STABLE PRINCIPAL BUNDLES AND REDUCTION OF STRUCTURE GROUP

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Abstract. Let E_G be a stable principal G–bundle over a compact connected Kähler manifold, where G is a connected reductive linear algebraic group defined over $\mathbb{C}$. Let $H \subset G$ be a complex reductive subgroup which is not necessarily connected, and let $E_H \subset E_G$ be a holomorphic reduction of structure group. We prove that E_H is preserved by the Einstein–Hermitian connection on E_G. Using this we show that if E_H is a minimal reductive reduction in the sense that there is no complex reductive proper subgroup of H to which E_H admits a holomorphic reduction of structure group, then E_H is unique in the following sense: For any other minimal reductive reduction $(H', E_{H'})$ of E_G, there is some element $g \in G$ such that $H' = g^{-1}Hg$ and $E_{H'} = E_H g$. As an application, we give an affirmative answer to a question posed in [BK].

1. Introduction

In [BK], the notion of algebraic holonomy for a stable vector bundle defined over a normal complex projective variety is introduced. Let X be a simply connected smooth complex projective variety equipped with a Kähler–Einstein metric. In [BK, Question 9] it is asked whether the algebraic holonomy of TM coincides with the complexification of the differential geometric holonomy of M. The present work started by trying to answer it. We prove that for any stable vector bundle E defined over a complex projective manifold, the algebraic holonomy coincides with the complexification of the differential geometric holonomy of the Einstein–Hermitian connection on E. Since a Kähler–Einstein connection is also an Einstein–Hermitian connection, this answers the above question affirmatively.

Let M be a compact connected Kähler manifold equipped with a Kähler form. Let G be a connected reductive linear algebraic group defined over $\mathbb{C}$. It is known that any stable principal G–bundle over M admits a unique Einstein–Hermitian connection.

Our main result is the following (Theorem 2.3):

Theorem 1.1. *Let E_G be a stable G–bundle over M. Let H be a complex reductive subgroup of G which is not necessarily connected, and let $E_H \subset E_G$ be a holomorphic reduction of structure group of E_G to H. Then the Einstein–Hermitian connection on E_G is induced by a connection on E_H.*

Let H be a complex reductive subgroup of G which is not necessarily connected, and let $E_H \subset E_G$ be a holomorphic reduction of structure group to H of a stable G–bundle E_G defined over M. Assume that there is no complex reductive proper subgroup of H to which E_H admits a holomorphic reduction of structure group. Such reductions will be called the minimal reductive ones. Theorem 1.1 says that E_H is preserved by the Einstein–Hermitian connection on E_G.

Fix a point $x_0 \in M$, and also choose a point in the fiber $z \in (E_G)_z$. Taking parallel translations, for the Einstein–Hermitian connection on E_G, of z along piecewise smooth paths in M based at x_0 we get a subset of E_G. The topological closure, in E_G, of this subset gives a smooth reduction of structure group of E_G to a compact subgroup $\overline{K}_E \subset G$. The corresponding smooth reduction of structure group $E^{z_0}_{\overline{K}^{\mathbb{C}}_E}$ of E_G to the Zariski closure $\overline{K}^{\mathbb{C}}_E$ of $\overline{K}_E$ in G is actually holomorphic. We prove the following (see Theorem 3.1):

Theorem 1.2. *There is a point $z_0 \in (E_G)_z$ such that the above minimal reductive reduction E_H coincides with $E^{z_0}_{\overline{K}^{\mathbb{C}}_E}$. In particular, H coincides with $\overline{K}^{\mathbb{C}}_E$ for such a base point z_0.*

Consequently, if $(H\,, E_H)$ and $(H'\,, E_{H'})$ are two minimal reductive reductions of E_G, then there is an element $g \in G$ such that $H' = g^{-1}Hg$ and $E_{H'} = E_Hg$. This gives us the following corollary (Corollary 3.2):

Corollary 1.3. *Let E_G be a stable principal G–bundle over M. Then there is a unique holomorphic sub–fiber bundle $\mathcal{G}_{E_G}$ of the adjoint bundle* $\text{Ad}(E_G)$*, with fibers being subgroups, that satisfies the following condition: For any minimal reductive reduction E_H of E_G, the adjoint bundle* $\text{Ad}(E_H)$*, which is a sub–fiber bundle of* $\text{Ad}(E_G)$*, coincides with $\mathcal{G}_{E_G}$.*

The sub–fiber bundle $\mathcal{G}_{E_G} \subset \text{Ad}(E_G)$ in Corollary 1.3 is the complexification of the holonomy of the Einstein–Hermitian connection on E_G. While the stability condition does not depend on the choice of a Kähler form in a given Kähler class, the Einstein–Hermitian connection on a stable bundle depends on the choice of the Kähler form. We note that the sub–fiber bundle $\mathcal{G}_{E_G}$ does not depend either on the Kähler form or on the Kähler class as long as E_G remains stable. When M is a complex projective manifold, $\mathcal{G}_{E_G}$ coincides with the algebraic monodromy of E_G introduced in [BK].

2. Einstein–Hermitian connection and reduction of structure group

Let M be a compact connected Kähler manifold equipped with a Kähler form ω. The *degree* of a torsionfree coherent analytic sheaf V defined over a dense open subset $U \subset M$, whose complement $M \setminus U$ is an analytic subset of M of complex codimension at least two, is defined to be

$$\text{degree}(V) := \int_M c_1(\iota_* V)\omega^{\dim_{\mathbb{C}} M - 1} \in \mathbb{R}\,,$$

where $\iota : U \hookrightarrow M$ is the inclusion map; the codimension condition ensures that $\iota_* V$ is a coherent analytic sheaf.

Let G be a connected reductive linear algebraic group defined over $\mathbb{C}$. Since the degree has been defined, we have the notions of a stable G–bundle and a polystable G–bundle over M. See [RS], [AB], [Ra] for definitions of stable and polystable principal G–bundles.

Fix a maximal torus $T \subset G$ and a Borel subgroup $B \subset G$ containing T. We also fix a maximal compact subgroup K of G. By a parabolic subgroup of G we will mean one containing B. Therefore, the Levi quotient $L(Q)$ of any parabolic subgroup $Q \subset G$ is also a subgroup of Q. Let $Z_0(G)$ denote the connected component of the center of G that contains the identity element.

Take any polystable principal G–bundle E_G over M. Therefore, there is a Levi subgroup $L(P)$ associated to some parabolic subgroup $P \subset G$ (the subgroup P need not be proper) along with a holomorphic reduction of structure group $E_{L(P)} \subset E_G$ to $L(P)$, such that

- the principal $L(P)$–bundle $E_{L(P)}$ is stable, and
- the principal P–bundle $E_P := E_{L(P)}(P)$, obtained by extending the structure group of $E_{L(P)}$ using the inclusion $L(P)$ in P, is an admissible reduction of E_G.

Note that since $L(P) \subset P \subset G$, the P–bundle E_P is a reduction of structure group of E_G to P. The condition that E_P is an admissible reduction of structure group of E_G means that for each character χ of P, which is trivial on $Z_0(G)$, the associated line bundle $E_P(\chi)$ over M is of degree zero. The G–bundle E_G is stable if and only if $P = G$.

Since E_G and $E_{L(P)}$ are polystable, they admit unique Einstein–Hermitian connections [AB, p. 208, Theorem 0.1]. It should be clarified that the Einstein–Hermitian reduction of structure to a maximal compact subgroup depends on the choice of the maximal compact subgroup. Even for a fixed maximal compact subgroup, the Einstein–Hermitian reduction of structure group need not be unique. However, once the Kähler form on M is fixed, the Einstein–Hermitian connection on a polystable principal bundle over M is unique. Let $\nabla^{L(P)}$ be the Einstein–Hermitian connection on the stable $L(P)$–bundle $E_{L(P)}$. Let ∇^G be the connection on E_G induced by $\nabla^{L(P)}$. Note that a connection on $L(P)$ induces a connection on any fiber bundle associated to $E_{L(P)}$. The principal G–bundle E_G is associated to $E_{L(P)}$ for the left translation action of $L(P)$ on G, and ∇^G is the connection on it obtained from $\nabla^{L(P)}$.

Proposition 2.1. *The induced connection ∇^G on E_G coincides with the unique Einstein–Hermitian connection on E_G.*

Proof. Since the inclusion map $L(P) \hookrightarrow G$ need not take the connected component, containing the identity element, of the center of $L(P)$ into $Z_0(G)$, the proposition does not follow immediately from [AB, p. 208, Theorem 0.1]. The connection $\nabla^{L(P)}$ on $E_{L(P)}$ is induced by a connection on a smooth reduction of structure group of $E_{L(P)}$ to a maximal compact subgroup of $L(P)$ (this is a part of the definition of a Einstein–Hermitian connection). A maximal compact subgroup of $L(P)$ is contained in a maximal compact subgroup of G. Consequently, the connection ∇^G on E_G is induced by a connection on a smooth reduction of structure group of E_G to a maximal compact subgroup of G. Therefore, to prove that ∇^G is the Einstein–Hermitian connection on E_G it suffices to show that ∇^G satisfies the Einstein–Hermitian equation.

Let $\mathfrak{z}(\mathfrak{l}(P))$ be the Lie algebra of the center of $L(P)$. For any $\theta \in \mathfrak{z}(\mathfrak{l}(P))$, the holomorphic section of the adjoint bundle $\mathrm{ad}(E_{L(P)})$ given by θ will be denoted by $\widehat{\theta}$; the vector bundle $\mathrm{ad}(E_{L(P)})$ is associated to $E_{L(P)}$ for the adjoint action of $L(P)$ on its Lie algebra. Let Λ_ω be the adjoint of multiplication by ω of differential form on M. That the connection $\nabla^{L(P)}$ is Einstein–Hermitian means that there is an element $\theta \in \mathfrak{z}(\mathfrak{l}(P))$ such that the Einstein–Hermitian equation

$$\Lambda_\omega \mathcal{K}_{\nabla^{L(P)}} \, = \, \widehat{\theta} \tag{1}$$

holds.

If θ in eqn. (1) is in the Lie algebra $\mathfrak{z}(G)$ of $Z_0(G)$, then ∇^G is a Einstein–Hermitian connection on E_G. Therefore, in that case the proposition is proved. Assume that

$$\theta \, \notin \, \mathfrak{z}(G)\, . \tag{2}$$

Fix a character χ of P which is trivial on $Z_0(G)$ but satisfies the following condition: the homomorphism of Lie algebras

$$\mathrm{d}\chi \, : \, \mathfrak{p} \, \longrightarrow \, \mathbb{C} \tag{3}$$

given by χ, where $\mathfrak{p}$ is the Lie algebra of P, is nonzero on θ. (It is easy to check that the group of characters of P coincides with the group of characters of $L(P)$.)

Consider the holomorphic line bundle $L_\chi \, := \, E_P(\chi)$ over M associated to E_P for χ. Let ∇^χ be the connection on L_χ induced by the connection on E_P given by $\nabla^{L(P)}$. Since $\nabla^{L(P)}$ is a Einstein–Hermitian connection, the connection ∇^χ is also Einstein–Hermitian. Indeed, if $\mathcal{K}(\nabla^\chi)$ is the curvature of ∇^χ, then

$$\Lambda_\omega \mathcal{K}(\nabla^\chi) \, = \, \mathrm{d}\chi(\theta)\, ,$$

where $\mathrm{d}\chi$ is the homomorphism in eqn. (3) and θ is the element in eqn. (1). Therefore,

$$\mathrm{degree}(L_\chi) \, = \, \frac{\mathrm{d}\chi(\theta)\sqrt{-1}}{2\pi d} \int_M \omega^d\, , \tag{4}$$

where $d \, = \, \dim_{\mathbb{C}} M$; see [Ko, p. 103, Proposition 2.1].

The condition that $E_P \subset E_G$ is an admissible reduction of structure group says that

$$\mathrm{degree}(L_\chi) \, = \, 0\, .$$

This, in view of the assumption in eqn. (2) that $\mathrm{d}\chi(\theta) \, \neq \, 0$, contradicts eqn. (4). Therefore, we conclude that $\theta \in \mathfrak{z}(G)$. This immediately implies that the connection ∇^G on E_G is Einstein–Hermitian. This completes the proof of the proposition. □

Let E_G be a holomorphic principal G–bundle over M and ∇' a C^∞ connection on E_G compatible with the holomorphic structure of E_G. Such a connection is called a complex connection; see [AB, p. 230, Definition 3.1(1)] for the precise definition of a complex connection. A connection $\widetilde{\nabla}$ is complex if and only if the $(0\,,2)$–Hodge type component of the curvature of $\widetilde{\nabla}$ vanishes.

Definition 2.2. Let H be a closed complex subgroup of G and $E_H \subset E_G$ a C^∞ reduction of structure group of E_G to H. We will say that E_H is *preserved* by the connection ∇' if ∇' induces a connection on E_H.

It follows immediately that E_H is preserved by ∇' if and only if there is a smooth connection on E_H that induces the connection ∇' on E_G. It is easy to see that E_H is preserved by ∇' if and only if for each point $z \in E_H$, the horizontal subspace in T_zE_G for the connection ∇' is contained in T_zE_H. If E_H is preserved by ∇', then E_H is a holomorphic reduction of structure group of E_G.

Theorem 2.3. *Let E_G be a stable principal G–bundle over M and ∇^G the Einstein–Hermitian connection on E_G. Let H be a complex reductive subgroup of G which is not necessarily connected, and let $E_H \subset E_G$ be a holomorphic reduction of structure group of E_G to H. Then E_H is preserved by the connection ∇^G.*

Proof. Let $H_0 \subset H$ be the connected component containing the identity element. Set

$$X := E_H/H_0\,,$$

which is finite étale Galois cover of M with Galois group H/H_0. Let

$$p : X \longrightarrow M \tag{5}$$

be the projection. We note that p^*E_H has a canonical reduction of structure group to the subgroup $H_0 \subset H$. Set

$$F_G := p^*E_G\,.$$

Let $F_{H_0} \subset F_G$ be the reduction of structure group to H_0 obtained from the canonical reduction of structure group of p^*E_H to H_0.

Equip X with the Kähler form $p^*\omega$. The Einstein–Hermitian connection on E_G pulls back to a Einstein–Hermitian connection on F_G for the Kähler form $p^*\omega$. Therefore, F_G is polystable with respect to $p^*\omega$. Let $\mathrm{ad}(F_G)$ be the adjoint bundle over X. We recall that $\mathrm{ad}(F_G)$ is associated to F_G for the adjoint action of G on its Lie algebra. The connection on the vector bundle $\mathrm{ad}(F_G)$ induced by the Einstein–Hermitian connection of F_G is clearly Einstein–Hermitian. Hence the adjoint vector bundle $\mathrm{ad}(F_G)$ is polystable.

Let $Z(G)$ be the center of G. Set

$$H' := H_0/(H_0 \cap Z(G))\,.$$

Therefore, H' is a complex reductive subgroup of the complex semisimple group $G' := G/Z(G)$. Let $\mathfrak{g}$ (respectively, $\mathfrak{h}_0$) be the Lie algebra of G (respectively, H_0). Note that the adjoint action makes $\mathfrak{g}$ (respectively, $\mathfrak{h}_0$) a G'–module (respectively, H'–module). We will also consider $\mathfrak{g}$ as a H'–module using the inclusion of H' in G'.

Since $\mathfrak{g}$ is a faithful G'–module, and H' is a reductive subgroup of G', there is a positive integer N and nonnegative integers a_i, b_i, $i \in [1\,,N]$, such that the H'–module $\mathfrak{h}_0$ is a direct summand of the H'–module

$$\bigoplus_{i=1}^{N} \mathfrak{g}^{\otimes a_i} \otimes (\mathfrak{g}^*)^{\otimes b_i} \;=\; \bigoplus_{i=1}^{N} \mathfrak{g}^{\otimes a_i} \otimes \mathfrak{g}^{\otimes b_i} \tag{6}$$

[De, p. 40, Proposition 3.1]; since H' is complex reductive, any exact sequence of H'–modules splits; also, $\mathfrak{g} = \mathfrak{g}^*$ as G is reductive. Therefore, the adjoint vector bundle $\mathrm{ad}(F_{H_0})$ is a direct summand of the vector bundle

$$\bigoplus_{i=1}^{N} \mathrm{ad}(F_G)^{\otimes a_i} \otimes \mathrm{ad}(F_G)^{\otimes b_i}\,. \tag{7}$$

We note that the vector bundle in eqn. (7) is associated to F_{H_0} for the H_0–module in eqn. (6).

Since the adjoint vector bundle $\mathrm{ad}(F_G)$ is polystable of degree zero (recall that $\mathrm{ad}(F_G) = \mathrm{ad}(F_G)^*$), the vector bundle

$$\mathrm{ad}(F_G)^{\otimes a} \otimes \mathrm{ad}(F_G)^{\otimes b}$$

is polystable of degree zero for all $a, b \in \mathbb{N}$ [AB, p. 224, Theorem 3.9]. Therefore, the vector bundle in eqn. (7) is polystable of degree zero. Since $\mathrm{ad}(F_{H_0})$ is a direct summand of it of degree zero, we conclude that $\mathrm{ad}(F_{H_0})$ is also polystable. Consequently, the principal H_0–bundle F_{H_0} is polystable [AB, p. 224, Corollary 3.8]. Hence F_{H_0} admits a unique Einstein–Hermitian connection [AB, p. 208, Theorem 0.1]. Let ∇^{H_0} be the Einstein–Hermitian connection on F_{H_0}. So, there is an element $\nu \in \mathfrak{h}$ such that

$$\Lambda_{p^*\omega}\mathcal{K}(\nabla^{H_0}) \;=\; \widehat{\nu}\,, \tag{8}$$

where $\mathcal{K}(\nabla^{H_0})$ is the curvature of ∇^{H_0}, and $\widehat{\nu}$ is the holomorphic section of $\mathrm{ad}(p^*E_H) = \mathrm{ad}(F_{H_0})$ given by ν.

Since H/H_0 is a finite group, giving a connection on a principal H_0–bundle is equivalent to giving a connection on the principal H–bundle obtained from it by extension of structure group. From the uniqueness of the Einstein–Hermitian connection on F_{H_0} it follows that the corresponding connection on p^*E_H is left invariant by the action of the Galois group H/H_0. Consequently, the connection on p^*E_H given by ∇^{H_0} descends to a connection on E_H. Let ∇^H denote the connection on E_H obtained this way. It is clear that ∇^H is a complex connection.

Let ∇ be the complex connection on E_G induced by the above connection ∇^H on E_H. We will first show that ∇ is unitary, which means that ∇ is induced by connection on a smooth reduction of structure group of E_G to a maximal compact subgroup of G. Then we will show that ∇ satisfies the Einstein–Hermitian equation.

To prove that ∇ is unitary, fix a point $x_0 \in M$, and also fix a point $z_0 \in (E_G)_{x_0}$ in the fiber of E_G over x_0. Taking parallel translations of z_0, with respect to the connection ∇, along piecewise smooth paths in M based at x_0 we get a subset $\mathcal{S}$ of E_G. Sending any $g \in G$ to the point $z_0 g \in (E_G)_{x_0}$ we get an isomorphism $G \longrightarrow (E_G)_{x_0}$. Using this isomorphism, the intersection $(E_G)_{x_0} \bigcap \mathcal{S}$ is a subgroup K_E of G. The condition that the connection ∇ is unitary is equivalent to the condition that K_E is contained in some compact subgroup of G.

Fix a point $x \in p^{-1}(x_0)$, and also fix a point $z \in (p^*E_G)_x$ that projects to z, where p is the covering map in eqn. (5). Consider parallel translations of z, with respect to the connection $p^*\nabla$ on $p^*E_G =: F_G$, along piecewise smooth paths in X based at x. As before, we get a subgroup $K_F \subset G$ from the resulting subset of F_G. It is easy to see that K_F is a finite index subgroup of the group K_E constructed above.

We note that the connection $p^*\nabla$ on F_G is induced by the unitary connection ∇^{H_0} on the reduction F_{H_0} of F_G. Therefore, the connection $p^*\nabla$ is unitary. Consequently, the subgroup $K_F \subset G$ is contained in a compact subgroup of G. Using this together with the observation that K_F is a finite index subgroup of the subgroup $K_E \subset G$ we conclude that K_E is also contained in a compact subgroup of G. Thus the connection ∇ is unitary.

We will now show that ∇ satisfies the Einstein–Hermitian equation.

Let $\mathcal{K}(\nabla)$ be the curvature of ∇. From eqn. (8) it follows immediately that $\Lambda_{p^*\omega}\mathcal{K}(\nabla^{H_0})$ is a holomorphic section of $\text{ad}(p^*E_H)$. Consequently, $\Lambda_\omega\mathcal{K}(\nabla)$ is a holomorphic section of the adjoint vector bundle $\text{ad}(E_G)$. Since E_G is stable, all holomorphic sections of $\text{ad}(E_G)$ are given by the Lie algebra $\mathfrak{z}(G)$ of $Z_0(G)$ [BG, Proposition 3.3]. In other words, there is an element $\theta \in \mathfrak{z}(G)$ such that $\Lambda_\omega\mathcal{K}(\nabla)$ coincides with the section of $\text{ad}(E_G)$ given by θ. Thus, we conclude that the connection ∇ is the unique Einstein–Hermitian connection on E_G. Since ∇ is induced by a connection on E_H, the proof of the theorem is complete. $\square$

Proposition 2.1 and Theorem 2.3 together have the following corollary.

Corollary 2.4. *Let E_G be a polystable G–bundle over M. Take $E_{L(P)}$ as in Proposition 2.1. Let H be a complex reductive subgroup of $L(P)$ (not necessarily connected), and let $E_H \subset E_{L(P)}$ be a holomorphic reduction of structure group of $E_{L(P)}$ to H. Then E_H is preserved by the Einstein–Hermitian connection on E_G.*

3. Properties of a smallest reduction

Let E_G be a stable principal G–bundle over M. Fix a point $x_0 \in M$, and also fix a point $z_0 \in (E_G)_{x_0}$ in the fiber over x_0. Let ∇^G be the Einstein–Hermitian connection on E_G.

Taking parallel translations of z_0, with respect to ∇^G, along piecewise smooth paths in M based at x_0 we get a subset $\mathcal{S}$ of E_G. Let $\overline{\mathcal{S}}$ be the topological closure of $\mathcal{S}$ in E_G. The map $G \longrightarrow (E_G)_{x_0}$ defined by $g \longmapsto z_0 g$ is an isomorphism. Using this isomorphism, the intersection $(E_G)_{x_0} \bigcap \overline{\mathcal{S}}$ gives a compact subgroup $\overline{K}_E \subset G$. The subset $\overline{\mathcal{S}} \subset E_G$ is a C^∞ reduction of structure group of E_G to the subgroup $\overline{K}_E$. Let $\overline{K}^{\mathbb{C}}_E$ be the complex reductive subgroup of G obtained by taking the Zariski closure of $\overline{K}_E$ in G. The subset

$$E^{z_0}_{\overline{K}^{\mathbb{C}}_E} := \mathcal{S}\overline{K}^{\mathbb{C}}_E \subset E_G \tag{9}$$

is a holomorphic reduction of structure group of E_G to $\overline{K}^{\mathbb{C}}_E$; see [Bi, Section 3] for the details. It is easy to see that the principal $\overline{K}^{\mathbb{C}}_E$–bundle $E^{z_0}_{\overline{K}^{\mathbb{C}}_E}$ is the extension of structure group of the principal $\overline{K}_E$–bundle $\overline{\mathcal{S}}$.

We note that in [Bi], the point z_0 is taken to be in the subset of E_G given by an Einstein–Hermitian reduction of structure group (see [Bi, p. 71, (3.20)]). This was only to make $\overline{K}_E$ lie inside a fixed maximal compact subgroup of G. If we replace the base point z_0 by z_0g, where g is any point of G, then it is easy to see that the subset $\mathcal{S}$ constructed above using z_0 gets replaced by $\mathcal{S}g$. Therefore, the subgroup $\overline{K}^{\mathbb{C}}_E$ in eqn. (9) gets replaced by $g^{-1}\overline{K}^{\mathbb{C}}_E g$, and the reduction $E^{z_0}_{\overline{K}^{\mathbb{C}}_E} \subset E_G$ in eqn. (9) gets replaced by $E^{z_0}_{\overline{K}^{\mathbb{C}}_E} g$.

We note that $(\overline{K}^{\mathbb{C}}_E, E^{z_0}_{\overline{K}^{\mathbb{C}}_E})$ in eqn. (9) is a minimal complex reduction of E_G in the following sense: There is no complex proper subgroup of $\overline{K}^{\mathbb{C}}_E$ to which $E^{z_0}_{\overline{K}^{\mathbb{C}}_E}$ admits a holomorphic reduction of structure group which is preserved by the connection ∇^G. Indeed, this follows immediately from the construction of $E^{z_0}_{\overline{K}^{\mathbb{C}}_E}$. Furthermore, if $E_{H'} \subset E_G$ is a holomorphic reduction of structure group of E_G, to a complex subgroup $H' \subset G$, satisfying the two conditions:

- $E_{H'}$ is preserved by ∇^G, and
- there is no complex proper subgroup of H' to which $E_{H'}$ admits a holomorphic reduction of structure group which is also preserved by ∇^G,

then it follows immediately that there is a point $z_0 \in (E_G)_{x_0}$ such that $E_{H'} = E^{z_0}_{\overline{K}^{\mathbb{C}}_E}$. Indeed, z_0 can be taken to be any point of the fiber $(E_{H'})_{x_0}$.

Let H be a complex reductive subgroup of G which is not necessarily connected, and let $E_H \subset E_G$ be a holomorphic reduction of structure group to H of the stable G–bundle E_G. This reduction E_H will be called a *minimal reductive reduction* of E_G if there is no complex reductive proper subgroup of H to which E_H admits a holomorphic reduction of structure group.

In view of the above observations, using Theorem 2.3 get the following theorem:

Theorem 3.1. *Let E_G be a stable principal G–bundle over a compact connected Kähler manifold M, where G is a connected reductive complex linear algebraic group. Let $E_H \subset E_G$ be a minimal reductive reduction of E_G. Fix a point $x_0 \in M$. Then there is a point z_0 in the fiber $(E_G)_{x_0}$ such that $E^{z_0}_{\overline{K}^{\mathbb{C}}_E}$ (defined in eqn. (9)) coincides with E_H. In particular, the subgroup H coincides with $\overline{K}^{\mathbb{C}}_E$ for such a base point z_0.*

Let $\mathrm{Ad}(E_G)$ be the adjoint bundle for E_G. So $\mathrm{Ad}(E_G)$ is the fiber bundle over M associated to E_G for the adjoint action of G on itself. The fibers of $\mathrm{Ad}(E_G)$ are groups isomorphic to G. Let $\mathrm{Ad}(E^{z_0}_{\overline{K}^{\mathbb{C}}_E})$ be the adjoint bundle of the principal $\overline{K}^{\mathbb{C}}_E$–bundle $E^{z_0}_{\overline{K}^{\mathbb{C}}_E}$ defined in eqn. (9). Since $E^{z_0}_{\overline{K}^{\mathbb{C}}_E}$ is a holomorphic reduction of structure group of E_G, the fiber bundle $\mathrm{Ad}(E^{z_0}_{\overline{K}^{\mathbb{C}}_E})$ is a holomorphic sub–fiber bundle of $\mathrm{Ad}(E_G)$ with the fibers of $\mathrm{Ad}(E^{z_0}_{\overline{K}^{\mathbb{C}}_E})$ being subgroups of the fibers of $\mathrm{Ad}(E_G)$.

We noted earlier that if z_0 is replaced by $z_0 g$, where $g \in G$, then the subgroup $\overline{K}^{\mathbb{C}}_E$ gets replaced by $g^{-1}\overline{K}^{\mathbb{C}}_E g$, and $E^{z_0}_{\overline{K}^{\mathbb{C}}_E}$ gets replaced by $E^{z_0}_{\overline{K}^{\mathbb{C}}_E} g$. From this it follows immediately that the subbundle

$$\mathrm{Ad}(E^{z_0}_{\overline{K}^{\mathbb{C}}_E}) \subset \mathrm{Ad}(E_G)$$

is independent of the choice of the base point z_0.

Therefore, from Theorem 3.1 we have the following corollary:

Corollary 3.2. *Let E_G be a stable principal G–bundle over a compact connected Kähler manifold M, where G is a connected reductive complex linear algebraic group. Then there is a unique holomorphic sub–fiber bundle $\mathcal{G}_{E_G}$ of the adjoint bundle* $\mathrm{Ad}(E_G)$*, with fibers being subgroups, that satisfies the following condition: For any minimal reductive reduction E_H of E_G, the adjoint bundle* $\mathrm{Ad}(E_H)$*, which is a sub–fiber bundle of* $\mathrm{Ad}(E_G)$*, coincides with $\mathcal{G}_{E_G}$.*

From Theorem 3.1 it follows that the sub–fiber bundle $\mathcal{G}_{E_G} \subset \mathrm{Ad}(E_G)$ in Corollary 3.2 is the complexification of the holonomy of the Einstein–Hermitian connection on E_G. We note that for defining stable bundles we need only the class $[\omega] \in H^2(M, \mathbb{R})$. In other words, the stability condition does not depend on the choice of the Kähler form in a given Kähler class. On the other hand, the Einstein–Hermitian connection on a stable bundle depends on the choice of the Kähler form. From Corollary 3.2 it follows that the sub–fiber bundle $\mathcal{G}_{E_G}$ does not depend on the choice of the Kähler form in a given Kähler class. In fact, it does not even depend on the Kähler class as long as E_G remains stable.

Using Corollary 3.2 and [BK, Theorem 20] it follows that the algebraic monodromy of E_G coincides with the fiber $(\mathcal{G}_{E_G})_{x_0}$ when M is a complex projective manifold. (See [BK] for algebraic monodromy where it is introduced.)

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