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On the Self-dual Representations of a $\mathfrak{p}$-adic Group

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In an earlier paper [P1], we studied self-dual complex representations of a finite group of Lie type. In this paper, we make an analogous study in the p-adic case. We begin by recalling the main result of that paper.

Let $G(\mathbf{F})$ be the group of $\mathbf{F}$ rational points of a connected reductive algebraic group G over a finite field $\mathbf{F}$. Fix a Borel subgroup B of G, defined over $\mathbf{F}$, which always exists by Lang's theorem. Let $B = TU$ be a Levi decomposition of the Borel subgroup B. Suppose that there is an element $t_0 \in T(\mathbf{F})$ that operates by -1 on all the simple roots in U with respect to the maximal split torus in T. It can be seen that t_0^2 belongs to the center of G. We proved in [P1] that t_0^2 operates by 1 on a self-dual irreducible complex representation π, which has a Whittaker model, if and only if π carries a symmetric G-invariant bilinear form. If π carries a symmetric G-invariant bilinear form, then we call π an orthogonal representation.

If π is an irreducible admissible self-dual representation of a p-adic group G, then there exists a nondegenerate G-invariant bilinear form $B\colon\ \pi \times \pi \to \mathbb{C}$. This form is unique up to scalars by a simple application of Schur's lemma, and it is either symmetric or skew-symmetric. The aim of this paper is to provide for a criterion to decide which of the two possibilities holds in the context of p-adic groups similar to our work in the finite field case in [P1]. This partly answers a question raised by Serre (see [P1]). We are able to say nothing about representations not admitting a Whittaker model. Our method, however, works in some cases even when there is no element in T which operates on each simple root by -1. This is the case, for instance, in some cases of SL_n and Sp_n, where we work instead with the similitude group where such an element exists; then we deduce information on SL_n and Sp_n.

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We also provide some complements to [P1], which were motivated by a similar result for algebraic representations of reductive groups (see [St, Lemma 78]). The criterion in [P1] and [St] therefore give two elements in the center of G such that the action of one of them determines whether a self-dual algebraic representation of G(**F**) is orthogonal, whereas the other element determines whether a self-dual complex representation of G(**F**) is orthogonal. We prove that these two elements in the center of G are actually the same. This also gives us an opportunity to give a proof of the result for algebraic representations of G(**F**) following the methods of [P1]. Finally, we extend another observation of the author from the context of finite groups to p-adic groups, which gives a method of checking whether a representation is self-dual.

1 Orthogonality criterion for algebraic represention**s**

We recall that the theorem characterising self-dual representations of finite groups of Lie type in [P1] was proved using the following elementary result whose proof we sketch for the sake of completeness.

Lemma 1. Let H be a subgroup of a finite group G. Let s be an element of G which normalises H, and whose square belongs to the center of G. Let ψ: H $\to$ $\mathbf{C}^*$ be a 1-dimensional representation of H which is taken to its inverse by the inner conjugation action of s on H. Let π be an irreducible representation of G in which the character ψ appears with multiplicity 1. Then, if π is self-dual, it is orthogonal if and only if the element s^2 belonging to the center of G operates by 1 on π. □

Proof. Let v be a vector in π on which H operates via ψ. Clearly, H operates via the character ψ^{-1} on the vector sv, and the space spanned by v and sv is a nondegenerate subspace for the unique G-invariant bilinear form on π. It has dimension 1 or 2. The inner product of v with sv must be nonzero, but since the inner product is G-invariant, and in particular, s-invariant, the conclusion of the lemma follows. ■

Remark. In the context of G = G(**F**), the lemma is used with H = U(**F**) and ψ a *nondegenerate* character on U. The character ψ appears with multiplicity ≤ 1 by a theorem of Gelfand and Graev for G = GL_n(**F**), which was generalised by Steinberg to all reductive groups.

Lemma 2. Let G be a reductive algebraic group over an algebraically closed field k. Then there exists an element z_0 in the center of G of order ≤ 2, which operates on an irreducible, self-dual algebraic representation π of G by 1 if and only if the representation π is orthogonal. □

Proof. We offer two proofs: one is in the spirit of the proof of the earlier lemma, and the other is more classical.

Let $B = TU$ be a Levi decomposition of a Borel subgroup B in G. The choice of the Borel subgroup defines an ordering on the set of roots of G with respect to T. Let w_0 be an element in G representing the element in the Weyl group, which takes all the positive roots to negative roots. By the theory of highest weights, irreducible algebraic representations of G are in bijective correspondence with dominant integral weights. It follows that an irreducible representation π corresponding to dominant weight λ is self-dual if and only if $\lambda = -w_0(\lambda)$.

We apply Lemma 1, or rather its analogue for algebraic representations of algebraic groups, to the highest weight λ of T in π. Let v_0 be a vector in π on which T operates via λ. The vector v_0 is unique up to scalars. The square of the Weyl group element w_0, which takes all the positive roots to negative roots, is the identity element in the Weyl group of G; and in fact, we can choose a representative for w_0 in G such that w_0^2 belongs to the center of G. (The author thanks Hung Yean Loke for confirming this for the group E_6.) If $w_0 = -1$, then it is easy to see that for any choice of a representative x in G for $w_0 \in N(T)/T$, where N(T) is the normaliser of T, x^2 is independent of x, and belongs to the center of G. We let $w_0^2 = t_0$, where t_0 is an element in the center of G.

Clearly $w_0(\lambda) = -\lambda$ is also a weight of π. Therefore, by the complete reducibility of π as a T module, the subspace generated by v_0 and $w_0(v_0)$ is a nondegenerate subspace of π for the unique G-invariant bilinear form on π. We have

$$\langle v_0, w_0(v_0)\rangle = \langle w_0(v_0), w_0^2(v_0)\rangle = \langle w_0(v_0), t_0 v_0\rangle.$$

It follows that t_0 operates by 1 on an irreducible self-dual representation π if and only if π is an orthogonal representation.

We now turn to the second proof, which is the standard proof given, for example, in Bourbaki.

Let u_0 be a regular unipotent element in U (i.e., an element in U whose *component* in each U_α, α simple, is nontrivial). Let $j\colon SL_2 \to G$ be a *principal* SL_2 in G that takes the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ to u_0 and the diagonal torus in SL_2 to T. (The mapping j need not be injective.)

An irreducible representation π of G with highest weight λ, when restricted to SL_2 via the mapping j, is a sum of certain irreducible representations of SL_2. It can be seen that the representation of SL_2 with highest weight the restriction of λ to the diagonal torus in SL_2 appears with multiplicity 1. Therefore, if π is self-dual, then it is orthogonal or symplectic depending exactly on whether this representation of SL_2 is orthogonal or symplectic.

Let $k = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, where i is a fourth root of 1. The inner conjugation action of k on the unique positive simple root of SL_2 is by -1. This implies that the inner conjugation action of k on every simple root of G is by -1. This proves that $j(k^2)$ belongs to the center of G, and its action on a self-dual irreducible representation of G is by 1, if and only if the representation is orthogonal.

Since the two proofs must define the same element in the center of the group, we obtain the following corollary. (It needs to be noted that, under the assumption of Corollary 1 below, since $w_0 = -1$, all the algebraic representations of G are self-dual. In particular, for any element of order 2 in the center of G, there is a faithful self-dual representation of G nontrivial on that element.) ■

Corollary 1. Let G be a simple algebraic group with center Z for which $w_0 = -1$. For any choice of w_0 as an element in G, w_0^2 is an element in the center of G and is independent of the representative in G for w_o. Let T be a maximal torus in G, and let t_0 be an element in T which operates on all the simple roots of G by -1. Then t_0^2 is also in the center of G, and it is a well-defined element in Z/Z^2. The two elements w_0^2 and t_0^2, thought of as elements in Z/Z^2, are equal. □

The following corollary is clear from the second proof of the lemma above.

Corollary 2. The element in the center of G that determines whether a self-dual algebraic representation of G is orthogonal, is the same as the element in the center of G that determines when a self-dual generic complex representation of $G(\mathbf{F})$ is orthogonal. Both the elements in the center are considered up to squares in the center. □

2 Orthogonality criterion for p-adic groups

Let $G = G(k)$ be the group of k-rational points of a quasi-split reductive group over a local field k. Let $B = TU$ be a Levi decomposition of a Borel subgroup of G. Fix a nondegenerate character $\psi\colon U(k) \to \mathbf{C}^*$. By a theorem of Gelfand and Kazhdan for GL_n and Shalika [Sh] for general quasi-split reductive groups, there exists at most one dimensional space of linear forms $\ell\colon \pi \to \mathbf{C}$ on any irreducible admissible representation π of $G(k)$ that transforms via ψ when restricted to $U(k)$. Assume that there is an element $t_0 \in T(k)$ such that the inner conjugation action of t_0 is by -1 on all the simple roots in U. Therefore, the inner conjugation action of t_0 on U takes ψ to ψ^{-1}. Since the group $U(k)$ is noncompact, one cannot use Lemma 1 in this context. However, a very nice idea of Rodier enables us to work with a compact *approximation* of (U, ψ), which we briefly describe now. We refer the reader to Rodier's article [R] for details.

Let U^- denote the unipotent radical of the Borel subgroup B^- of G which is opposite to B. Fix an integral structure on G so that one can speak about $G(\mathfrak{p}^n)$, where $\mathfrak{p}$ is the maximal ideal in the maximal compact subring $\mathcal{O}$ in k generated by an element $\pi \in \mathfrak{p}$. If G is a quasi-split group over k which splits over an unramified extension of k, then G has a natural integral structure whose reduction modulo $\mathfrak{p}$ is reductive. However, for our purposes, all we need is for the groups $G(\mathfrak{p}^n)$ to have the Iwahori factorisation,

$$G(\mathfrak{p}^n) = U^-(\mathfrak{p}^n)T(\mathfrak{p}^n)U(\mathfrak{p}^n).$$

We assume that the character ψ on $U(k)$ has been so normalised that it is trivial on $\mathcal{O}$-rational points of every simple root, but not on $\pi^{-1}\mathcal{O}$-rational points of any simple root. Define a character ψ_n on $G(\mathfrak{p}^n)$ by

$$\psi_n(u^- t u) = \psi_n(u) = \psi(\pi^{-2n}u),$$

where we leave the task of making the meaning of $\pi^{-2n}u$ precise to the reader. Rodier proves the following result in [R]. Actually, Rodier works only with split groups, but his work extends easily to quasi-split groups.

Proposition 1. Let π be an irreducible admissible representation of $G(k)$. Then,

$$\dim \mathrm{Hom}_{G(\mathfrak{p}^n)}(\pi, \psi_n) \le \dim \mathrm{Hom}_{G(\mathfrak{p}^{n+1})}(\pi, \psi_{n+1}) \le \dim \mathrm{Hom}_U(\pi, \psi).$$

Moreover, $\dim \mathrm{Hom}_{G(\mathfrak{p}^n)}(\pi, \psi_n) = \dim \mathrm{Hom}_U(\pi, \psi)$ for all n large enough. Therefore, if π has a Whittaker model, then for some n, $\dim \mathrm{Hom}_{G(\mathfrak{p}^n)}(\pi, \psi_n) = 1$. □

Now the analogue of Lemma 1 can be applied to the subgroup $H = G(\mathfrak{p}^n)$ with the character ψ_n on it to deduce the following proposition. (We note that since the element $t_0 \in T(k)$ operates by -1 on all the simple roots of G, it preserves $G(\mathfrak{p}^n) = U^-(\mathfrak{p}^n)T(\mathfrak{p}^n)U(\mathfrak{p}^n)$ and takes the character ψ_n to ψ_n^{-1}.)

Proposition 2. Let G be a quasi-split reductive algebraic group over a p-adic field k with $B = TU$ a Levi decomposition of a Borel subgroup of G. Assume that $T(k)$ has an element t_0 which operates by -1 on all the simple roots in U. Then $z_0 = t_0^2$ belongs to the center of G and operates on an irreducible, self-dual generic representation π of G by 1 if and only if the representation π is orthogonal. □

3 Examples

In this section, we list some groups for which Proposition 2 applies to give a criterion for orthogonality of self-dual generic representations. As observed in [P1] for the case of finite fields, here is what the proposition says for the split groups.

(1) GL_n: In this case, all the self-dual generic representations are orthogonal.

(2) SL_n: If $n \not\equiv 2 \bmod 4$, then all the self-dual generic representations are orthogonal; if $n \equiv 2 \bmod 4$, and the local field k has a 4th root of 1, then a self-dual generic representation is orthogonal if and only if -1 operates by 1.

(3) $Sp(2n)$: In this case, a self-dual generic representation is orthogonal if and only if -1 operates by 1.

(4) $SO(n)$: In this case, a self-dual generic representation is always orthogonal.

(5) Simply connected exceptional groups: In all cases except for E_7, all self-dual generic representation is orthogonal. In the case of E_7, a self-dual generic representation is orthogonal if and only if the center, which is $\mathbf{Z}/2$, operates trivially.

Remark. In the case of $Sp(2n)$, Proposition 2 does not directly apply, as there may not be an element in the group which operates on all simple roots by -1. However, as in [P1], one can give a proof by going to the symplectic similitude group, and again using Rodier's method of approximating (U, ψ) by compact groups.

Remark. In the case of $GL_n(\mathbf{F}_q)$, we were able to prove that *any*, not necessarily generic, self-dual representation of $GL_n(\mathbf{F}_q)$ is orthogonal. We have not been able to do this here in the p-adic case, though we believe it should be true. We also remark that there is another proof about the orthogonality of self-dual representations of GL_n in the p-adic in [PRam] that depended on the theory of new forms due to Jacquet, Piatetski-Shapiro, and Shalika. But the theory of new forms is also available only for generic representations.

Remark. In [P1], we constructed an example of a self-dual representation of $SL(6, \mathbf{F}_q)$ with $q \equiv 3 \bmod 4$ for which -1 acts trivially, even though the representation is symplectic. We also constructed an example of a self-dual representation of $SL(6, \mathbf{F}_q)$ with $q \equiv 3 \bmod 4$ for which -1 acts nontrivially, even though the representation is orthogonal. This construction remains valid for any p-adic field of residue field cardinality $q \equiv 3 \bmod 4$, and it works for any $SL(4m + 2)$.

4 A criterion for self-dual representations

The following lemma was observed in [P2].

Lemma 3. If G is a finite group, and H is a subgroup of G such that $g \to g^{-1}$ takes every double coset of G with respect to H to itself, then every irreducible representation of G with an H fixed vector is self-dual, and is in fact an orthogonal representation. □

Proof. Let π be an irreducible representation of G with an H fixed vector $v \in \pi$. Fix a G-invariant Hermitian form on π, and consider the matrix coefficient

$$f(g) = \langle gv, v \rangle.$$

Clearly, $f(h_1 g h_2) = f(g)$ for all h_1, h_2 in H. Therefore, since $g \to g^{-1}$ takes every double coset of G with respect to H to itself, it follows that $f(g) = f(g^{-1})$. However, $f(g^{-1})$ is a matrix coefficient of π^*, and by orthogonality relations, matrix coefficients of distinct representations are orthogonal. Therefore, $\pi \cong \pi^*$.

Since $g \to g^{-1}$ takes every double coset of G with respect to H to itself, it follows that (G, H) is a Gelfand pair, and the space of H-invariant vectors is a 1-dimensional nondegenerate subspace of π, and hence π, must be an orthogonal representation. ■

We extend this lemma to p-adic groups as follows. We have not been able to prove the last part of the previous lemma, which deals with orthogonal representations.

Lemma 4. Let H be a closed subgroup of a p-adic group G. Assume that any distribution on G that is H-bi-invariant, is invariant under the involution $g \to g^{-1}$. Then every irreducible admissible representation π of G with an H-invariant linear form on both π and π^* is self-dual. □

Proof. Let

$$\ell\colon \pi \to \mathbb{C}$$

$$\ell'\colon \pi^* \to \mathbb{C}$$

be nonzero H-invariant linear forms on π and its dual.

Let f be a locally constant, compactly supported function on G. The space of such functions operates on π in the standard way and operates also on the space of all linear forms on π. In particular, it makes sense to talk about $\pi(f)\ell$. It is easy to see that $\pi(f)\ell$ in fact belongs to the smooth dual π^* of π, and therefore it makes sense to define the distribution B on G by

$$B(f) = \ell'(\pi(f)\ell).$$

The distribution B is easily seen to be H-bi-invariant. The distribution B is called the relative character of G with respect to H. It was introduced by H. Jacquet.

Reversing the roles of π and π^*, one can define another distribution B' on G by

$$B'(f) = \ell(\pi(f)\ell').$$

If for a function f on G, we define $f^\vee$ to be the function

$$f^\vee(g) = f(g^{-1}),$$

it can be seen that

$$B'(f) = B(f^\vee).$$

However, by hypothesis, an H-bi-invariant distribution on G is invariant under the involution $g \to g^{-1}$, and therefore, $B(f) = B(f^\vee)$. So,

$$B'(f) = B(f).$$

Now we appeal to the following proposition, which says that the relative character of a representation of G characterises the representation to deduce that $\pi \cong \pi^*$.

Proposition 3. Let π_1 and π_2 be two irreducible admissible representations of a p-adic group G. Assume that for a subgroup H of G, all the representations $\pi_1, \pi_1^*, \pi_2, \pi_2^*$ have H-invariant linear forms. Then if a relative character of π_1 is equal to a relative character of π_2, then π_1 is isomorphic to π_2. □

Proof. We give a proof of this result, as it does not exist in the literature. (The author thanks J. Hakim for supplying this proof.) We prove that π_1 and π_2 are isomorphic by using a variant of the theorem that two irreducible representations are isomorphic if and only if a matrix coefficient of π_1 is also a matrix coefficient of π_2. The variant we use is to look at the distribution $v_1'(fv_1)$, where f belongs to the space $\mathcal{S}(G)$ of locally constant, compactly supported functions on G, and v_1 and v_1' are *nonzero* vectors in π_1 and π_1^*; the vector fv_1 is the result of the natural action of $f \in \mathcal{S}(G)$ on $v_1 \in \pi_1$. We call this distribution a *generalised matrix coefficient* of π_1. The variant we use is that if a generalised matrix coefficient of π_1 (which is a distribution on G) is the same as for π_2, then π_1 and π_2 are isomorphic. We outline the simple proof of this. For this, let $F_{v_1,v_1'}(g) = v_1'(gv_1)$ be the corresponding matrix coefficient on G. Clearly, the distribution we called a generalised matrix coefficient is given by

$$\phi \to \int F_{v_1,v_1'}\phi \, dg, \quad \phi \in \mathcal{S}(G).$$

Now it is clear that if a generalised matrix coefficient of π_1 is the same as one for π_2, then π_1 and π_2 share a matrix coefficient, and hence they are isomorphic.

Let

$$\ell_1\colon \pi_1 \to \mathbb{C}$$
$$\ell_1'\colon \pi_1^* \to \mathbb{C}$$
$$\ell_2\colon \pi_2 \to \mathbb{C}$$
$$\ell_2'\colon \pi_2^* \to \mathbb{C}$$

be nonzero H-invariant linear forms. We assume that

$$\ell_1'(\pi_1(f)\ell_1) = \ell_2'(\pi_2(f)\ell_2)$$

for all f in the space $\mathcal{S}(G)$ of locally constant, compactly supported functions on G. Observe that for any $f_1 \in \mathcal{S}(G)$, $v_1' = \pi_1(f_1)\ell_1 \in \pi_1^*$ and for any $f_2 \in \mathcal{S}(G)$, $v_1 = \pi_1^*(f_2)\ell_1' \in \pi_1$. Therefore, it makes sense to talk about $\left[\pi_1(f)v_1'\right](v_1)$. We choose f_1 and f_1' so that v_1 and v_1' are nonzero. We have

$$\begin{aligned}\left[\pi_1(f)v_1'\right](v_1) &= \left[\pi_1(f)\pi_1(f_1)\ell_1\right](\pi_1^*(f_2)\ell_1') \\ &= \left[\pi_1(f_2^\vee * f * f_1)\ell_1\right](\ell_1').\end{aligned}$$

Similarly, define $v_2' = \pi_2(f_1)\ell_2 \in \pi_2^*$ and $v_2 = \pi_2^*(f_2)\ell_2' \in \pi_2$. We have as before,

$$\left[\pi_2(f)v_2'\right](v_2) = \left[\pi_2(f_2^\vee * f * f_1)\ell_2\right](\ell_2').$$

Therefore, for all $f \in \mathcal{S}(G)$,

$$\left[\pi_1(f)v_1'\right](v_1) = \left[\pi_2(f)v_2'\right](v_2),$$

where v_1 and v_1' are nonzero vectors in π_1 and π_1^*. From the remark at the beginning of the proof of this proposition, π_1 and π_2 are isomorphic. ■

Corollary 3. An irreducible admissible representation of $GL_2(D)$, where D is a division algebra over a local field k that has a nonzero invariant form for the subgroup $H = D^* \times D^*$ sitting diagonally in $GL_2(D)$, must be self-dual and orthogonal. □

Proof. It was proved in [P2] that a distribution on G which is H bi-invariant is invariant under the involution $g \to g^{-1}$. It therefore suffices to check that whenever a representation of $GL_2(D)$ has an H-invariant linear form, its contragredient also has an H-invariant linear form. This is a simple consequence of the Kirillov theory developed in [PR].

We now prove that a self-dual representation π of $GL_2(D)$ containing an invariant form for the subgroup $H = D^* \times D^*$ must be orthogonal. For this, we use the result proved in [PR] that if π has a $D^* \times D^*$ invariant form, then it also has a Shalika form which is unique up to scalars. We recall that π is said to have a Shalika form $\ell\colon \pi \to \mathbb{C}$ if ℓ is left invariant by the diagonal matrices of the form (x, x), $x \in D^*$, and transforms under the upper triangular unipotent matrices $\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}$ by $\psi(\mathrm{tr}(X))$, where ψ is a nontrivial character on k. The method of Rodier, as also observed in [PR], applies to $GL_2(D)$, and gives a compact approximation to the pair $(U, \psi \cdot \mathrm{tr})$, and therefore for some compact group U_n, there is a character ψ_n that appears in π with multiplicity 1. This implies orthogonality of π by Lemma 1. ∎

We end the paper with the following question.

Question. Let π be an irreducible admissible *orthogonal* representation of $GL_2(D)$, where D is a division algebra over a local field k. Does π have an invariant form for the subgroup $H = D^* \times D^*$ sitting diagonally in $GL_2(D)$, and therefore a Shalika model?

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References

[P1] D. Prasad, *On the self-dual representations of a finite group of Lie type*, J. Algebra **210** (1998), 298–310.

[P2] ———, *Some remarks on representations of division algebras and Galois groups of local fields*, J. Number Theory **74** (1999), 73–97.

[PR] D. Prasad and A. Raghuram, *Kirillov theory for* $GL_2(D)$ *where* D *is a division algebra over a non-Archimedean local field*, preprint, 1999.

[PRam] D. Prasad and D. Ramakrishnan, *Lifting orthogonal representations to spin groups and local root numbers*, Proc. Indian Acad. (Math. Sci.) **105** (1995), 259–267.

[R] F. Rodier, "Modèle de Whittaker et charactères de représentations" in *Non-commutative Harmonic Analysis (Actes Colloq., Marseille-Luminy, 1974)*, Lecture Notes in Math. **466**, Springer-Verlag, Berlin, 1975, 151–171.

[Sh] J. Shalika, *The multiplicity one theorem for* GL_n, Ann. of Math. **100** (1974), 171–193.

[St] R. Steinberg, *Lectures on Chevalley Groups*, notes prepared by John Faulkner and Robert Wilson, Yale University, New Haven, CT, 1968.

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