that our conjecture is internally consistent with changing ψ_0.

On the other hand, if we replace μ by μ', then $\mu' = \mu \cdot \mu_0$ for some character $\mu_0 : k^\times/k_0^\times \to \mathbb{C}^\times$. Moreover, we have [HKS]

$$\nu_{\mu',\psi_0} \cong \mu_0 \cdot \nu_{\mu,\psi_0}.$$

Hence

$$\mathrm{Hom}_H(\pi \otimes \overline{\nu}_{\mu',\psi_0}, \mathbb{C}) \cong \mathrm{Hom}_H((\pi \cdot \mu_0^{-1}) \otimes \overline{\nu}_{\mu,\psi_0}, \mathbb{C}).$$

Now our conjecture for μ' says that the left hand side of the above is nonzero if and only if π has Vogan parameter $(M, N, \chi_{M,N,\mu'})$ with

$$\begin{aligned}\chi_{M,N,\mu'}(a,b) &= \epsilon(M^a \otimes N(\mu'^{-1}), \psi) \cdot \epsilon(M \otimes N(\mu'^{-1})^b, \psi)\\ &= \epsilon(M^a \otimes (N \cdot \mu_0^{-1})(\mu^{-1}), \psi) \cdot \epsilon(M \otimes (N \cdot \mu_0^{-1})(\mu^{-1})^b, \psi)\\ &= \chi_{M,N(\mu_0^{-1}),\mu}(a,b).\end{aligned}$$

On the other hand, our conjecture for μ says that the right hand side is nonzero if and only if $\pi \cdot \mu_0^{-1}$ has Vogan parameter $(M, N(\mu_0)^{-1}, \chi_{M,N(\mu_0^{-1}),\mu})$. Thus, our conjecture for μ' is equivalent to that for μ.

Finally, consider the case when $\dim W \equiv \dim V \equiv 0 \mod 2$. In this case, we need the additive character $\psi_0 : k_0 \to \mathbb{S}^1$ to specify the Vogan parametrization, and both ψ_0 and μ to define the representation $\nu_{W,V,\psi_0,\mu}$. The character χ, on the other hand, is independent of ψ_0 but depends on μ.

Suppose that under the ψ_0-Vogan parametrization, the representation π corresponding to the character χ is one for the group $G(V) \times G(W)$, so that our conjecture for ψ_0 predicts that

$$\mathrm{Hom}_{H_{W,V}}(\pi \otimes \overline{\nu}_{W,V,\psi_0,\mu}, \mathbb{C}) \neq 0.$$

If we replace the additive character ψ_0 by ψ_0^t with $t \in k_0^\times$ but $t \notin \mathbb{N}k^\times$, then under the ψ_0^t-Vogan parametrization, the character χ corresponds to the conjugated representation π^t (using an element in the similitude group with similitude t). So our conjecture for ψ_0^t predicts that

$$\mathrm{Hom}_{H_{W,V}}(\pi^t \otimes \overline{\nu}_{W,V,\psi_0^t,\mu}, \mathbb{C}) \neq 0.$$

But one may check that

$$\nu_{W,V,\psi_0^t,\mu} \cong \nu_{W,V,\psi_0,\mu}^t,$$

so that the two predictions (for ψ_0 and ψ_0^t) are consistent with each other. The consistency check when changing μ is similar to the analogous situation treated above; so we omit the details.

19. Reduction to basic cases

In this section, we shall show:

Theorem 19.1. *Assume that k is a non-archimedean local field. Then Conjectures 17.1 and 17.3 follow from the basic cases where* $\dim W^\perp = 0$ *or* 1.

Proof. As we shall explain, this is a simple consequence of Theorems 15.1 and 16.1. We treat the two cases separately.

We first consider the orthogonal and hermitian cases. Suppose that $W \subset V$ with

$$W^\perp = X + X^\vee + E$$

where $X = \langle v_1, \cdots, v_n \rangle$ is nonzero isotropic and E is a non-isotropic line. Let M and N be L-parameters for $G(V)$ and $G(W)$ respectively. We would like to verify Conjectures 17.1 and 17.3 for the associated Vogan packet $\Pi_M \times \Pi_N$ of $G = G(V) \times G(W)$. We shall exploit Theorem 15.1 for this purpose.

Recall the setting of Theorem 15.1, where we have set

$$W' = V \oplus (-E) = V \oplus k \cdot f$$

and

$$Y = \langle v_1, \cdots, v_n, v_{n+1} \rangle$$

with $v_{n+1} = e + f$. Then we have

$$W' = Y + Y^\vee + W.$$

Let τ be an irreducible supercuspidal representation of $\mathrm{GL}(Y)$ with L-parameter N_τ. We may assume that for any $\pi_W \in \Pi_N$, the induced representation

$$I(\tau, \pi_W) = \mathrm{Ind}_{P(Y)}^{G(W')}(\tau \boxtimes \pi_W)$$

is irreducible. Then the set

$$\{I(\tau, \pi_W) : \pi_W \in \Pi_N\}$$

is simply the Vogan packet associated to the parameter

$$N' = N_\tau + N + (N_\tau^\sigma)^\vee.$$

Moreover, there is a canonical isomorphism

$$A_N \cong A_{N'}$$

and the representations π_W and $I(\tau, \pi_W)$ are indexed by the same character of $A_N^+ \cong A_{N'}^+$.

We may further assume that τ is chosen so that the conditions of Theorem 15.1 are met. Then by Theorem 15.1, we see that for any $\pi_V \in \Pi_M$,

$$\mathrm{Hom}_{G(V)}(I(\tau, \pi_W) \otimes \pi_V, \mathbb{C}) = \mathrm{Hom}_H(\pi_V \otimes \pi_W, \nu).$$

Thus, Conjecture 17.1 holds for $\Pi_M \times \Pi_N$ if it holds for $\Pi_{N'} \times \Pi_M$. To see that the same implication holds for Conjecture 17.3, it suffices to check that the character

$$\chi_N \times \chi_M \quad \text{of } A_M^+ \times A_N^+$$

agrees with the character

$$\chi_{N'} \times \chi_M \quad \text{of } A_M^+ \times A_{N'}^+.$$

For $a \in A_M^+$, we see from definition that

$$\chi_{N'}(a) = \chi_N(a) \cdot \chi_{N_\tau + (N_\tau^\sigma)^\vee}(a),$$

and it follows by Proposition 5.1

$$\chi_{N_\tau + (N_\tau^\sigma)^\vee}(a) = 1 \quad \text{for any } a \in A_M^+.$$

This establishes Theorem 19.1 in the orthogonal and hermitian cases.

The symplectic and skew-hermitian cases are handled in a similar way, using Theorem 16.1; we omit the details. □

20. Variant of the local conjecture

In this section, we give a variant of the local conjecture 17.3. This variant does not require the precise parametrization of the members of a Vogan L-packet by the characters of the component group, which can be a very delicate issue. This conjecture is typically what is checked in practice.

Suppose that $W \subset V$ and we are given an L-parameter M of $G(V)$, so that M is a selfdual or conjugate-dual representation of the Weil-Deligne group $WD(k)$ of k with a given sign. As described in §4, the component group A_M is an elementary abelian 2-group with a canonical basis $\{a_i\}$, indexed by the distinct isomorphism classes of irreducible summands M_i of M which are of the same type as M. Hence, we have a canonical isomorphism

$$A_M \cong \mathbb{Z}/2\mathbb{Z} \cdot a_1 \times \cdots \times \mathbb{Z}/2\mathbb{Z} \cdot a_k.$$

Here, the elements a_i are such that

$$M^{a_i} \cong M_i.$$

Now consider a representation π of $G(V)$ in the Vogan packet Π_M. Let $\mathcal{L}_W(\pi)$ denote the set of generic L-parameters N for $G(W)$ such that

$$d(\pi, N) := \sum_{\pi' \in \Pi_N} \dim \mathrm{Hom}_H(\pi \boxtimes \pi', \nu) \neq 0.$$

According to our Conjecture 17.1, one has a partition

$$\{\text{generic } L\text{-parameters of } G(W)\} = \bigcup_{\pi \in \Pi_M} \mathcal{L}_W(\pi).$$

In this context, we have the following variant of Conjecture 17.3.

Conjecture 20.1. *(1) Fix $\pi \in \Pi_M$. For any two L-parameters N and N' in $\mathcal{L}_W(\pi)$, we have:*

(i) in the orthogonal case,

$$\epsilon(M_i \otimes N, \psi) = \epsilon(M_i \otimes N', \psi) \in \{\pm 1\} \quad \textit{for any } i,$$

where ψ is any nontrivial additive character of k;

(ii) in the hermitian case,

$$\epsilon(M_i \otimes N, \psi) = \epsilon(M_i \otimes N', \psi) \in \{\pm 1\} \quad \textit{for any } i,$$

for any nontrivial additive character ψ of k/k_0.

(iii) in the symplectic case, with $\nu = \nu_\psi$,

$$\epsilon(M_i \otimes N, \psi) = \epsilon(M_i \otimes N', \psi) \in \{\pm 1\} \quad \textit{for any } i.$$

(iv) in the skew-hermitian case, with $\nu = \nu_{\psi_0,\mu}$ (for an additive character ψ_0 of k_0),

$$\epsilon(M_i \otimes N(\mu^{-1}), \psi) = \epsilon(M_i \otimes N'(\mu^{-1}), \psi) \in \{\pm 1\} \quad \text{for any } i,$$

where ψ is any nontrivial additive character of k/k_0.

In particular, π determines a character χ_π on A_M, defined by

$$\chi_\pi(a_i) = \epsilon(M_i \otimes N, \psi)$$

for any $N \in \mathcal{L}_W(\pi)$ and a fixed ψ appropriate for each of the cases above.

(2) The map $\pi \mapsto \chi_\pi$ gives a bijection (depending on the choice of a character ψ)

$$\Pi_M \longleftrightarrow Irr(A_M).$$

(3) One has the analogs of (1) and (2) above with the roles of V and W exchanged.

In effect, the above conjecture says that one can exploit the restriction problem for $W \subset V$ and use the collection of epsilon factors described above to serve as parameters for elements of Π_M. In the hermitian and skew-hermitian cases, the character χ_π associated to a given $\pi \in \Pi_M$ is equal to the character χ_N of A_M defined in §6, for any $N \in \mathcal{L}_W(\pi)$, provided the additive character ψ is appropriately chosen. In the orthogonal and symplectic cases, these two characters may differ.

21. Unramified parameters

In this section, we assume that k is a non-archimedean local field with ring of integers A, uniformizing element ϖ and finite residue field $A/\varpi A$. We will also assume that $A/\varpi A$ has characteristic $p > 2$, so that the group $A^\times/A^{\times 2}$ has order 2.

In the case when k has a nontrivial involution σ, we will assume that the action of σ on $A/\varpi A$ is also nontrivial. Then k is unramified over k_0 and every unit in the subring A_0 of A fixed by σ is the norm of a unit of A.

In addition, we will only consider additive characters ψ of k which are trivial on A but not on $\varpi^{-1}A$. Then ψ is determined up to translation by a unit in A. If we insist that $\psi^\sigma = \psi^{\pm 1}$, then ψ is determined up to translation by a unit in A_0. We call such additive characters of k unramified.

Let $WD(k) = W(k) \times \mathrm{SL}_2(\mathbb{C})$ be the Weil-Deligne group of k. A representation

$$\varphi : WD(k) \longrightarrow \mathrm{GL}(M)$$

is *unramified* if φ is trivial on SL_2 and on the inertia subgroup I of $W(k)$. An unramified representation is determined by the semisimple conjugacy class $\varphi(F)$ in $\mathrm{GL}(M)$. Let $\mathbb{C}(s)$ denote the one dimensional unramified representation of $WD(k)$ with $\varphi(F) =$

$s \in \mathbb{C}^\times$. Then any unramified representation M of $WD(k)$ is isomorphic to a direct sum of the form

$$M = \bigoplus_{i=1}^{n} \mathbb{C}(s_i), \quad \text{with } n = \dim M.$$

We now determine which unramified representations of $WD(k)$ are selfdual or conjugate-dual (with respect to the unramified involution σ of k).

Proposition 21.1. *Assume that M is an unramified representation of $WD(k)$ and is either selfdual or conjugate-dual. Then M is isomorphic to a direct sum of the form:*

$$M \cong \oplus_i(\mathbb{C}(s_i) + \mathbb{C}(s_i^{-1})) \oplus m \cdot \mathbb{C}(-1) \oplus n \cdot \mathbb{C}(1),$$

with $s_i \neq s_i^{-1}$ in $\mathbb{C}^\times$ and $m, n \geq 0$ in $\mathbb{Z}$.

If M has this form, then we have the following cases:

(i) M is orthogonal and its centralizer in $\mathrm{SO}(M)$ has component group

$$A_M^+ = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \textit{if both } m, n > 0, \\ 1, & \textit{otherwise.} \end{cases}$$

(ii) M is symplectic if and only if $m \equiv n \equiv 0 \mod 2$, in which case its centralizer in $\mathrm{Sp}(M)$ has component group

$$A_M = 1.$$

(iii) M is conjugate-orthogonal if and only if $m \equiv 0 \mod 2$, in which case its centralizer in $\mathrm{Aut}(M, B)$ has component group

$$A_M = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \textit{if } n > 0, \\ 1, & \textit{otherwise.} \end{cases}$$

(iv) M is conjugate-symplectic if and only if $n \equiv 0 \mod 2$, in which case its centralizer in $\mathrm{Aut}(M, B)$ has component group

$$A_M = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \textit{if } m > 0, \\ 1, & \textit{otherwise.} \end{cases}$$

Proof. Since $\mathbb{C}(s)^\vee \cong \mathbb{C}(s^{-1})$ and $\mathbb{C}(s)^\sigma \cong \mathbb{C}(s)$, the one dimensional representation $\mathbb{C}(s)$ is selfdual or conjugate-dual if and only if $s^2 = 1$. In the selfdual case, both $\mathbb{C}(-1)$ and $\mathbb{C}(1)$ are orthogonal. In the conjugate-dual case, $\mathbb{C}(1)$ is conjugate-orthogonal and $\mathbb{C}(-1)$ is conjugate-symplectic. Indeed, the unramified character

$$\mu : k^\times/\mathbb{N}k^\times \longrightarrow \langle \pm 1 \rangle$$

defined by

$$\mu(\alpha) = (-1)^{\mathrm{ord}_\varpi(\alpha)}$$

is nontrivial on $k_0^\times$. The proposition follows easily. $\square$

Proposition 21.2. *(i) If M and N are two selfdual unramified representations of $WD(k)$ of even dimension, with signs c_M and c_N respectively, then the character*

$$\chi_N : A_M^+ \longrightarrow \langle \pm 1 \rangle$$

defined by

$$\chi_N(a) = \epsilon(M^a \otimes N, \psi) \cdot \det M^a(-1)^{\frac{1}{2}\dim N} \cdot \det N(-1)^{\frac{1}{2}\dim M^a}$$

is trivial.

(ii) If M and N are two conjugate-dual unramified representations with signs c_M and c_N respectively and $\psi^\sigma = \psi^{-1}$, then the character

$$\chi_N : A_M^+ \longrightarrow \langle \pm 1 \rangle$$

defined by

$$\chi_N(a) = \epsilon(M^a \otimes N, \psi)$$

is trivial.

Proof. If M is any unramified representation of $WD(k)$ and ψ is an unramified additive character, then we have the formulae:

$$\epsilon(M, \psi) = 1 \quad \text{and} \quad \det M(-1) = 1.$$

The proposition follows easily from these facts. □

We now turn to the restriction conjectures for unramified generic parameters φ. Since $\chi_N \times \chi_M$ is the trivial character of A_φ, the unique representation in the associated Vogan packet which supports a nonzero Hom space should be the one indexed by the trivial character. In that case, for the purpose of global applications, we can make our conjectures more refined.

Recall that $W \subset V$ is a pair of nondegenerate spaces for the sesquilinear form $\langle -, - \rangle$. and that $W^\perp$ is split. We say that an A-lattice $L \subset V$ is nondegenerate if

(1) $\langle -, - \rangle : L \times L \to A$;

(2) the map $L \to \mathrm{Hom}(L, A)$ defined by mapping w to $f_w(v) = \langle v, w \rangle$ is an isomorphism of A_0-modules.

We assume henceforth that there is a nondegenerate A-lattice $L \subset V$ with the additional property that $L_W = L \cap W$ is a nondegenerate A-lattice in W. Then the orthogonal complement $L_W^\perp$ of L_W in L is a nondegenerate lattice in $W^\perp$, so that $L_W^\perp$ has the form

$$Y + Y^\vee \quad \text{or} \quad Y + Y^\vee + Ae$$

with Y isotropic and $Y^\vee \cong \mathrm{Hom}(Y, A)$. Moreover, $L = L_W + L_W^\perp$.

Under this assumption, the group $G = G(V) \times G(W)$ is quasi-split and split by an unramified extension of k_0. Indeed, the subgroup $J = \mathrm{Aut}(L) \times \mathrm{Aut}(L_W)$ is a hyperspecial maximal compact subgroup of G. We now construct the subgroup H of G and the unitary representation ν of H using this unramified data.

Write $L = L_W + L_W^{\perp}$, and define the parabolic subgroup P_A and its unipotent radical N_A using a complete A-flag in the isotropic subspace $Y \subset L_W^{\perp}$. Then

$$H_A = N_A \cdot \mathrm{Aut}(L_W)$$

gives a model of H over A_0 and

$$H_A = J \cap H_A(k_0) = J \cap H.$$

In the orthogonal and hermitian cases, the one dimensional representation ν of H associated to the decomposition

$$L_W^{\perp} = Y + Y^{\perp} + Ae$$

and a suitable unramified additive character ψ has trivial restriction to the subgroup $J \cap H$. In the metaplectic case, we can define $\nu = \nu_\psi$ using an unramified additive character ψ (there are two choices, up to translation by $A^{\times 2}$). In the skew-hermitian case, we define $\nu = \nu_{\psi,\mu}$ using an unramified character ψ with $\psi^\sigma = \psi$ (which is unique up to translation by $\mathbb{N}A^\times$) and the unramified symplectic character μ associated to the representation $\mathbb{C}(-1)$. Then in all cases, the representation ν of H is $J \cap H$-spherical; it has a unique line fixed by the compact open subgroup $J \cap H$.

Since the group G is quasi-split over k_0, we can also define unramified generic characters θ of the unipotent radical U of a Borel subgroup, using the pair $L_W \subset L$ of nondegenerate lattices and a suitable unramified additive character ψ. Again, the T-orbit of θ is unique except in the metaplectic case when there are two unramified orbits. In all cases, the restriction of θ to the compact open subgroup $J \cap U$ is trivial.

To summarize, if we use unramified data to define the representations θ of U and ν of H, then the complex vector spaces

$$\mathrm{Hom}_{J\cap U}(\mathbb{C}, \theta) \quad \text{and} \quad \mathrm{Hom}_{J\cap H}(\mathbb{C}, \nu)$$

both have dimension equal to 1.

Now let φ be an unramified generic parameter and let π be the unique θ-generic element in the Vogan packet π_φ. Then the formula of Casselman and Shalika [CS] shows that

(i) $\mathrm{Hom}_J(\mathbb{C}, \pi)$ has dimension 1;

(ii) the pairing of one-dimensional complex vector spaces

$$\mathrm{Hom}_J(\mathbb{C}, \pi) \otimes \mathrm{Hom}_U(\pi, \theta) \longrightarrow \mathrm{Hom}_{J\cap U}(\mathbb{C}, \theta) = \mathbb{C}$$

is nondegenerate.

We conjecture that the same is true for the representation ν of H.

Conjecture 21.3. *Let π be the unique J-spherical representation in the Vogan packet Π_φ. Then*

(i) $\mathrm{Hom}_H(\pi, \nu)$ *has dimension* 1*;*

(ii) the pairing of one-dimensional complex vector spaces

$$\mathrm{Hom}_J(\mathbb{C}, \pi) \otimes \mathrm{Hom}_H(\pi, \nu) \longrightarrow \mathrm{Hom}_{J\cap H}(\mathbb{C}, \nu)$$

is nondegenerate.

Besides the cases treated by Casselman-Shalika [CS], this conjecture has been verified in a large number of cases, which we summarize below.

Theorem 21.4. *Conjecture 21.3 is known in the following cases:*

(i) the special orthogonal and hermitian cases;

(ii) the general linear case, with $\dim W^\perp = 1$*;*

(iii) the symplectic case, with $\dim W^\perp = 2$*;*

Proof. The orthogonal case is due to Kato-Murase-Sugano [KMS]. Their proof is extended to the unitary case by Khouri in his Ohio-State PhD thesis [Kh]. Parts (ii) and (iii) are both due to Murase-Sugano [MS1, MS2]. □

22. Automorphic forms and L-functions

The remainder of this paper is devoted to formulating global analogs of our local conjectures.

Let F be a global field with ring of adèles $\mathbb{A}$ and let G be a reductive algebraic group over F. Then $G(F)$ is a discrete subgroup of the locally compact group $G(\mathbb{A})$. For simplicity, we shall further assume that the identity component of the center of G is anisotropic, so that the quotient space $G(F)\backslash G(\mathbb{A})$ has finite measure.

We shall consider the space $\mathcal{A}(G)$ of automorphic forms on G, which consists of smooth functions

$$f : G(F)\backslash G(\mathbb{A}) \longrightarrow \mathbb{C}$$

satisfying the usual finiteness conditions [MW2, I.2.17, Pg. 37], except that we do not impose the condition of K_∞-finiteness at the archimedean places. For each open compact $K_f \subset G(\mathbb{A}_f)$, the space $\mathcal{A}(G)^{K_f}$ has a natural topology, giving it the structure of an LF-space (see [W2]) with respect to which the action of $G(F \otimes \mathbb{R})$ is smooth.

Let $\mathcal{A}_0(G) \subset \mathcal{A}(G)$ denote the subspace of cusp forms. An irreducible admissible representation $\pi = \pi_\infty \otimes \pi_f$ of $G(\mathbb{A})$ is cuspidal if it admits a continuous embedding

$$\pi \hookrightarrow \mathcal{A}_0(G).$$

The multiplicity of π in $\mathcal{A}_0(G)$ is the dimension of the space $\mathrm{Hom}_{G(\mathbb{A})}(\pi, \mathcal{A}_0(G))$, which is necessarily finite.

Suppose now that G is quasi-split, with a Borel subgroup $B = T \cdot U$ defined over F. A homomorphism $\lambda : U \longrightarrow \mathbb{G}_a$ is generic if its centralizer in T is equal to the center of G. Composing $\lambda_{\mathbb{A}}$ with a nontrivial additive character ψ of $\mathbb{A}/F$ gives an automorphic generic character $\theta = \psi \circ \lambda_{\mathbb{A}}$. Now one may consider the map

$$F(\theta) : \mathcal{A}(G) \longrightarrow \mathbb{C}(\theta)$$

defined by

$$f \mapsto \int_{U(F)\backslash U(\mathbb{A})} f(u) \cdot \overline{\theta(u)}\, du.$$

The map $F(\theta)$ is a nonzero continuous homomorphism of $U(\mathbb{A})$-modules, which is known as the *θ-Fourier coefficient*. If $F(\theta)$ is nonzero when restricted to the π-isotypic component in $\mathcal{A}(G)$, we say that π is globally generic with respect to θ.

The notion of automorphic forms can also be defined for nonlinear finite covers $\widetilde{G}(\mathbb{A})$ of $G(\mathbb{A})$, which are split over the discrete subgroup $G(F)$; see [MW]. For the purpose of this paper, we only need to consider this in the context of the metaplectic double cover of $\mathrm{Sp}(W)(\mathbb{A})$ and so we give a brief description in this case.

Assume that the characteristic of F is not two. For each place v of F, we have a unique nonlinear double cover $\widetilde{\mathrm{Sp}}(W)(F_v)$ of $\mathrm{Sp}(W)(F_v)$. If the residual characteristic of F_v is odd, then this cover splits uniquely over a hyperspecial maximal compact subgroup F_v of $\mathrm{Sp}(W)(F_v)$. Hence, one may form the restricted direct product

$$\prod_{K_v} \widetilde{\mathrm{Sp}}(W)(F_v),$$

which contains a central subgroup $Z = \oplus_v \mu_{2,v}$. If Z^+ denotes the index two subgroup of Z consisting of elements with an even number of components equal to -1, then the group

$$\widetilde{\mathrm{Sp}}(W)(\mathbb{A}) := \left(\prod_{K_v} \widetilde{\mathrm{Sp}}(W)(F_v) \right) / Z^+$$

is a nonlinear double cover of $\mathrm{Sp}(W)(\mathbb{A})$. It is a result of Weil that this double cover splits uniquely over the subgroup $\mathrm{Sp}(W)(F)$, so that we may speak of automorphic forms on $\widetilde{\mathrm{Sp}}(W)(\mathbb{A})$. An automorphic form f on $\widetilde{\mathrm{Sp}}(W)(\mathbb{A})$ is said to be genuine if it satisfies

$$f(\epsilon \cdot g) = \epsilon \cdot f(g) \quad \text{for } \epsilon \in \mu_2.$$

We denote the space of genuine automorphic forms on $\widetilde{\mathrm{Sp}}(W)(\mathbb{A})$ by $\mathcal{A}(\widetilde{\mathrm{Sp}}(W))$.

If $B = T \cdot U$ is a Borel subgroup of $\mathrm{Sp}(W)$, then the double covering splits uniquely over $U(F_v)$ for each v. Hence, in the adelic setting, there is a unique splitting of the double cover over $U(\mathbb{A})$, and more generally over the adelic group of the unipotent radical of any parabolic subgroup of $\mathrm{Sp}(W)$. As a result, one can define the notion of cusp forms as in the linear case, and we denote the space of such cusp forms by $\mathcal{A}_0(\widetilde{\mathrm{Sp}}(W))$. Moreover, if θ is a generic automorphic character of U, then one can define the θ-Fourier coefficient of f in the same way as before.

For the global analog of our restriction problems, we also need to discuss the notion of automorphic forms on the non-reductive group

$$J_W = \mathrm{Sp}(W) \ltimes H(W)$$

where $H(W) = W \oplus F$ is the Heisenberg group associated to W. The group J_W is called the Jacobi group associated to W and we shall consider its double cover $\widetilde{J}_W(\mathbb{A}) = \widetilde{\mathrm{Sp}}(W)(\mathbb{A}) \cdot H(W)(\mathbb{A})$. For a given additive character ψ of $F\backslash\mathbb{A}$, one has the the space of automorphic forms $\mathcal{A}_\psi(\widetilde{J}_W)$ on $\widetilde{J}_W(\mathbb{A})$, which consists of certain smooth functions on $J_W(F)\backslash\widetilde{J}_W(\mathbb{A})$ with central character ψ and is usually called the space of Jacobi forms.

For our applications, we are interested in a particular automorphic representation of $\widetilde{J}_W(\mathbb{A})$, namely the automorphic realization of the global Weil representation associated to $\psi = \prod_v \psi_v$. Recall that for each place v, the group

$$\widetilde{J}_W(F_v) = \widetilde{\mathrm{Sp}}(W)(F_v) \cdot H(W)(F_v)$$

has a local Weil representation ω_{ψ_v} whose restriction to $H(W)(F_v)$ is the unique irreducible representation with central character ψ_v. The restricted tensor product

$$\omega_\psi = \widehat{\otimes}_v \omega_{\psi_v}$$

is the global Weil representation associated to ψ. One of the main results of Weil [We] is that there is a unique (up to scaling) continuous embedding

$$\theta_\psi : \omega_\psi \hookrightarrow \mathcal{A}_\psi(\widetilde{J}_W).$$

Composing θ_ψ with the restriction of functions from $\widetilde{J}_W(\mathbb{A})$ to $\widetilde{\mathrm{Sp}}(W)(\mathbb{A})$ gives a $\widetilde{\mathrm{Sp}}(W)(\mathbb{A})$-equivariant (but not injective) map

$$\omega_\psi \longrightarrow \mathcal{A}(\widetilde{\mathrm{Sp}}(W)).$$

We now come to the global L-functions and epsilon factors associated to an irreducible cuspidal representation π, following Langlands. To define an L-function or epsilon factor, one needs the extra data of a finite dimensional representation R of the L-group ${}^LG$. If $\pi = \otimes_v \pi_v$ is an irreducible automorphic representation and we assume the local Langlands-Vogan correspondence for $G(F_v)$, then each π_v determines a local L-parameter

$$\phi_v : WD(F_v) \longrightarrow {}^LG.$$

Hence, one has the local L-factors $L(R \circ \phi_v, s)$ and one defines the global L-function

$$L(\pi, R, s) = \prod_v L(R \circ \phi_v, s),$$

which converges when $Re(s)$ is sufficiently large. Similarly, one has the local epsilon factors

$$\epsilon_v(\pi, R, \psi, s) = \epsilon(R \circ \phi_v, \psi_v, s),$$

and one defines the global epsilon factor by

$$\epsilon(\pi, R, s) = \prod_v \epsilon_v(\pi, R, \psi, s).$$

It is a finite product independent of the additive character ψ of $\mathbb{A}/F$. One expects that the L-function above has meromorphic continuation to the whole complex plane and satisfies a functional equation of a standard type, taking s to $1 - s$, so that the center of the critical strip is $s = 1/2$.

The following table gives some examples of R and their associated L-functions which appear in this paper. When the cuspidal representation π is globally generic, the meromorphic continuation of these L-functions are known.

G	${}^L G$	R
$\mathrm{GL}(V)$	$\mathrm{GL}(M)$	$\mathrm{Sym}^2 M$
$\mathrm{GL}(V)$	$\mathrm{GL}(M)$	$\wedge^2 M$
$\mathrm{GL}(V/E)$, E/F quadratic	$(\mathrm{GL}(M) \times \mathrm{GL}(M)) \cdot \mathrm{Gal}(E/F)$	$\mathrm{As}^{\pm}(M)$
$\mathrm{GL}(V) \times \mathrm{GL}(W)$	$\mathrm{GL}(M) \times \mathrm{GL}(N)$	$M \otimes N$
$\mathrm{SO}(W) \times \mathrm{SO}(V)$, $\dim W^{\perp}$ odd	$\mathrm{O}(M) \times \mathrm{Sp}(N)$, $\dim M$ even	$M \otimes N$
$\mathrm{Sp}(W) \times \widetilde{\mathrm{Sp}}(V)$	$\mathrm{SO}(M) \times \mathrm{Sp}(N)$, $\dim M$ odd	$M \otimes N$
$\mathrm{U}(W) \times \mathrm{U}(V)$	$(\mathrm{GL}(M) \times \mathrm{GL}(N)) \cdot \mathrm{Gal}(E/F)$	$\mathrm{Ind}_{\widehat{G}}^{{}^L G}(M \otimes N)$

23. Global Restriction Problems

We are now ready to formulate the global restriction problems. We shall change notations slightly from the earlier part of the paper, by replacing the pair of fields $k_0 \subset k$ in the local setting by $F \subset E$ in the global setting, with the characteristic of F different from 2. Hence σ is an involution (possibly trivial) on E with $E^{\sigma} = F$, and V is a vector space over E equipped with a sesquilinear form $\langle -, - \rangle$ of the relevant type. The group $G = G(V)$ is then an algebraic group over F. Also, we shall include the case $E = F \times F$ in our discussion.

Suppose that we have a pair of vector spaces $W \subset V$ over E equipped with a sesquilinear form of sign ϵ, such that $W^{\perp}$ is split and $\epsilon \cdot (-1)^{\dim W^{\perp}} = -1$. Then we have the groups

$$\begin{cases} G = G(V) \times G(W); \\ H = N \cdot G(W) \end{cases}$$

over F, as defined in §2.

The groups of F-points $G(F)$ and $H(F)$ are discrete subgroups of the locally compact adelic groups $G(\mathbb{A})$ and $H(\mathbb{A})$ respectively, where $\mathbb{A}$ is the ring of adèles of F. In the orthogonal case, we assume that if V or W has dimension 2, then it is not split. Then the quotient spaces $G(F)\backslash G(\mathbb{A})$ and $H(F)\backslash H(\mathbb{A})$ both have finite measure. We may then consider the space of automorphic forms and cusp forms for the group G, as in §22.

In this section, we will consider irreducible tempered representations π of $G(\mathbb{A})$ which occur in the space of cusp forms $\mathcal{A}_0(G)$ on $G(F)\backslash G(\mathbb{A})$ and study their restriction to $H(\mathbb{A})$. As in the local setting, when G is quasi-split, we need to introduce an automorphic generic character

$$\theta : U(F)\backslash U(\mathbb{A}) \longrightarrow \mathbb{S}^1$$

for the group G; this serves to fix the local Langlands-Vogan parametrization at all places v of F. In addition, we need to construct an automorphic representation ν on $H(F)\backslash H(\mathbb{A})$ for the restriction problem.

Assume in this paragraph that $G = G(V) \times G(W)$ is quasi-split. In the orthogonal or symplectic case, we use the spaces $W \subset V$ to naturally define a generic F-homomorphism

$$\lambda : U \longrightarrow \mathbb{G}_a$$

as in §12. This defines $\lambda_{\mathbb{A}} : U(\mathbb{A}) \longrightarrow \mathbb{A}$; now composing $\lambda_{\mathbb{A}}$ with a nontrivial additive character

$$\psi : \mathbb{A}/F \longrightarrow \mathbb{S}^1$$

gives an automorphic generic character of $U(\mathbb{A})$:

$$\theta = \psi \circ \lambda_{\mathbb{A}}.$$

In the hermitian or skew-hermitian case, we use the spaces $W \subset V$ to construct a generic homomorphism

$$\lambda : U \longrightarrow \mathrm{Res}_{E/F}(\mathbb{G}_a).$$

Then, in the hermitian case, we compose $\lambda_{\mathbb{A}}$ with a fixed nontrivial additive character

$$\psi : \mathbb{A}_E/(E + \mathbb{A}) \longrightarrow \mathbb{S}^1$$

to obtain an automorphic generic character $\theta_0 = \psi \circ \lambda_{\mathbb{A}}$. We then set

$$\theta(x) = \theta_0(-2 \cdot \delta \cdot \lambda_{\mathbb{A}}(x))$$

where δ is the discriminant of the odd hermitian space. In the skew-hermitian case, we take a nontrivial additive character

$$\psi : \mathbb{A}/F \longrightarrow \mathbb{S}^1$$

and set

$$\theta(x) = \psi(2 \cdot Tr_{E/F}(\lambda_{\mathbb{A}}(x)).$$

We stress that these definitions are global analogs of our definitions in the local setting.

Next, we need to define an automorphic version of the representation ν of $H(\mathbb{A}) = N(\mathbb{A})\cdot G(W)(\mathbb{A})$. (The group $G(W)$ is not assumed to be quasi-split.) In the orthogonal and hermitian cases, we define ν by composing the generic $G(W)$-invariant map

$$l : N \longrightarrow \mathrm{Res}_{E/F}(\mathbb{G}_a),$$

constructed in §12 using the spaces $W \subset V$, with a nontrivial additive character $\psi : \mathbb{A}_E/E \to \mathbb{S}^1$ and then extending this trivially on $G(W)(\mathbb{A})$:

$$\nu = \psi \circ l_{\mathbb{A}}.$$

As in the local case, the choice of ψ is unimportant. Then we define:

$$F(\nu) : \mathcal{A}_0(G) \longrightarrow \mathbb{C}(\nu)$$

by

$$f \mapsto \int_{H(F)\backslash H(\mathbb{A})} f(h) \cdot \overline{\nu(h)} \cdot dh.$$

The map $F(\nu)$ is called a Bessel coefficient.

In the symplectic and skew-hermitian cases, the representation ν is infinite-dimensional; so the situation is slightly more involved. Recall from §12 that, using the spaces $W \subset V$, we have defined a $G(W)$-invariant generic linear form

$$l : N \longrightarrow \mathbb{G}_a,$$

which gives rise to a continuous linear map $l_{\mathbb{A}} : N(\mathbb{A}) \longrightarrow \mathbb{A}$. Composing this with a nontrivial additive character $\psi : \mathbb{A}/F \to \mathbb{S}^1$, and extending trivially to $G(W)(\mathbb{A})$, we obtain an automorphic character

$$\Lambda = \psi \circ l_{\mathbb{A}}$$

of $H(\mathbb{A})$. On the other hand, we have also defined a homomorphism

$$N \longrightarrow H(W)$$

where

$$H(W) = F \oplus \mathrm{Res}_{E/F} W$$

is the Heisenberg group associated to $\mathrm{Res}_{E/F}(W)$. Thus, we have a homomorphism

$$H = G(W) \cdot N \longrightarrow J_W := \mathrm{Sp}(\mathrm{Res}_{E/F}(W)) \cdot H(W).$$

As discussed in §22, the group $\widetilde{J}_W(\mathbb{A})$ has a global Weil representation ω_ψ with central character ψ, and one has a canonical automorphic realization

$$\theta_\psi : \omega_\psi \hookrightarrow \mathcal{A}(\widetilde{J}_W).$$

It will now be convenient to consider the symplectic and skew-hermitian cases separately.

In the symplectic case, the map $H \longrightarrow J_W$ defined above gives rise to a map

$$\widetilde{H}(\mathbb{A}) \longrightarrow \widetilde{J}_W(\mathbb{A}).$$

By pulling back, one can thus regard ω_ψ as a representation of $\widetilde{H}(\mathbb{A})$. Moreover, the above map gives rise to a natural inclusion

$$\mathcal{A}(\widetilde{J}_W) \hookrightarrow \mathcal{A}(\widetilde{H}).$$

Composing the automorphic realization θ_ψ with this inclusion realizes ω_ψ as a submodule in $\mathcal{A}(\widetilde{H})$. Multiplying by the automorphic character Λ of H, one obtains an automorphic realization

$$\theta_\psi : \nu_\psi = \omega_\psi \otimes \Lambda \hookrightarrow \mathcal{A}(\widetilde{H}).$$

Now we can define the map

$$F(\nu_\psi) : \mathcal{A}_0(G) \otimes \overline{\nu_\psi} \longrightarrow \mathbb{C}$$

by

$$f \otimes \phi \mapsto \int_{H(F)\backslash H(\mathbb{A})} f(h) \cdot \overline{\theta_\psi(\phi)(h)}\, dh.$$

The map $F(\nu_\psi)$ is called a Fourier-Jacobi coefficient.

In the skew-hermitian case, we choose an automorphic character

$$\mu : \mathbb{A}_E^\times / E^\times \longrightarrow \mathbb{C}^\times$$

satisfying

$$\mu|_{\mathbb{A}^\times} = \omega_{E/F}.$$

Then one obtains a splitting homomorphism (see [K2] and [HKS])

$$s_{\psi,\mu} : H(\mathbb{A}) \longrightarrow \widetilde{J}_W(\mathbb{A}).$$

Using $s_{\psi,\mu}$, one may pull back the global Weil representation ω_ψ to obtain a representation $\omega_{\psi,\mu}$ of $H(\mathbb{A})$. As above, one also obtains an automorphic realization

$$\theta_{\psi,\mu} : \nu_{\psi,\mu} = \omega_{\psi,\mu} \otimes \Lambda \longrightarrow \mathcal{A}(H).$$

Thus, we can define the map

$$F(\nu_{\psi,\mu}) : \mathcal{A}_0(G) \otimes \overline{\nu_{\psi,\mu}} \longrightarrow \mathbb{C}$$

by

$$f \otimes \phi \mapsto \int_{H(F)\backslash H(\mathbb{A})} f(h) \cdot \overline{\theta_{\psi,\mu}(\phi)(h)}\, dh.$$

The map $F(\nu_{\psi,\mu})$ is called a Fourier-Jacobi coefficient in the context of unitary groups.

Now the global restriction problem is:

Determine whether the map $F(\nu)$ defined in the various cases above is nonzero when restricted to a tempered cuspidal representation π of $G(\mathbb{A})$.

24. Global conjectures: central values of L-functions

To formulate our global conjectures to the restriction problem of the previous section, we need to introduce a distinguished symplectic representation R of the L-group ${}^LG$ over F. Recall from §7 that either the L-group of a classical group or its identity component comes equipped with a standard representation. Thus, with $G = G(V) \times G(W)$, either ${}^LG$ or its identity component $\widehat{G}$ has a standard representation $M \otimes N$. We set

$$R = \begin{cases} M \otimes N & \text{in the orthogonal and symplectic cases;} \\ \\ \mathrm{Ind}_{\widehat{G}}^{{}^LG}(M \otimes N), & \text{in the hermitian case;} \\ \\ \mathrm{Ind}_{\widehat{G}\times W(E)}^{\widehat{G}\rtimes W(F)}((M \otimes N) \boxtimes \mu^{-1}), & \text{in the skew-hermitian case.} \end{cases}$$

This representation R was already introduced in the table at the end of §22, except for the skew-hermitian case. In the skew-hermitian case, we have incorporated the character μ used in the definition of ν. In doing so, we need to work with the L-group $\widehat{G} \rtimes W(F)$ rather than the version $\widehat{G} \rtimes \mathrm{Gal}(E/F)$ which we have been using in the rest of the paper. It is this twist by μ which makes R a symplectic representation of $\widehat{G} \rtimes W(F)$ (by Lemmas 3.4 and 3.5). As explained in §22, we can then speak of the L-function $L(\pi, R, s)$ and the global epsilon factor $\epsilon(\pi, R, s)$ for any automorphic representation π of G. We recall that these L-functions are normalized so that the functional equation takes s to $1 - s$ and the center of the critical strip is $s = 1/2$.

The first form of our global conjecture is:

Conjecture 24.1. *Let π be an irreducible tempered representation of $G(\mathbb{A})$ which occurs with multiplicity one in the space $\mathcal{A}_0(G)$ of cusp forms on $G(F)\backslash G(\mathbb{A})$. Let ν be the automorphic representation of $H(\mathbb{A})$ introduced in §23. Then the following are equivalent:*

(i) the restriction of the linear form $F(\nu)$ to π is nonzero;

(ii) the complex vector space $\mathrm{Hom}_{H(\mathbb{A})}(\pi, \nu)$ is nonzero and the L-function $L(\pi, R, s)$ does not vanish at $s = 1/2$, which is the center of the critical strip;

(iii) the complex vector spaces $\mathrm{Hom}_{H(F_v)}(\pi_v, \nu_v)$ are nonzero for all places v of F and $L(\pi, R, 1/2) \neq 0$.

Let us make some remarks about this conjecture.

Remarks:

(i) The equivalence of (ii) and (iii) is clear, provided one knows Conjecture 21.3.

(ii) When $E = F \times F$, with $G \cong \mathrm{GL}(V_0) \times \mathrm{GL}(W_0)$, then the L-function $L(\pi, R, s)$ is the product

$$L(\pi, R, s) = L(\pi_V \otimes \pi_W, s) \cdot L(\pi_V^\vee \times \pi_W^\vee, s)$$

of two Rankin-Selberg L-functions, so that

$$L(\pi, R, 1/2) = |L(\pi_v \times \pi_W, 1/2)|^2.$$

In this case, the conjecture is known when $\dim W^\perp = 1$. Indeed, this is an immediate consequence of the integral representation of the global Rankin-Selberg L-function $L(\pi_V \times \pi_W, 1/2)$ [JPSS]. The general case seems to be open.

(iii) More generally, under the assumption that π is globally generic and has a cuspidal functorial lift to the appropriate general linear group, the implication (i) $\Longrightarrow$ (ii) has been shown by Ginzburg-Jiang-Rallis in a series of papers for the various cases [GJR1,2,3]. Moreover, in the hermitian case with $\dim W^\perp = 1$, an approach to this conjecture via the relative trace formula has been developed by Jacquet and Rallis [JR].

(iv) One expects a refinement of Conjecture 24.1 in the form of an exact formula relating $|F(\nu)|^2$ with the central value $L(\pi, R, 1/2)$. Such a refinement has been formulated by Ichino-Ikeda [II] in the orthogonal case, with $\dim W^\perp = 1$. In the analogous setting for the hermitian case, the formulation of this refined conjecture is the ongoing thesis work of N. Harris at UCSD.

As formulated, the global conjecture 24.1 is essentially independent of the local conjectures 17.1 and 17.3. Rather, they complement each other, since the local non-vanishing in Conjecture 24.1 is governed by our local conjectures. From this point of view, the appearance of the particular central L-value $L(\pi, R, 1/2)$ may not seem very well-motivated. However, as we shall explain in the next two sections, if we examine the implications of our local conjectures in the framework of the Langlands-Arthur conjecture on the automorphic discrete spectrum, the appearance of $L(\pi, R, 1/2)$ is very natural.

For example, observe that the global conjecture 24.1 in the symplectic/metaplectic case involves the central L-value of the symplectic representation $R = M \otimes N$ with M symplectic and N odd orthogonal, whereas in the local conjecture 17.3, it is the epsilon factor associated to $M \otimes (N \oplus \mathbb{C})$ which appears. So in some sense, the global conjecture is less subtle than the local one. The explanation of this can be found in §26, as a consequence of the Langlands-Arthur conjecture (or rather its extension to the metaplectic case).

Finally, we highlight a particular case of the conjecture. As we explained in §12, special cases of the data (H, ν) are automorphic generic characters $\nu = \theta$ on $H = U$.

These cases are highlighted in the following table, and arise when the smaller space W is either 0 or 1-dimensional.

$G(V)$	$\dim W$	$\widehat{G(W)}$	N
odd orthogonal	0	SO(0)	0
even orthogonal	1	Sp(0)	0
symplectic	0	Sp(0)	0
metaplectic	0	SO(1)	$\mathbb{C}$
odd hermitian	0	GL(0)	0
even skew hermitian	0	GL(0)	0

As one sees from the table, in all except the metaplectic case, $N = 0$ so that $R = 0$ and $L(\pi, R, s)$ is identically 1. In the metaplectic case, $N = \mathbb{C}$ so that $R = M$ and $L(\pi, R, s)$ is the standard L-function $L(\pi, s)$. Hence Conjecture 24.1 specializes to the following two conjectures in these degenerate cases.

Conjecture 24.2. *Let π be an irreducible tempered representation of $G(\mathbb{A})$ which occurs with multiplicity one in the space $\mathcal{A}_0(G)$ of cusp forms on $G(F)\backslash G(\mathbb{A})$ and let θ be an automorphic generic character for G. Then, when G is a linear group, the following are equivalent:*

(i) the restriction of the map $F(\theta)$ to π is nonzero;

(ii) the complex vector spaces $\mathrm{Hom}_{U(F_v)}(\pi_v, \theta_v)$ are nonzero for all places v of F.

When $\widetilde{G}(\mathbb{A}) = \widetilde{\mathrm{Sp}}(V)(\mathbb{A})$ is metaplectic, we fix an additive character ψ of $\mathbb{A}/F$ which determines an automorphic generic character θ and also gives the notion of Langlands-Vogan parameters. For any element $c \in F^\times/F^{\times 2}$, let χ_c be the associated quadratic character of $\mathbb{A}^\times/F^\times$ and let θ_c denote the generic character associated to the additive character $\psi_c(x) = \psi(cx)$. Then we have:

Conjecture 24.3. *Let π be an irreducible tempered representation of $\widetilde{G}(\mathbb{A}) = \widetilde{\mathrm{Sp}}(V)(\mathbb{A})$ which occurs with multiplicity one in the space $\mathcal{A}_0(G)$ of cusp forms on $G(F)\backslash\widetilde{G}(\mathbb{A})$ and let θ be an automorphic generic character for G. Then the following are equivalent:*

(a) the restriction of the map $F(\theta_c)$ to π is nonzero;

(b) the complex vector spaces $\mathrm{Hom}_{U(F_v)}(\pi_v, \theta_{c,v})$ *are nonzero for all places* v *of* F *and* $L(\pi \otimes \chi_c, 1/2) \neq 0$.

In the metaplectic case, with $\dim V = 2$, the above conjecture is known by the work of Waldspurger [Wa1-2].

Note that if the conditions in the above two conjectures hold, the space $\mathrm{Hom}_{U(\mathbb{A})}(\pi, \theta)$ has dimension 1 and $F(\theta)$ is a basis. Moreover, since π is tempered in the conjecture, the adjoint L-function $L(\pi, Ad, s)$ of π is expected to be regular and nonzero at $s = 1$ (which is the edge of the critical strip). Indeed, just as the holomorphy of the local adjoint L-factor at $s = 1$ characterizes the generic L-packets in the local part of this paper, the tempered cuspidal representations considered in the global conjectures of this section should be characterized by the analytic properties of their global adjoint L-function. More precisely, we have:

Conjecture 24.4. *Let* G *be a connected reductive group over* F *(or the metaplectic group* $\widetilde{\mathrm{Sp}}(V)(\mathbb{A})$*) and let* $\overline{Ad}$ *be the adjoint representation of the* L*-group* ${}^LG$ *on* $\mathrm{Lie}(\widehat{G}/Z(\widehat{G}))$.

(a) Let π *be a tempered automorphic representation of* $G(\mathbb{A})$*. Then the following are equivalent:*

(i) The representation π *is cuspidal;*

(ii) The adjoint L*-function* $L(\pi, \overline{Ad}, s)$ *is holomorphic at* $s = 1$.

(b) Let π *be a cuspidal representation of* $G(\mathbb{A})$*. Then the following are equivalent:*

(i) The representation π *is tempered;*

(ii) The partial adjoint L*-function* $L^S(\pi, \overline{Ad}, s)$ *is holomorphic in* $Re(s) \geq 1$ *(for* S *a finite set of places containing all archimedean places and finite places* v *where* $G \times_F F_v$ *or* π_v *is ramified).*

The rationale for part (a) of the conjecture is that one expects the conjectural L-parameter of a tempered representation π to have finite centralizer modulo $Z(\widehat{G})$ if and only if π is cuspidal, and further, the holomorphy of the adjoint L-function of a tempered π at $s = 1$ detects the finiteness (modulo $Z(\widehat{G})$) of the centralizer. The rationale for part (b) is similar, taking into account the conjecture of Arthur [A1-2] which describes the non-tempered part of the cuspidal spectrum in terms of A-parameters.

The implication (i) $\Longrightarrow$ (ii) (in both (a) and (b)) will follow from known analytic properties of Rankin-Selberg L-functions of GL_n once the functorial lifting from classical groups to GL_n is established. In general, this conjecture should be a consequence of the Ramanujan conjecture and the Arthur conjecture [A1-2].

25. Global L-parameters and Multiplicity Formula

In this section, we review the notion of global L-parameters in the context of a fundamental conjecture of Langlands and Arthur [A1, A2], concerning multiplicities of representations in the automorphic discrete spectrum. We will only present this conjecture for tempered representations, and will also discuss its simplification for the classical groups considered in this paper. In the next section, we shall re-examine the global conjecture 24.1 in the framework of the Langlands-Arthur conjecture. We henceforth assume that the F-algebra E is a field.

Let G be a connected reductive group over the global field F, and assume that the quotient space $G(F)\backslash G(\mathbb{A})$ has finite volume. The Langlands-Arthur conjecture gives a description of the decomposition of the discrete spectrum $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}))$ or equivalently the space $\mathcal{A}^2(G)$ of square-integrable automorphic forms. We shall only describe this conjecture for the tempered part of the discrete spectrum, which we denote by $L^2_{\mathrm{disc,temp}}(G)$. Note that $L^2_{\mathrm{disc,temp}}(G)$ is necessarily contained in the cuspidal spectrum by a result of Wallach [W3].

Suppose that G_0 is the quasi-split inner form of G over F, with a Borel-subgroup $B = T\cdot U$. Fix an automorphic generic character $\theta = \otimes_v \theta_v$ of U as in the previous section. We fix an integral structure on G, which determines a hyperspecial maximal compact subgroup $J_v \subset G_0(F_v)$ for almost all finite places v, as in §21. If $G = G(V)$ is a classical group, such an integral structure is given by fixing a lattice $L \subset V$.

<u>The Langlands-Arthur Conjecture</u>

(1) For any pure inner form G of G_0, there is a decomposition

$$L^2_{\mathrm{disc,temp}}(G) = \widehat{\bigoplus}_{\phi} L^2_{\phi}(G),$$

where ϕ runs over the discrete global L-parameters and each L^2_{ϕ} is a $G(\mathbb{A})$-submodule. The precise definitions of these objects are given as follows.

By definition, a discrete global L-parameter is a homomorphism

$$\phi : L_F \longrightarrow {}^L G = {}^L G_0 = \widehat{G} \rtimes W_F$$

such that its centralizer in the dual group $\widehat{G}$ is finite. These parameters are taken up to conjugacy by the dual group $\widehat{G}$. Moreover, L_F is the hypothetical Langlands group of F – the global analog of the Weil-Deligne group $WD(F_v)$ – whose existence is only conjectural at this point. One postulates however that there is a natural surjective map

$$L_F \longrightarrow W_F \quad \text{(the Weil group of } F\text{)},$$

and the projection of ϕ to the second factor W_F in ${}^L G$ is required to be this natural surjection. Moreover, one postulates that for each place v of F, there is a natural conjugacy class of embedding

$$WD(F_v) \longrightarrow L_F.$$

Assuming the above, one may attach the following data to a given discrete global L-parameter ϕ:

(i) a global component group

$$A_\phi = Z_{\widehat{G}}(\mathrm{Im}(\phi)),$$

which is finite by assumption.

(ii) for each place v of F, a local L-parameter

$$\phi_v : WD(F_v) \longrightarrow L_F \xrightarrow{\phi} {}^L G_0$$

for the local group $G_{0,v}$, such that for almost all v, ϕ_v is unramified. This gives rise to a natural map of component groups:

$$A_\phi \longrightarrow A_{\phi_v}.$$

One thus has a diagonal map

$$\Delta : A_\phi \longrightarrow \prod_v A_{\phi_v}.$$

(iii) for each place v, the local Vogan packet Π_{ϕ_v} of irreducible representations of the pure inner forms $G(F_v)$, together with a bijection

$$J(\theta_v) : \Pi_{\phi_v} \leftrightarrow \mathrm{Irr}(A_{\phi_v})$$

specified by the local component θ_v of the automorphic generic character θ. For an irreducible character η_v of A_{ϕ_v}, we denote the corresponding representation in Π_{ϕ_v} by π_{η_v}. In particular, the representation corresponding to the trivial character of A_{ϕ_v} is a representation of $G_0(F_v)$ and for almost all v, it is spherical with respect to the hyperspecial maximal compact subgroup J_v.

(iv) a global Vogan packet

$$\Pi_\phi = \{\pi_\eta = \bigotimes_v \pi_{\eta_v} : \pi_{\eta_v} \in \Pi_{\phi_v} \text{ and } \eta_v \text{ is trivial for almost all } v\}.$$

In particular, the representations in the global packet are indexed by irreducible characters

$$\eta = \otimes_v \eta_v \quad \text{of} \quad \prod_v A_{\phi_v}.$$

If π_{η_v} is a representation of $G_{\eta_v}(F_v)$, then π_η is a representation of the restricted direct product

$$G_\eta := \prod_{J_v} G_{\eta_v}(F_v).$$

Note, however, that the group G_η need not be the adelic group of a pure inner form of G_0. For example, in the classical group case, G_η need not be associated to a space V equipped with a relevant sesquilinear form over F.

If $G_\eta = G(\mathbb{A})$ for a pure inner form of G_0, we shall call the representation $\eta = \prod_v \eta_v$ *coherent.* This notion can be explicated as follows. It was shown by Kottwitz [Ko, Cor. 2.5 and Prop. 2.6] that one has a natural map

$$\begin{array}{ccc} \bigoplus_v H^1(F_v, G_0) & \longrightarrow & \bigoplus_v \mathrm{Hom}(\pi_0(Z(\widehat{G_0})^{W(F_v)}), \mathbb{C}^\times) \\ & & \downarrow \\ & & \mathrm{Hom}(\pi_0(Z(\widehat{G_0})^{W(F)}), \mathbb{C}^\times) \end{array}$$

and the kernel of this map is the image of the natural map

$$H^1(F, G_0) \longrightarrow \bigoplus_v H^1(F_v, G_0).$$

Now the character $\eta = \prod_v \eta_v$ gives rise to an element in $\oplus_v H^1(F_v, G_0)$, and η is coherent if and only if this element is in the image of $H^1(F, G_0)$. Thus, we see that η is coherent if and only if η is trivial when restricted to $\pi_0(Z(\widehat{G_0})^{W(F)})$.

(v) for each $\pi_\eta \in \Pi_\phi$, a non-negative integer

$$m_\eta = \langle \Delta^*(\eta), 1 \rangle_{A_\phi},$$

where the expression on the right denotes the inner product of the two characters of the finite group A_ϕ. Thus m_η is the multiplicity of the trivial character of A_ϕ, in the representation obtained by restriction of the tensor product of the representations η_v to the diagonal. If η is not coherent, then one sees that m_η is equal to zero, since η is nontrivial when restricted to $\pi_0(Z(\widehat{G_0})^{W(F)})$ and hence on A_ϕ. When η is coherent, so the adelic group G_η is defined over F, the Langlands-Arthur conjecture for tempered representations predicts that m_η is the multiplicity of the representation π_η in the discrete spectrum of G_η

With the above data, we have:

(2) As G runs over all pure inner forms of G_0 over F, there is an equivariant decomposition:

$$\bigoplus_G L^2_\phi(G) = \bigoplus_\eta m_\eta \cdot \pi_\eta.$$

We denote the representation in Π_ϕ associated to the trivial character by π_0. It is a representation of $G_0(\mathbb{A})$ and is the unique representation in Π_ϕ which is abstractly θ-generic. According to the multiplicity formula in (v), its multiplicity in $L^2_\phi(G_0)$ is 1, and the conjecture 24.2 then says that π_0 has a nonzero θ-Fourier coefficient.

Though the above conjecture of Langlands and Arthur is extremely elegant, it has a serious drawback: the group L_F is not known to exist. However, in the case of the classical groups considered in this paper, one can present the conjecture on multiplicities in a way that avoids mentioning the group L_F. We do this below. In the case of classical groups, there is a further simplification, as the component groups A_{ϕ_v} are all elementary abelian 2-groups. In particular, the representations η_v are all 1-dimensional, so their restricted tensor product η also has dimension 1. Hence the predicted multiplicity m_η is either zero or one, the latter case occurring when η has trivial restriction to the diagonal. In the general case, the groups A_{ϕ_v} can be non-abelian, and both the dimension of the representation η and the dimension m_η of its A_ϕ-invariants can be arbitrarily large.

We now specialize to the case where $G = G(V)$ is a classical group. Let $G_0 = G(V_0)$ be the quasi-split inner form. Arguing exactly as we did in the local case, one sees that giving a global L-parameter for G

$$\phi : L_F \longrightarrow {}^L G_0$$

is equivalent to giving a representation

$$\varphi : L_F \longrightarrow \mathrm{GL}(M)$$

which is selfdual or conjugate-dual with a specific sign b. The requirement that ϕ is discrete then translates to the requirement that as a representation of L_E,

$$M \cong \bigoplus_i M_i$$

where each M_i is selfdual or conjugate-dual with the same sign b as M and $M_i \ncong M_j$ if $i \neq j$. In this case, the global component group is the 2-group

$$A_\phi = A_\varphi^+ = (\prod_i (\mathbb{Z}/2\mathbb{Z})_{M_i})^+,$$

where the superscript + is needed only when φ is selfdual.

Now to remove the mention of the hypothetical group L_E, observe that when specialized to the case $G = \mathrm{GL}(V)$, with $\dim V = n$, the Langlands-Arthur conjecture simply says that there is a natural bijection

$$\begin{array}{c}\{\text{irreducible cuspidal representations of } \mathrm{GL}(V)\} \\ \updownarrow \\ \{\text{irreducible } n\text{-dimensional representations of } L_F\}\,.\end{array}$$

Thus, one may suppress the mention of L_F by replacing the latter set with the former. Hence, in the context of the classical groups $G(V)$, one replaces the data of each M_i by an irreducible cuspidal representation π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$, with $n_i = \dim M_i$. Moreover, in view of Proposition 7.5 and its analog for symplectic and orthogonal groups, the selfduality or conjugate-duality of M_i with sign b can be described invariant theoretically and hence can be captured by the following L-function condition:

(a) an irreducible cuspidal representation π of $\mathrm{GL}_n(\mathbb{A})$ is selfdual of sign

$$\begin{cases} +1 \text{ if its symmetric square } L\text{-function } L(\pi, \mathrm{Sym}^2, s) \text{ has a pole at } s = 1; \\ -1 \text{ if its exterior square } L\text{-function } L^S(\pi, \wedge^2, s) \text{ has a pole at } s = 1. \end{cases}$$

(b) an irreducible cuspidal representation π of $\mathrm{GL}_n(\mathbb{A}_E)$ is conjugate-dual of sign

$$\begin{cases} +1, \text{ if the Asai } L\text{-function } L(\pi, \mathrm{As}^+, s) \text{ has a pole at } s = 1; \\ -1, \text{ if the Asai } L\text{-function } L(\pi, \mathrm{As}^-, s) \text{ has a pole at } s = 1. \end{cases}$$

To summarize, a discrete global L-parameter φ for $G_0 = G(V_0)$ is the data of a number of *inequivalent* cuspidal representations π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$, with $\sum_i n_i = \dim M$, satisfying the above L-function conditions for *each* i. The point of this reformulation is that given such a global L-parameter, one still has the data given in (i) -(v) above. More precisely, one has:

(i) The global component group A_φ is simply the 2-group $\prod_i (\mathbb{Z}/2\mathbb{Z})_{\pi_i}$ with a canonical basis indexed by the π_i's.

(ii) For each v, the associated local L-parameter is the representation

$$\varphi_v = \bigoplus_i \varphi_{i,v}$$

of $WD(k_v)$, where $\varphi_{i,v}$ is the local L-parameter of the local component $\pi_{i,v}$ of π_i. The L-function condition presumably forces each $\varphi_{i,v}$ to be selfdual or conjugate-dual with the given sign b. Moreover, one has a natural homomorphism

$$A_\varphi \longrightarrow A_{\varphi_v} = \prod_i A_{\varphi_{i,v}}$$

arising from the natural map

$$(\mathbb{Z}/2\mathbb{Z})_{\pi_i} \to C_{\varphi_{i,v}} \to A_{\varphi_{i,v}},$$

obtained by sending $(\mathbb{Z}/2\mathbb{Z})_{\pi_i}$ to the central subgroup $\langle \pm 1 \rangle$ in the centralizer $C_{\varphi_{i,v}}$. Thus, one continues to have the diagonal map Δ.

(iii) For each place v, the local parameter thus gives rise to a local Vogan packet Π_{φ_v} as before.

(iv) One can now define the global Vogan packet as before.

(v) The formula for m_η is as given above.

The formulation of the Langlands-Arthur conjecture given above amounts to a description of the discrete spectrum of classical groups in terms of the automorphic representations of GL_n. The proof has been promised in a forthcoming book [A3].

In the remainder of this section, we formulate an extension of the Langlands-Arthur conjecture to the case of the metaplectic groups $\widetilde{\mathrm{Sp}}(W)$.

Motivated by Theorem 11.1, one expects that discrete global L-parameters for $\widetilde{\mathrm{Sp}}(W)$ should be discrete global L-parameters for $\mathrm{SO}(2n+1)$ with $2n = \dim W$. Thus, a discrete global L-parameter of $\widetilde{\mathrm{Sp}}(W)$ should be a multiplicity free $2n$-dimensional symplectic representation of L_F:

$$M = M_1 \oplus \cdots \oplus M_r$$

with each irreducible summand M_i also symplectic. In the reformulation of L-parameters given above, it is thus given by the data of a collection of pairwise inequivalent cuspidal representations π_i of $\mathrm{GL}_{2n_i}(\mathbb{A})$ with

(a) $\sum_i n_i = n$ and

(b) $L(\pi_i, \wedge^2, s)$ having a pole at $s = 1$ for each i.

Using Theorem 11.1 and Corollary 11.2, one sees that such a global L-parameter φ continues to give rise to the data (i) - (iv) above in the context of $\widetilde{\mathrm{Sp}}(W)$. In particular, one obtains a global Vogan packet Π_φ of irreducible genuine representations of $\widetilde{\mathrm{Sp}}(W)(\mathbb{A})$ with a bijection

$$\Pi_\varphi \longleftrightarrow \mathrm{Irr}\left(\prod_v A_{\varphi_v}\right).$$

However, the multiplicity formula given in (v) above needs to be modified. Motivated by results of Waldspurger in the case when $\dim W = 2$, we make the following conjecture.

Conjecture 25.1. *Let (φ, M) be a discrete global L-parameter for $\widetilde{\mathrm{Sp}}(W)$ with associated global Vogan packet Π_φ. Let χ_φ be the character on the global component group A_φ defined by*

$$\chi_\varphi(a) = \epsilon(M^a, 1/2).$$

More concretely, if $a_i \in A_\varphi$ is the basis element associated to the factor (M_i, π_i) in M, then $\chi_\varphi(a_i) = \epsilon(\pi_i, 1/2)$. Then

$$L^2_\varphi(\widetilde{\mathrm{Sp}}(W)) \cong \bigoplus_\eta m_\eta \pi_\eta$$

where

$$m_\eta = \langle \Delta^*(\eta), \chi_\varphi \rangle.$$

We note that Arthur has also introduced nontrivial quadratic characters of the global component group A_φ in his conjectures for the multiplicities of non-tempered representations of linear groups.

We conclude this section with some ramifications of Conjecture 25.1. Given a discrete global L-parameter (φ, M) (relative to a fixed additive character ψ) for $\widetilde{\mathrm{Sp}}(W)$, with $\dim W = 2n$, note that M is also a discrete global L-parameter for $\mathrm{SO}(V)$ with $\dim V = 2n+1$. For each place v, the elements in the associated Vogan packets

$$\begin{cases} \Pi_{\varphi_v}(V) \text{ of } \mathrm{SO}(V) \\ \Pi_{\varphi_v}(W) \text{ of } \widetilde{\mathrm{Sp}}(W) \end{cases}$$

are both indexed by $\mathrm{Irr}(A_{\varphi_v})$. For a character η_v of A_{φ_v}, let

$$\pi_{\eta_v} \in \Pi_{\varphi_v}(V) \quad \text{and} \quad \sigma_{\eta_v} \in \Pi_{\varphi_v}(W)$$

be the corresponding representations. By construction, π_{η_v} and σ_{η_v} are local theta lifts (with respect to ψ) of each other. One might expect that, globally, the submodule $L^2_\varphi(\widetilde{\mathrm{Sp}}(W))$ of the discrete spectrum can be obtained from $\bigoplus_{V'} L^2_\varphi(\mathrm{SO}(V'))$ using global theta correspondence. As we explain below, this is not always the case.

More precisely, if $\eta = \otimes_v \eta_v$ is a character of $\prod_v A_{\varphi_v}$, then the corresponding representations

$$\pi_\eta \in \Pi_\varphi(V) \quad \text{and} \quad \sigma_\eta \in \Pi_\varphi(W)$$

may or may not be global theta lifts of each other. Indeed,

$$\pi_\eta \text{ occurs in the discrete spectrum of some } \mathrm{SO}(V') \Longleftrightarrow \Delta^*(\eta) = 1$$

whereas

$$\sigma_\eta \text{ occurs in the discrete spectrum of } \widetilde{\mathrm{Sp}}(W) \Longleftrightarrow \Delta^*(\eta) = \chi_\varphi.$$

Thus, if χ_φ is nontrivial, then the subset of η's which indexes automorphic representations for $\mathrm{SO}(V)$ will be disjoint from that which indexes automorphic representations of $\widetilde{\mathrm{Sp}}(W)$. In such cases, there is clearly no way of obtaining the automorphic elements in $\Pi_\varphi(W)$ from those of $\Pi_\varphi(V)$ via global theta correspondence (with respect to ψ).

Suppose, on the other hand, that χ_φ is the trivial character, so that

$$\epsilon(M_i, 1/2) = 1$$

for all the irreducible symplectic summands M_i of M. Then the automorphic elements in $\Pi_\varphi(W)$ and $\Pi_\varphi(V)$ are indexed by the same subset of η's and are abstract theta lifts of each other. However, to construct $L^2_\varphi(\widetilde{\mathrm{Sp}}(W))$ from $L^2_\varphi(\mathrm{SO}(V))$ via global theta correspondence, there is still an issue with the non-vanishing of global theta liftings. In this case, the non-vanishing of the global theta lifting is controlled by the non-vanishing of the central L-value

$$L(M, 1/2) = \prod_i L(M_i, 1/2).$$

Only when $L(M, 1/2)$ is nonzero does one know that $L^2_\varphi(\widetilde{\mathrm{Sp}}(W))$ can be obtained from $\bigoplus_{V'} L^2_\varphi(\mathrm{SO}(V'))$ by global theta lifting (with respect to ψ).

Another observation is that while the packet $\Pi_\varphi(V)$ always contains automorphic elements (for example the representation corresponding to $\eta = 1$), it is possible that none of the elements in the packet $\Pi_\varphi(W)$ are automorphic.

We give two examples which illustrate Conjecture 25.1 and the phenomena noted above, in the case $\dim W = 2$. In this case, the conjecture is known by the work of Waldspurger [Wa1, 2].

Example 1: Suppose that (φ, M) is a discrete global L-parameter for $\mathrm{SO}(3) \cong \mathrm{PGL}(2)$ and $\widetilde{\mathrm{Sp}}(2)$, so that

$$A_\varphi = \mathbb{Z}/2\mathbb{Z}.$$

Suppose that for 3 places v_1, v_2 and v_3, the local L-parameter φ_{v_i} corresponds to the Steinberg representation of PGL(2), and φ_v is unramified for all other v. Then

$$\epsilon(M, 1/2) = -1,$$

so that χ_φ is the nontrivial character of A_φ.

In this case, the local Vogan packets (for both SO(3) and $\widetilde{\mathrm{Sp}}(2)$) have size 2 at the 3 places v_i, and we label the representations by

$$\Pi_{\varphi_v}(\mathrm{SO}(3)) = \{\pi_v^+, \pi_v^-\} \quad \text{and} \quad \Pi_{\varphi_v}(\widetilde{\mathrm{Sp}}(2)) = \{\sigma_v^+, \sigma_v^-\},$$

with the minus sign indicating the nontrivial character of A_{φ_v}. At all other places, the local packets are singletons. Thus, the global L-packet of SO(3) has 8 elements, which we can label as π^{+++}, π^{++-} and so on. Similarly, the global L-packet for $\widetilde{\mathrm{Sp}}(2)$ also has 8 elements, denoted by σ^{+++}, σ^{++-} and so on.

Now observe that a representation in the global Vogan packet for SO(3) is automorphic if and only if it has an even number of minus signs in its label, whereas a representation in the global Vogan packet for $\widetilde{\mathrm{Sp}}(2)$ is automorphic if and only if it has an odd number of minus signs in its label.

Example 2: Suppose again that (φ, M) is a discrete global parameter for SO(3) and $\widetilde{\mathrm{Sp}}(2)$, but now assume that φ_v is reducible for all v. Moreover, suppose that

$$\epsilon(M, 1/2) = -1,$$

so that χ_φ is the nontrivial character of $A_\varphi = \mathbb{Z}/2\mathbb{Z}$. These conditions can be arranged.

In this case, the global Vogan packets for SO(3) and $\widetilde{\mathrm{Sp}}(2)$ are both singletons, containing the representation π and σ respectively, which are indexed by the trivial character η of

$$\prod_v A_{\varphi_v} = 1.$$

Hence, $\Delta^*(\eta)$ is the trivial character of A_φ. In particular, π is automorphic for SO(3), whereas σ is not automorphic for $\widetilde{\mathrm{Sp}}(2)$.

26. Revisiting the global conjecture

In this section, we shall revisit the global conjecture formulated in §24. In particular, we shall approach the restriction problem using the framework of the Langlands-Arthur conjecture reviewed in §25.

We start with a pair of spaces $W_0 \subset V_0$ which gives rise to a quasi-split group $G_0 = G(V_0) \times G(W_0)$ over F and fix an automorphic generic character θ of U as in the previous section; in particular, θ may depend on the choice of an appropriate additive character ψ in various cases. Given a discrete global L-parameter (φ, M, N) for G_0, there is a corresponding submodule $\mathcal{A}^2_\varphi$ in the automorphic discrete spectrum and we are interested in the restriction of the linear functional $F(\nu)$ to this submodule.

Recall that a natural symplectic representation R of ${}^LG$ plays a prominent role in the global conjecture 24.1. Using the global component group A_φ of the parameter φ, we may refine the associated L-function $L(\pi, R, s)$ for $\pi \in \Pi_\varphi$, as follows. If $a \in A_\varphi$, we may consider it as an element of A_{φ_v} for any v and then choose any semisimple element in C_{φ_v} projecting to it. Denoting any such element in $C_{\varphi_v} \subset \widehat{G}$ by a again, we see that the subspace R^a is a representation of $WD(F_v)$ under φ_v. Thus, one has the associated L-function

$$L(\pi, R^a, s) := \prod_v L(R^a, s)$$

and epsilon factor

$$\epsilon(\pi, R^a, s) := \prod_v \epsilon(R^a, \psi_v, s).$$

We are now ready to revisit the global restriction problem. Let us first draw some implications of the various local conjectures we have made so far.

(i) According to our local conjectures 17.1 and 17.3, there is a unique representation π_v in the local Vogan packet Π_{φ_v} such that

$$\mathrm{Hom}_{H(F_v)}(\pi_v \otimes \overline{\nu_v}, \mathbb{C}) \neq 0,$$

and this distinguished representation is indexed by a distinguished (relevant) character

$$\chi_v \quad \text{of } A_{M_v} \times A_{N_v}.$$

For each v, the representation π_{χ_v} is a representation of a pure inner form G_v of G_0 over F_v, associated to a pair of spaces $W_v \subset V_v$.

(ii) According to our unramified local conjecture 21.3, for almost all v, the distinguished character χ_v is trivial and the representations π_{χ_v}, θ_v and ν_v are all unramified. At these places, the pair of spaces $W_v \subset V_v$ is simply $W_{0,v} \subset V_{0,v}$ and the group G_v is simply $G_0(F_v)$. Thus, we can form the restricted direct product groups

$$\begin{cases} G_{\mathbb{A}} = \prod_{J_v} G_v; \\ H_{\mathbb{A}} = \prod_{J_v \cap H_v} H_v, \end{cases}$$

and representations

$$\begin{cases} \pi_\chi = \widehat{\otimes}_v \pi_{\chi_v} \text{ of } G_{\mathbb{A}}; \\ \nu \ \ = \widehat{\otimes}_v \nu_v \ \text{ of } H_{\mathbb{A}}, \end{cases}$$

which are restricted tensor products, defined using the unique line of J_v-invariant or $J_v \cap H_v$-invariant vectors for almost all v. The representation π_χ is simply the element in the global Vogan packet Π_φ indexed by the distinguished character

$$\chi = \otimes_v \chi_v$$

of the compact group $\prod_v A_{M_v} \times A_{N_v}$. It is the only (relevant) element in Π_φ such that

$$\mathrm{Hom}_{H(\mathbb{A})}(\pi_\chi \otimes \overline{\nu}, \mathbb{C}) \neq 0.$$

With these preliminaries out of the way, there are now three questions to address.

(1) Are $H_{\mathbb{A}} \hookrightarrow G_{\mathbb{A}}$ the adelic points of algebraic groups

$$H = N \cdot G(W) \hookrightarrow G = G(V) \times G(W)$$

defined over F, associated to a relevant pair of spaces $W \subset V$ over E?

In the symplectic case, this question clearly has a positive answer, since there is a unique symplectic vector space in any even dimension over any field. Hence, we focus on the other cases, where the issue is whether the collection $(W_v \subset V_v)$ of local spaces is coherent in the terminology of Kudla. Now these local spaces have the same rank as $W_0 \subset V_0$ and the same discriminant in the orthogonal case. Hence, they form a coherent collection if and only if we have changed the Hasse-Witt invariant or the hermitian/skew-hermitian discriminants at an even number of places v. This is equivalent to the identity

$$1 = \prod_v \chi_v(-1,1) = \begin{cases} \prod_v \epsilon(M_v \otimes N_v, \psi_v), & \text{in the orthogonal and hermitian cases;} \\ \\ \prod_v \epsilon(M_v \otimes N_v(\mu_v^{-1}), \psi_v), & \text{in the skew-hermitian case.} \end{cases}$$

Note that since $\chi_v(-1,1) = \chi_v(1,-1)$ for all v, the coherence condition for the collection of local quadratic spaces $\{V_v\}$ is the same as that for $\{W_v\}$. Thus, we will have a global pair of spaces $W \subset V$ with these localizations if and only if

$$\epsilon(\pi_\chi, R, 1/2) = 1.$$

Assuming this is the case, the second question is:

(2) Does the representation π_χ in the global Vogan packet Π_φ occur in the space $\mathcal{A}_0(G)$ of cusp forms?

To answer this question, we exploit the Langlands-Arthur conjecture discussed in §25. Thus, in the orthogonal, hermitian and skew-hermitian cases, we need to see if the distinguished character χ is trivial when restricted to the global component group A_φ via the diagonal map Δ. This amounts to the assertion that, for all $a \in A_\varphi$,

$$1 = \prod_v \chi_v(a) = \begin{cases} \prod_v \epsilon((M_v \otimes N_v)^a, \psi_v), & \text{in the orthogonal or hermitian cases;} \\ \\ \prod_v \epsilon(M_v \otimes N_v(\mu_v^{-1}))^a, \psi_v), & \text{in the skew-hermitian case,} \end{cases}$$

or equivalently, that

$$\epsilon(\pi_\chi, R^a, 1/2) = 1 \quad \text{for all } a \in A_\varphi.$$

On the other hand, in the symplectic case, using the multiplicity formula given in Conjecture 25.1, we see that we need the distinguished character to be equal on $A_\varphi = A_M \times A_N^+$ to the character $\chi_\varphi \times 1$ (assuming that M is symplectic and N orthogonal), where χ_φ is the character of A_M defined in Conjecture 25.1. This translates to the same condition

$$1 = \prod_v \epsilon((M_v \otimes N_v)^a, \psi_v) = \epsilon(\pi_\chi, R^a, 1/2).$$

Finally, assuming that $\epsilon(\pi_\chi, R^a, 1/2) = 1$ for all $a \in A_\varphi$, so that π_χ occurs in the space of cusp forms, we can ask:

(3) Does the linear form $F(\nu)$ have nonzero restriction to π_χ?

The point is that, when the above conditions on epsilon factors hold, there are no trivial reasons for the central critical L-value $L(\pi_\chi, R, 1/2)$ to vanish. Here then is the second form of our global conjecture:

Conjecture 26.1. *Let π_χ be the representation in the global Vogan packet Π_φ corresponding to the distinguished character χ. Then the following are equivalent:*

(i) π_χ *occurs with multiplicity one in* $\mathcal{A}_0(G)$ *and the linear form* $F(\nu)$ *is nonzero on* π_χ

(ii) $L(\pi_\chi, R, 1/2) \neq 0$.

27. The first derivative

We maintain the notation and setup of the previous section, so that $W_0 \subset V_0$ is a pair of spaces over E with quasi-split group $G_0 = G(V_0) \times G(W_0)$ over F. For a given discrete global L-parameter (φ, M, N) of G_0, we have a distinguished representation

$$\pi = \pi_\chi = \widehat{\otimes}_v \pi_v$$

in the global Vogan packet Π_φ, which is a representation of a restricted direct product

$$G_{\mathbb{A}} = \prod_{J_v} G_v(F_v)$$

and is the unique element in the packet such that

$$\mathrm{Hom}_{H_{\mathbb{A}}}(\pi \otimes \overline{\nu}, \mathbb{C}) \neq 0.$$

In this final section, we specialize to the orthogonal and hermitian cases (i.e. where $\epsilon = 1$) and assume that

$$\epsilon(\pi_0, R, 1/2) = -1 \quad \text{so that } L(\pi_0, R, 1/2) = 0,$$

where π_0 is the generic automorphic representation of $G_0(\mathbb{A})$ with parameter φ. In this case the group $G_{\mathbb{A}}$ does not arise from a pair of orthogonal or hermitian spaces $W \subset V$ over E. In Kudla's terminology, the local data $(W_v \subset V_v)$ is incoherent. Nevertheless, we can formulate a global conjecture in this case, provided that the following condition holds:

($*$) There is a non-empty set S of places of F, containing all archimedean primes, such that the groups $G_v(F_v)$ and $H_v(F_v)$ are *compact* for all places $v \in S$.

This condition has the following implications:

(i) $\dim W_0^{\perp} = 1$. Indeed, this follows from the fact that $H_v = N_v.G(W_v)$ and a nontrivial unipotent subgroup N_v cannot be compact. Hence, for all places v, we have

$$\dim V_v = \dim W_v + 1.$$

If we consider the orthogonal decomposition

$$V_0 = W_0 \oplus L$$

over E, then since $W_v \subset V_v$ is relevant for all v, we have

$$L_v = W_v^{\perp}.$$

Thus, though the collection $(W_v \subset V_v)$ is not coherent, the collection $(W_v^{\perp})$ is.

(ii) Any archimedean place v of F is real and the space V_v must be definite. In the hermitian case, we must have $E_v = \mathbb{C}$. Hence, in the number field case, F is totally real and, in the hermitian case, E is a CM field. Moreover, at all archimedean places v of F, the generic representation $\pi_{0,v}$ of $G_0(F_v)$ is in the discrete series, and π_v is a finite dimensional representation of the compact group

$$G_v(F_v) = \mathrm{SO}(n) \times \mathrm{SO}(n-1) \text{ or } \mathrm{U}(n) \times \mathrm{U}(n-1)$$

with a unique line fixed by $H(F_v) = \mathrm{SO}(n-1)$ or $\mathrm{U}(n-1)$.

(iii) At finite primes $v \in S$, we must have $\dim(V_v) \leq 4$. Indeed, a quadratic form of rank ≥ 5 over F_v represents 0. Hence, for function fields F, we have the following nontrivial cases:

$$(\dim V_v, \dim W_v) = \begin{cases} (3,2) \text{ or } (4,3) \text{ in the orthogonal case;} \\ (2,1) \text{ in the unitary case.} \end{cases}$$

For simplicity, we will assume that F is a totally real number field and S consists only of the archimedean places. In the hermitian case, the quadratic extension E of F is a CM field.

Suppose first that the spaces V_v are orthogonal of dimension $n \geq 3$. Fix a real place α. If V_α has signature $(n, 0)$, let

$$W_\alpha^* \subset V_\alpha^*$$

be the unique orthogonal spaces over $F_\alpha = \mathbb{R}$ with signatures

$$(n-3, 2) \subset (n-2, 2).$$

If V_α has signature $(0, n)$, let

$$W_\alpha^* \subset V_\alpha^*$$

have signatures

$$(2, n-3) \subset (2, n-2).$$

Since we have modified the Hasse-Witt invariant at a single place of F, and kept the discriminant of $W_\alpha^{*\perp} \simeq W_v^{\perp}$ equal to the discriminant of L_α, there are unique global spaces

$$W^\alpha \subset V^\alpha \quad \text{over } E$$

with localizations

$$\begin{cases} W_v \subset V_v \text{ for all } v \neq \alpha, \\ W_\alpha^* \subset V_\alpha^* \text{ at } \alpha. \end{cases}$$

We note that we can make such a pair of global spaces for *any* place α of F, having localizations $W_v \subset V_v$ for all $v \neq \alpha$, provided that $\dim W_\alpha \geq 3$. When $\dim W_\alpha = 2$, we can make such a global space provided that W_α is *not split* over F_α, i.e. for all primes α which are ramified or inert in the splitting field E of the 2-dimensional space W^0. The proof is similar to [Se, Prop 7].

We can use the global spaces $W^\alpha \subset V^\alpha$ so constructed to define the groups

$$H^\alpha \hookrightarrow G^\alpha = \mathrm{SO}(V^\alpha) \times \mathrm{SO}(W^\alpha)$$

over F. These have associated Shimura varieties

$$\Sigma(H^\alpha) \hookrightarrow \Sigma(G^\alpha)$$

over $\mathbb{C}$, of dimensions $n-3$ and $2n-5$ respectively, which are defined over the reflex field $E = F$, embedded in $\mathbb{C}$ via the place α. The varieties over F are independent of the choice of the real place α, so we denote them simply by

$$\Sigma(H) \hookrightarrow \Sigma(G),$$

suppressing the mention of α.

Next, suppose that the spaces V_v are hermitian over E_v of dimension $n \geq 2$. Fix a real place α and a complex embedding $z : E_\alpha \to \mathbb{C}$. If V_α has signature $(n, 0)$, let

$$W_\alpha^* \subset V_\alpha^*$$

be the unique hermitian spaces over E_α with signature

$$(n-2, 1) \subset (n-1, 1).$$

If V_α has signature $(0, n)$, let

$$W_\alpha^* \subset V_\alpha^*$$

be the unique hermitian spaces over E_α with signatures

$$(1, n-2) \subset (1, n-1).$$

Again, since we have modified the hermitian discriminants at a single place α of F, and kept $(W_\alpha^*)^\perp \simeq W_v^\perp$ constant, there is a unique pair of global spaces

$$W^\alpha \subset V^\alpha$$

over E with localizations

$$\begin{cases} W_v \subset V_v, \text{ for all } v \neq \alpha; \\ W_\alpha^* \subset {V_\alpha}^* \text{ at } \alpha. \end{cases}$$

Again, we can make such a modification at any place α of F which is not split in the quadratic extension E.

As before, we use the global spaces $W^\alpha \subset V^\alpha$ to define groups

$$H^\alpha \hookrightarrow G^\alpha = \mathrm{U}(V^\alpha) \times \mathrm{U}(W^\alpha)$$

over F. These have associated Shimura varieties

$$\Sigma(H^\alpha) \hookrightarrow \Sigma(G^\alpha)$$

over $\mathbb{C}$, of dimensions $(n-2)$ and $(2n-3)$ respectively, which are defined over the reflex field $= E$, embedded in $\mathbb{C}$ via the extension z of the place α. These varieties over E are independent of the choice of real place α of F, so we denote them simply by:

$$\Sigma(H) \hookrightarrow \Sigma(G).$$

We sketch the definition of the Shimura variety Σ of dimension $n-1$ associated to incoherent hermitian data $\{W_v\}$ of dimension n which is definite at all real places v of F; the orthogonal case is similar. Take the modified space W^α at a real place α, and let $G = Res_{F/\mathbb{Q}}\mathrm{U}(W^\alpha)$. We define a homomorphism

$$h : S_{\mathbb{R}} = Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}} = \prod_{v|\infty} \mathrm{U}(W_v^\alpha)$$

as follows. Let $\langle e_1, \cdots, e_n \rangle$ be an orthogonal basis of W_α^*, such that the definite space $e_1^\perp$ has the same sign as the definite space W_α. We set

$$h(z) = \begin{pmatrix} z/\overline{z} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \quad \text{in } \mathrm{U}(W_\alpha^*)$$

and

$$h(z) = 1 \text{ in all the other (compact) components } \prod_{v \neq \alpha} \mathrm{U}(W_v).$$

Let X be the $G_{\mathbb{R}}$-conjugacy class of h, which is isomorphic to the unit ball $\mathrm{U}_{n-1,1}(\mathbb{R})/[\mathrm{U}_{n-1}(\mathbb{R}) \times \mathrm{U}_1(\mathbb{R})]$ in $\mathbb{C}^{n-1}$. The pair (G, X) satisfies the axioms for a Shimura variety [De3, §2.1]. The composite homomorphism

$$w : (\mathbb{G}_m)_{\mathbb{R}} \to S_{\mathbb{R}} \to G_{\mathbb{R}}$$

is trivial, and the reflex field of $\Sigma(G) = M(G, X)$ is equal to E, embedded in $\mathbb{C}$ via the homomorphism z extending α. Indeed, the miniscule co-character $\mu : (\mathbb{G}_m)_{\mathbb{C}} \to G_{\mathbb{C}}$ is defined over E:

$$\mu(\alpha) = \begin{pmatrix} \alpha & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \times 1.$$

The complex points of $\Sigma(G)$ are:

$$\Sigma(G, \mathbb{C}) = G(\mathbb{Q}) \backslash [X \times G(\widehat{\mathbb{Q}})].$$

Over E, the variety $\Sigma(G)$ and the action of $G(\widehat{\mathbb{Q}}) = \prod_{v \text{ finite}} \mathrm{U}(W_v)$ on it depends only on the incoherent family $\{W_v\}$. If

$$\pi_\infty = \otimes_{v \text{ real}} \quad \pi_v$$

is any finite dimensional representation of the compact group $\prod_{v \text{ real}} \mathrm{U}(W_v)$, there is a local system $\mathcal{F}$ on $\Sigma(G)$ over E associated to π_∞.

We now return to the study of the L-function $L(\pi_0, R, s)$ at $s = 1/2$, using the arithmetic geometry of the cycle

$$\Sigma(H) \hookrightarrow \Sigma(G)$$

associated to the incoherent family $(W_v \subset V_v)$. The representation

$$\pi_\infty = \otimes_{v \text{ real}} \quad \pi_v \quad \text{of} \quad \prod_{v \text{ real}} G_v(F_v)$$

gives a local system $\mathcal{F}$ on $\Sigma(G)$ which contains the trivial local system $\mathbb{C}$ when restricted to the cycle $\Sigma(H)$.

To get the appropriate representation $\pi_f = \widehat{\otimes}_{v \text{ finite}} \pi_v$ of $G(\widehat{\mathbb{Q}})$ on the Chow group of $\Sigma(G)$ with coefficients in $\mathcal{F}$, we need to find this representation in the middle dimensional cohomology of $\Sigma(G)$ with coefficients in $\mathcal{F}$ (which is the "tangent space" of the Chow group). Hence we need

$$\mathrm{Hom}_{G(\widehat{\mathbb{Q}})}(\pi_f, H^d(\Sigma(G), \mathcal{F})) \neq 0$$

with $d = \dim \Sigma(G)$. We put

$$n = \begin{cases} \dim V^\alpha, & \text{in the hermitian case;} \\ \dim V^\alpha - 1, & \text{in the orthogonal case,} \end{cases}$$

so that $n \geq 2$ in all cases. We have

$$\begin{cases} \dim \Sigma(G) = d = 2n - 3 \\ \dim \Sigma(H) = n - 2 \\ \text{codim } \Sigma(H) = n - 1. \end{cases}$$

Now Matsushima's formula for cohomology shows that

$$\dim \mathrm{Hom}_{G(\widehat{\mathbb{Q}})}(\pi_f, H^d(\Sigma(G), \mathcal{F}))$$

is equal to the sum of multiplicities in the cuspidal spectrum

$$\sum_{\pi^\alpha} m(\pi^\alpha \otimes (\otimes_{\beta \neq \alpha \text{ real}} \pi_\beta) \otimes \pi_f) \cdot (G(\mathbb{R}_\alpha) : G(\mathbb{R}_\alpha)^0)$$

over the discrete series representations π^α of $G(\mathbb{R}_\alpha)$ with the same infinitesimal and central character as π_α. If all of the multiplicities are 1, the middle cohomology of $\Sigma(G)$ with coefficients in $\mathcal{F}$ will contain the motive $M \otimes N$ over F or E, associated to the parameter of the L-packet of π_0.

On the other hand, the conjecture of Birch and Swinnerton-Dyer, as extended by Bloch and Beilinson, predicts that

$$\dim \mathrm{Hom}_{G(\widehat{\mathbb{Q}})}(\pi_f, CH^{n-1}(\Sigma(G), \mathcal{F}))$$

is equal to the order of vanishing of the L-function

$$L(\pi_0, R, s)$$

at the central critical point $s = 1/2$. If the first derivative is nonzero, we should have an embedding, unique up to scaling

$$\pi_f \hookrightarrow CH^{n-1}(\Sigma(G), \mathcal{F})$$

and the Chow group of codim$(n-1)$ cycles plays the role of the space of automorphic forms in §26.

The height pairing against the codimension $(n-1)$ cycle $\Sigma(H)$, on which $\mathcal{F}$ has a unique trivial system, should give a nonzero linear form

$$F : CH^{n-1}(\Sigma(G), \mathcal{F}) \to \mathbb{C}$$

analogous to the integration of automorphic forms over $H(F)\backslash H(\mathbb{A})$. This form is $H(\widehat{\mathbb{Q}})$-invariant, and our global conjecture in this setting is:

Conjecture 27.1. *The following are equivalent:*

(i) The representation π_f occurs in $CH^{n-1}(\Sigma(G), \mathcal{F})$ with multiplicity one and the linear form F is nonzero on π_f;

(ii) $L'(\pi_0, R, 1/2) \neq 0$.

Remark : Just as the cohomology of a pro-Shimura variety associated to a reductive group G over $\mathbb{Q}$ carries an admissible, automorphic action of $G(\widehat{\mathbb{Q}})$, it is reasonable to expect that the Chow groups of cycles defined over the reflex field E will also be admissible and automorphic. We note that this is true for the Shimura curves associated to inner forms G of $\mathrm{GL}_2(\mathbb{Q})$: the action of $G(\widehat{\mathbb{Q}})$ on the Chow group of zero cycles of degree 0 is the Hecke action on the Mordell-Weil group of the Jacobian over $\mathbb{Q}$, which factors through the action of endomorphisms on the differential forms. Here the multiplicity of a representation π_f of $G(\widehat{\mathbb{Q}})$ on the Chow group in the tower is conjecturally equal to the order of zero of the standard L-function associated to π_f at the central critical point.

As in the global conjecture in central value case, one expects a refinement of the above conjecture, in the form of an exact formula relating the pairing

$$\langle \Sigma(H)(\pi_f), \Sigma(H)(\pi_f) \rangle$$

to the first derivative $L'(\pi_0, R, 1/2)$. This would generalize the formula of Gross-Zagier [GZ], as completed by Yuan-Zhang-Zhang [YZZ], which is the case $n = 2$ where the codimension of the cycle is 1. Such a refined formula in higher dimensions has been proposed in a recent preprint of W. Zhang [Zh].

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